A Rotating Frame Paradox in Quantum Mechanics

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Abstract

We consider a one particle quantum rotating system. We expect the probability densities at a point to be the same for a stationary and rotating frames of reference. We show this is not the case.

1 Introduction

Consider a frame of reference $F'$ with coordinates $r', t'$ rotating with constant angular velocity $\omega$ about the z axis of a frame of reference $F$ with coordinates $r, t$. The coordinates are related by

$$
\rho' = \rho \quad \varphi' = \varphi - \omega t \quad z' = z \quad t' = t
$$

With respect to $F$ let there be a quantum system of a particle with mass $m$ in a potential $V(r)$. For the wave function $\psi(r, t)$ with respect to $F$ let $\psi'(r', t')$ be the corresponding wave function with respect to $F'$. We expect the probability densities in the two frames are equal hence [1]

$$
|\psi'(r', t')|^2 = |\psi(r, t)|^2
$$

Consequently there is a real valued function $\beta(r, t)$ such that

$$
\psi'(r', t') = e^{-\frac{i}{\hbar}\beta(r, t)}\psi(r, t)
$$

2 Schrödinger Equations

With respect to $F$ the wave function $\psi(r, t)$ satisfies the Schrödinger equation

$$
-\frac{\hbar^2}{2m}\nabla^2 \psi(r, t) + V(r)\psi(r, t) = i\hbar\frac{\partial}{\partial t}\psi(r', t')
$$

The Lagrangian with respect to $F'$ is

$$
L' = \frac{1}{2}mv'^2 + mv' \cdot \omega \times r' + \frac{m}{2}(\omega \times r')^2 - V'(r')
$$

Construct the Hamiltonian from $L'$. The wave function $\psi'(r', t')$ then satisfies the Schrödinger equation

$$
-\frac{\hbar^2}{2m}\nabla'^2 \psi'(r', t') - \frac{1}{2}m\omega^2 r'^2 \psi'(r', t') + V'(r')\psi'(r', t') = i\hbar\frac{\partial \psi'}{\partial t'}(r', t')
$$

Now

$$
V'(r') = V(r) \quad \nabla' = \nabla \quad \frac{\partial}{\partial \varphi'} = \frac{\partial}{\partial \varphi} \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \varphi}
$$
On substituting (3) in (6) and using (1), (4), and (7) we have
\[
\left[ \frac{i\hbar}{2m} \nabla^2 \beta + \frac{1}{2m} \left( \nabla \beta \right)^2 - \frac{\omega}{2} \frac{\partial \beta}{\partial \varphi} - \frac{1}{2} m \omega^2 \rho^2 - \frac{\partial \beta}{\partial t} \right] \psi - i\hbar \frac{\partial \psi}{\partial \varphi} + i\hbar \frac{\nabla \psi}{m} = 0
\] (8)

Adding and subtracting (8), multiplied by \( \psi^* \), and its complex conjugate gives the two equations
\[
2 \left[ \frac{1}{2m} \left( \nabla \beta \right)^2 - \frac{\omega}{2} \frac{\partial \beta}{\partial \varphi} - \frac{1}{2} m \omega^2 \rho^2 - \frac{\partial \beta}{\partial t} \right] \psi \psi^* + \frac{i\hbar}{m} \nabla \beta \cdot \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right)

- i\hbar \omega \left( \psi^* \frac{\partial \psi}{\partial \varphi} - \psi \frac{\partial \psi^*}{\partial \varphi} \right) = 0
\] (9)
\[
\nabla \cdot \left( \psi \psi^* \nabla \beta \right) = m \omega \frac{\partial (\psi \psi^*)}{\partial \varphi}
\] (10)

3 No Solution to Equations

Choose \( V \) and \( \psi \) so that \( \psi \) has form \( \psi(\rho, z, t) \) and at \( z \) and \( t \) if \( \psi(\rho, z, t) \) is zero it is zero at a decrete set of \( \rho \). Assume there is a point \( p_0 = (\rho_0, z_0, t_0) \) such that \( \nabla \beta(p_0) \neq 0 \). We can also choose \( p_0 \) so that also \( \psi(p_0) \neq 0 \). We then have \( \psi(p_0) \psi^*(p_0) \nabla \beta(p_0) \neq 0 \). There is a curve with tangent vector \( \nabla \beta \) and containing \( p_0 \). Since the system is symmetric about the \( z \) axis following this curve from \( p_0 \) along the direction of \( \nabla \beta \) or in the opposite direction we will reach a point \( p_1 \) such that \( \nabla \beta(p_1) = 0 \).

From (10) and \( \partial \psi / \partial \varphi = 0 \) we have
\[
\nabla \cdot \left( \psi \psi^* \nabla \beta \right) = 0
\] (11)

hence
\[
\frac{\partial}{\partial s} \left[ \psi(s) \psi^*(s) \frac{\partial \beta}{\partial s}(s) \right] = 0
\] (12)

where \( s \) is the coordinate along \( \nabla \beta \). This implies
\[
\psi(p_0) \psi^*(p_0) \nabla \beta(p_0) = \psi(p_1) \psi(p_1) \nabla \beta(p_1) = 0
\] (13)

This is a contradiction hence \( \nabla \beta = 0 \). There is then a function \( f(t) \) such that \( \beta(r, t) = f(t) \). By (9) and form of \( \psi \)
\[
- \frac{1}{2} m \omega^2 \rho^2 - \dot{f} = 0
\] (14)

which does not hold. This \( \psi \) has then no solution for \( \beta \).

4 Conclusion

No solution implies that (2) does not hold. Consequently measuring position of the mass can give the mass is at a point in the stationary frame but is not at that point in the rotating frame of reference.

References