Ontology, epistemology, and quantum reality

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Abstract: The emergence of quantum mechanics in 1920s opened an intense discussion, which continues to these days, about its interpretation. This article aims to contribute to this discussion. First, a definition of ontic (really existing) and epistemic (pertaining to knowledge) states of a quantum system is proposed. Based on these definitions, the key concepts and postulates of quantum mechanics such as quantum state collapse, measurements and system properties, statistical inference, and key properties of quantum probability calculus are discussed. An alternative interpretation of degenerate ontic states is presented. The proposed ontological and epistemological framework for quantum mechanics is applied to explain Schrödinger's cat paradox, to redefine quantum entanglement, to illustrate quantum entanglement based on the Bohm's variant of the Einstein-Podolsky-Rosen (EPR) paradox, and to substantiate the principle of local causality. This framework is further compared with the quantum histories approach, quantum information approach, and spontaneous collapse approach.

Keywords: ontic and epistemic states, state collapse, measurement, quantum probability

1. Introduction

Despite spectacular successes of quantum physics in explaining observed phenomena, despite a leading role of quantum mechanics in modern technologies, especially in microelectronics, there is still no consensus among physicists and philosophers on how to interpret quantum theory. Thus, the number of papers attempting to provide a viable interpretation of quantum mechanics is still growing. An overview of these interpretations was provided by Herbert [13] and, more recently, by Bricmont [4]. One of the key questions in the on-going discussion of quantum theory is the question about the nature of quantum state, whether it represents an objective description of physical reality or just a tool for explaining observed phenomena. Another point that is hotly discussed is how to understand measurements performed on quantum systems and how to interpret their outcomes.

The struggle to understand quantum mechanics is rooted in two classical branches of philosophy: ontology and epistemology. While ontology relates to the existence, nature, and behavior of entities and their properties independently of any empirical information about them, epistemology is concerned with acquiring information (knowledge) about the world and using this information to gain contextual understanding of observed phenomena. According to Atmanspacher and Primas [2], "[f]*or a proper discussion of interpretations of quantum theory, Scheibe* [18] *introduced the notions of epistemic and ontic states of a system.*" Since then, many authors have used the terms ontic or epistemic states to provide an ontological or epistemological description of quantum systems. An overview of such descriptions can be found in [15].

In classical mechanics, to describe a system, for example, a system of N point-like particles interacting with each other, possibly influenced by external forces acting on them, one needs to explain how they behave in the $6N$-dimensional space $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$. Coordinates in the first factor, the configuration space, correspond to three spatial (position) coordinates of each particle. Coordinates in the second factor, the momentum space, correspond to three momentum components of each particle. The Cartesian product of the configuration space and momentum space defines the system space. In fact, the system may be identified with its system space.

The motion of particles can be described using Hamilton's equations defined by the Hamiltonian of the system. Hamilton's equations may impose constraints on the system. For example, the positions of particles may be constrained to points on a smooth manifold M in the configuration space. The phase space of such system is the cotangent bundle T^*M [1]. Each point of the phase space or the corresponding Dirac measure centered on the phase-space point is considered an ontic state of the system, and Hamilton's equations describe the behavior of phase space points. Probability measures on the phase space or on the system space are considered epistemic states. They describe the knowledge about the system. Obviously, it may happen that a Dirac measure is also an epistemic state when the exact positions and momenta of all particles are known.

Most physicist and philosophers agree that quantum states can be represented by normalized vectors in or density matrices on a Hilbert space associated with a quantum system. However, "[w]*hen a quantum state* $|\psi\rangle$ *is assigned to a physical system, does this mean that there is some independently existing property of the individual system that is in one-to-one correspondence with* $|\psi\rangle$ *(up to a global phase), or is* $|\psi\rangle$ *simply a mathematical tool for determining probabilities, existing only in the minds and calculations of quantum theorists?*" [15]. The first interpretation of a quantum state is an ontic interpretation supported by realists who seek to describe the nature and behavior of a quantum system independently of the information about the system. The second interpretation is an epistemic interpretation.

While in classical Hamilton mechanics, the space of ontic states (the phase space) is defined based on the system Hamiltonian, it appears that in quantum mechanics attempts are often made to define ontic states without taking the Hamiltonian of a quantum system and the resulting constraints into account. In this article, another approach is proposed: only the eigenspaces of the system Hamiltonian (or the corresponding orthogonal projection operators) are considered ontic states of the system. To avoid mathematical complexity, the Hilbert space associated with the system is assumed to be finite-dimensional throughout this article.

In section 2, the framework for the proposed ontological and epistemological interpretation of non-relativistic quantum mechanics is presented. The main concepts and postulates of quantum mechanics including (i) ontic and epistemic states, (ii) ontic and epistemic state collapse, (iii) measurements and system properties, (iv) statistical inference and key properties of quantum probability calculus, and (v) degenerate ontic states, are defined and discussed in section 3. In section 4, the proposed approach is applied to (i) interpret Schrödinger's cat paradox, (ii) examine exemplary quantum systems, (iii) redefine quantum entanglement, (iv) analyze EPR paradox, and (v) substantiate the principle of local causality. In section 5, each of (i) the quantum histories approach, (ii) the quantum information approach, and (iii) the spontaneous collapse approach is compared with the proposed interpretation of quantum mechanics.

2 Preliminaries

Consider a finite-dimensional Hilbert space $\mathcal{H}$ associated with a quantum system. In fact, the quantum system can be identified with its Hilbert space. The Hilbert space is a quantum analogue of the system space in classical mechanics. It provides a foundation for defining both the ontic and epistemic states of a quantum system.

Let $H(t)$ be a time-dependent Hamiltonian of the system and let $H_0, H_1, \dots, H_n$ be a sequence of Hamiltonians governing the system behavior in the respective time intervals $[t_0, t_1), [t_1, t_2), \dots$ $\dots, [t_n, t_{n+1})$, where $t_0 < t_1 < \cdots < t_{n+1}$. The change of the Hamiltonian may result from an interaction of the system with an external system, for example, with an external electric or magnetic field, or with particles of an external system. Further, the change of the Hamiltonian may occur when the system ceases to interact with an external system. An external system may be a measurement apparatus.

The change of the Hamiltonian $H(t)$ at t_m can be described by a term V_m defined by the interaction of the system with an external system:

$$H_m = H_{m-1} + V_m = H_0 + \sum_{k=1}^{m} V_k. \tag{1}$$

Sometimes, the interaction of a system of interest with an external system requires considering quantum states of the external system. An approach for dealing with such situations is outlined in the final paragraphs of section 4.3. For now, it is assumed that the Hilbert space $\mathcal{H}$ allows dealing with the states of the system of interest as well as the states of external systems interacting with it during some time intervals $[t_m, t_{m+1})$.

Each Hamiltonian H_m, $m = 0,1, \dots, n$, admits a spectral decomposition

$$H_m = \sum_{E \in \sigma(H_m)} E P_E^{H_m}, \tag{2}$$

where $\sigma(H_m)$ denotes the spectrum of H_m, $P_E^{H_m}$ denotes the projector (orthogonal projection operator) from $\mathcal{H}$ onto the eigenspace $\mathcal{H}_E^{H_m}$ associated with the eigenvalue $E \in \sigma(H_m)$, and $\sum_{E \in \sigma(H_m)} P_E^{H_m} = I$ is the identity operator on $\mathcal{H}$.

Let $|\varphi(t)\rangle$ be a normalized vector in the Hilbert space $\mathcal{H}$, a solution to the time-dependent Schrödinger equation satisfying an initial condition $|\varphi(t_0)\rangle$. At $t \in [t_m, t_{m+1})$, $0 \le m \le n$, this solution has the form

$$|\varphi(t)\rangle = U(t)|\varphi(t_0)\rangle = U_m(t - t_m)U_{m-1}(\Delta_{m-1}) \dots U_k(\Delta_k) \dots U_0(\Delta_0)|\varphi(t_0)\rangle, \tag{3}$$

where, for every $k = 0,1, \dots, m$, $\Delta_k = t_{k+1} - t_k$ and for every $\tau \in [0, \Delta_k)$,

$$U_k(\tau) = e^{-iH_k\tau/\hbar} \tag{4}$$

is a unitary operator on $\mathcal{H}$ describing the evolution of $|\varphi(t)\rangle$ during the time interval $[t_k, t_{k+1})$. Vector $|\varphi(t)\rangle$ is a continuous function of time such that for every $m = 1,2, \dots, n$

$$|\varphi(t_m)\rangle = \lim_{t \uparrow t_m} |\varphi(t)\rangle = U_{m-1}(\Delta_{m-1})|\varphi(t_{m-1})\rangle, \tag{5}$$

Let $S_1, \dots, S_n$ be subsets of the respective spectra $\sigma_1(H_1), \dots, \sigma_n(H_n)$. According to the Born rule, the probability $\mathbb{P}(H_n, S_n, \tau_n; \dots; H_1, S_1, \tau_1)$ that at $\tau_1 \in [t_1, t_2), \dots, \tau_n \in [t_n, t_{n+1})$ the system energies satisfy the conditions $E_1 \in S_1, \dots, E_n \in S_n$ is

$$\mathbb{P}(H_n, S_n, \tau_n; \dots; H_1, S_1, \tau_1|\varphi(t_0)) = \left\| P_{S_n}^{H_n} U_n(\tau_n - t_n) \dots \right.$$
$$\left. \dots U_2(t_3 - \tau_2) P_{S_2}^{H_2} U_2(\tau_2 - t_2) U_1(t_2 - \tau_1) P_{S_1}^{H_1} U_1(\tau_1 - t_1) U_0(\Delta_0)|\varphi(t_0)\rangle \right\|, \tag{6}$$

where

$$P_{S_m}^{H_m} = \sum_{E \in S_m} P_E^{H_m} \tag{7}$$

is the projector from $\mathcal{H}$ on the Hilbert subspace $\mathcal{H}_{S_m}^{H_m}$ of $\mathcal{H}$ which is the linear span of vectors comprised in the eigenspaces $\mathcal{H}_E^{H_m}$ such that $E \in S_m$. These probabilities are stationary in the sense that they are independent of the choice of times $\tau_1, \dots, \tau_n$ in the respective time intervals:

$$\mathbb{P}(H_n, S_n, \tau_n; \dots; H_1, S_1, \tau_1|\varphi(t_0)) = \mathbb{P}(H_n, S_n, t_n; \dots; H_1, S_1, t_1|\varphi(t_0))$$
$$= \left\| P_{S_n}^{H_n} U_{n-1}(\Delta_{n-1}) \dots P_{S_1}^{H_1} U_0(\Delta_0)|\varphi(t_0)\rangle \right\|^2. \tag{8}$$

3 Ontology and epistemology of quantum mechanics

3.1. Quantum ontic states and epistemic states

During each time interval $[t_m, t_{m+1})$, $0 \leq m \leq n$, the mutually orthogonal eigenspaces $\mathcal{H}_E^{H_m}$ associated with the eigenvalues $E \in \sigma(H_m)$ of the system Hamiltonian H_m are the ontic states of the quantum system. Such ontic states should be considered real states which exist independently of any knowledge about the system resulting from measurements. At any point of time, the system is in, i.e., "occupies" one and only one ontic state $\mathcal{H}_E^{H_m}$. When the system occupies an ontic state $\mathcal{H}_E^{H_m}$ at $t \in [t_m, t_{m+1})$, it occupies this state during the entire time interval $[t_m, t_{m+1})$. Ontic states are defined by solutions to the time-independent Schrödinger equation. For each time interval $[t_m, t_{m+1})$, the set of ontic states defines a Boolean algebra $\mathcal{B}_m$ with the usual operations of conjunction (intersection), disjunction (linear span), and negation (orthocomplementation). The elements of $\mathcal{B}_m$ are Hilbert subspaces $\mathcal{H}_S^{H_m}$ of the Hilbert space $\mathcal{H}$, which are the linear spans of vectors comprised in the eigenspaces $\mathcal{H}_E^{H_m}$ such that $E \in S$, where S is a subset of $\sigma(H_m)$ They may be called events. Then the ontic states $\mathcal{H}_E^{H_m}$ are considered elementary events (atoms) of $\mathcal{B}_m$. They are quantum analogues of points in the phase space of a classical mechanical system.

Normalized vectors in the Hilbert space $\mathcal{H}$ are epistemic states of the system. A subclass of epistemic states, the conditional epistemic states, is defined in section 3.4. At any point of time $t \in [t_m, t_{m+1})$, an epistemic state $|\varphi(t)\rangle$ should be interpreted based on its "ontic components" $P_E^{H_m}|\varphi(t)\rangle$ defined by the projectors $P_E^{H_m}$ in the spectral decomposition (2) of H_m:

$$|\varphi(t)\rangle = \sum_{E \in \sigma(H_m)} P_E^{H_m}|\varphi(t)\rangle. \tag{9}$$

Ontic components can be considered generalized probability amplitudes. Each ontic component defines the probability $\left\|P_E^{H_m}|\varphi(t)\rangle\right\|^2$ of finding the system described by an epistemic state $|\varphi(t)\rangle$ in the ontic state $\mathcal{H}_E^{H_m}$. These probabilities are stationary during the time interval $[t_m, t_{m+1})$ (Eq. (8)).

Epistemic states are solutions to the time-dependent Schrödinger equation. The solution in Eq. (3) is an unconditional epistemic state which is a continuous function of time (Eq. (5)). At any point of time $t \in [t_m, t_{m+1})$, an epistemic state depends on the initial epistemic state $|\varphi(t_0)\rangle$ and the system Hamiltonians $H_1, \dots, H_m$. Further, conditional epistemic states depend on the information obtained from measurements performed on the system (see section 3.4). Epistemic states describe one's knowledge about the system. They do not represent real (i.e., ontic) states. Nevertheless, it may happen that an epistemic state $|\varphi(t_m)\rangle = \left|\varphi_E^{H_m}\right\rangle$, where $\left|\varphi_E^{H_m}\right\rangle$ is an eigenvector of the system Hamiltonian H_m associated with an eigenvalue E. In this case, the epistemic state $|\varphi(t)\rangle$ has only one ontic component $P_E^{H_m}|\varphi(t)\rangle = \left|\varphi_E^{H_m}\right\rangle$, and the system is known to occupy the ontic state $\mathcal{H}_E^{H_m}$ with probability 1 during the time interval $[t_m, t_{m+1})$.

3.2. Ontic state collapse

An "ontic state collapse" is an essentially instantaneous process which may occur if and only if the Hamiltonian of the system changes. As a result of a change of the Hamiltonian at t_m, $1 \leq m \leq n$, ontic states $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$, $E_{m-1} \in \sigma(H_{m-1})$, one of which is occupied by the system during the time interval $[t_{m-1}, t_m)$, are replaced by ontic states $\mathcal{H}_{E_m}^{H_m}$, $E_m \in \sigma(H_m)$, one of which is occupied by the system during the time interval $[t_m, t_{m+1})$. If, during the time interval $[t_{m-1}, t_m)$, the system occupies an ontic state $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$ which is identical with an ontic state $\mathcal{H}_{E_m}^{H_m}$ for certain $E_m \in \sigma(H_m)$, then the system occupies this state $\mathcal{H}_{E_{m-1}}^{H_{m-1}} = \mathcal{H}_{E_m}^{H_m}$ during the subsequent time interval $[t_m, t_{m+1})$. There is no ontic state collapse. However, if $E_{m-1} \neq E_m$, the energy E_{m-1} of the ontic state $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$ is replaced by the energy E_m of the ontic state $\mathcal{H}_{E_m}^{H_m}$. Otherwise, if for every $E_m \in \sigma(H_m)$ the ontic state $\mathcal{H}_{E_m}^{H_m}$ is different from the ontic state $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$ occupied during the time interval $[t_{m-1}, t_m)$, the ontic state $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$ collapses at t_m to one of the ontic states $\mathcal{H}_{E_m}^{H_m}$. Such collapse means a real change of the ontic state of the system and system properties (see section 3.3). The probability of a collapse from a non-degenerate (i.e., one-dimensional) ontic state $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$ defined by an eigenvector $\left|\varphi_{E_{m-1}}^{H_{m-1}}\right\rangle$ to an ontic state $\mathcal{H}_{E_m}^{H_m}$ is

$$\mathbb{P}^m\left(\mathcal{H}_{E_m}^{H_m} | \mathcal{H}_{E_{m-1}}^{H_{m-1}}\right) = \left\| P_{E_m}^{H_m} \left|\varphi_{E_{m-1}}^{H_{m-1}}\right\rangle \right\|^2. \tag{10}$$

When $\mathcal{H}_{E_{m-1}}^{H_{m-1}}$ is degenerate, it is most often impossible to define the probability of a collapse from this ontic state to an ontic state $\mathcal{H}_{E_m}^{H_m}$. This is a fundamental limitation of quantum mechanics which makes it difficult to describe the behavior of a quantum system as a sequence of ontic states with well-defined transition probabilities, which indicates that transition probabilities are intrinsically epistemic.

An ontic state collapse at t_m has no effect on the epistemic state $|\varphi(t_m)\rangle$ of the system (Eq. (5)). However, since $H_m \neq H_{m-1}$, the change of the Hamiltonian may result in the way the epistemic state $|\varphi(t)\rangle$ evolves in time. As can be seen from Eq. (3), at t_m, the evolution operator U_{m-1} of the epistemic state $|\varphi(t)\rangle$ in the time interval $[t_{m-1}, t_m)$ is replaced by the evolution operator U_m of the epistemic state $|\varphi(t)\rangle$ in the time interval $[t_m, t_{m+1})$. In this way, the change of the Hamiltonian at t_m may affect the epistemic state at $t > t_m$, even when there is no state collapse.

3.3. Measurements and system properties

Observables of a quantum system are represented by self-adjoint operators on the Hilbert space $\mathcal{H}$ associated with the system and are identified with them. During each time interval $[t_m, t_{m+1})$,

$1 \leq m \leq n$, each property of a quantum system is an eigenvalue of an observable This property depends only on the ontic state $\mathcal{H}_E^{H_m}$ occupied by the system. Therefore, there must exist a function $\alpha: E \to \alpha(E)$ from $\sigma(H_m)$ onto the spectrum $\sigma(A)$ of A such that the spectral decomposition of A has the form

$$A = \sum_{E \in \sigma(H_m)} \alpha(E) P_E^{H_m}. \tag{11}$$

An operator A satisfying this condition is called "compatible with the Hamiltonian H_m." One should note that the compatibility condition proposed in Eq. (11) is stronger than the usual requirement that A and H_m commute with each other. The eigenvalue $\alpha(E)$ of an observable A compatible with H_m and corresponding to the ontic state $\mathcal{H}_E^{H_m}$ occupied by the system during the time interval $[t_m, t_{m+1})$ should be considered a real property of the system irrespective of whether it is known or not.

When a measurement of an observable incompatible with the Hamiltonian H_{m-1} starts at t_m, the interaction of the system with the measurement apparatus changes the system Hamiltonian. The new Hamiltonian H_m includes a new term V_m incompatible with H_{m-1}, describing the interaction of the system with the measurement apparatus (Eq. (1)). This can result in an ontic state collapse at t_m and leads to a change of system properties.

A measurement apparatus may include a readable memory for storing and pointers or displays for showing the outcome of a measurement, i.e., the measured system property. Properties which are stored in memory or shown by pointers or on displays are called "registered properties." The interaction of the system with the measurement apparatus occurs irrespective of whether a registered outcome of the measurement becomes known or not. The time at which an outcome of the measurement registered within the time interval $[t_m, t_{m+1})$ becomes known has no effect on the system. A registered system property may become known long after the end of the measurement process. When such property becomes known, it is known to be the system property during the entire time interval $[t_m, t_{m+1})$. A measurement apparatus may be arranged to register properties corresponding to many observables compatible with the Hamiltonian H_m.

It can be concluded based on Eq. (11) that the maximal number of linearly independent observables compatible with H_m is equal to the number of ontic states of the system. Such set will be called a "maximal set of observables compatible with H_m." The set of projectors $\{P_E^{H_m}: E \in \sigma(H_m)\}$ is an example of a maximal set of observables compatible with H_m. For each maximal set of observables compatible with H_m, every other observable compatible with H_m is a linear combination of observables from this set (see Eq. (11)). Each ontic state $\mathcal{H}_E^{H_m}$ of the system can be identified with (labelled by) its properties (quantum numbers), a unique set of eigenvalues of a suitable subset of a maximal set of linearly independent observables.

For some measurements one needs to consider quantum states of the measurement apparatus. For example, the system may undergo a transition from one ontic state to another due to the absorption of energy of a particle of the measurement apparatus interacting with the system of interest. In this case, the measurement apparatus must be included in the system. Dealing with such cases is discussed in the final paragraphs of section 4.3.

3.4. Statistical inference

An epistemic state $|\varphi(t)\rangle$ is determined by the initial epistemic state $|\varphi(t_0)\rangle$ and the system Hamiltonian $H(t)$. In addition, unlike ontic states, epistemic states may be further determined by known, registered outcomes of measurements performed on the system. Assume that the measurement of an observable A compatible with H_m, $1 \leq m \leq n$, gives an eigenvalue a. Then, during the time interval $[t_m, t_{m+1})$, the system is known to be in one of the ontic states $\mathcal{H}_E^{H_m}$

corresponding to an eigenvalue E of H_m such that $\alpha(E) = a$ (Eq. (11)), i.e., to have the property a. The knowledge of the measurement outcome allows replacing the epistemic state $|\varphi(t)\rangle$ at t_m with the "conditional epistemic state" $P_a^A|\varphi(t_m)\rangle/\|P_a^A|\varphi(t_m)\rangle\|$, where $P_a^A = \sum_{E\in\sigma(H_m):\alpha(E)=a} P_E^{H_m}$. The conditional epistemic state is defined by ontic components $P_E^{H_m}|\varphi(t)\rangle$ corresponding to ontic states $\mathcal{H}_E^{H_m}$ associated with the property a. During the time interval $[t_m, t_{m+1})$, $|\varphi(t)\rangle = U_m(t - t_m)\, P_a^A|\varphi(t_m)\rangle/\|P_a^A|\varphi(t_m)\rangle\|$. The replacement of the epistemic state $|\varphi(t_m)\rangle$ with the conditional epistemic state $P_a^A|\varphi(t_m)\rangle/\|P_a^A|\varphi(t_m)\rangle\|$ is often referred to as the reduction of state or the state collapse. In the present approach, it is called an "epistemic state collapse." An epistemic state collapse corresponds to a discontinuous evolution of the epistemic state. However, an epistemic state collapse does not represent a change of the ontic state or the properties of the system. It means an increase in the knowledge about the system resulting from learning a system property registered by the measurement apparatus. The probability of registering an eigenvalue a of A is determined by the ontic components $P_E^{H_m}|\varphi(t)\rangle$ of the epistemic state $|\varphi(t)\rangle$ of the system, corresponding to ontic states $\mathcal{H}_E^{H_m}$ associated with the property a: $\mathbb{P}(a) = \|P_a^A|\varphi(t)\rangle\|^2 = \|P_a^A|\varphi(t_m)\rangle\|^2$.

In general, for every $m = 1,2, \dots, n$, let $R_m = \left(a_{m1}, \dots, a_{mn_m}\right)$ be a sequence of n_m known eigenvalues of observables in the corresponding sequence $O_m = \left(A_{m1}, \dots, A_{mn_m}\right)$, registered by a measurement apparatus during the time interval $[t_m, t_{m+1})$. Then the information about the system at $t \in [t_n, t_{n+1})$ can be represented by the conditional epistemic state defined as

$$\left|\varphi\left(O_n, R_n, t; \dots O_1, R_1, t_1 \middle| \varphi(t_0)\right)\right\rangle = \frac{1}{N} U_n(t - t_n) P_{R_n}^{O_n} U_{\mathrm{n-1}}(\Delta_{n-1}) \dots P_{R_1}^{O_1} U_0(\Delta_0)|\varphi(t_0)\rangle, \qquad (12)$$

where

$$P_{R_m}^{O_m} = \begin{cases} \prod_{k=1}^{n_m} P_{a_{mk}}^{A_{mk}} \text{ when } n_m > 0 \\ I \text{ when } n_m = 0 \end{cases} \tag{13}$$

is the projector from $\mathcal{H}$ onto the intersection $\mathcal{H}_{a_{m1}}^{A_{m1}} \cap \ldots \cap \mathcal{H}_{a_{mk}}^{A_{mk}}$ when $n_m > 0$ or the identity operator on $\mathcal{H}$ when $n_m = 0$, i.e., when no system property becomes known,

$$P_{a_{mk}}^{A_{mk}} = \sum_{E \in \sigma(H_m): \alpha_{mk}(E) = a_{mk}} P_E^{H_m}, \tag{14}$$

is the projector from $\mathcal{H}$ onto $\mathcal{H}_{a_{mk}}^{A_{mk}}$, the linear span of subspaces $\mathcal{H}_E^{H_m}$ such that $\alpha_{mk}(E) = a_{mk}$, and

$$N = \left\| P_{R_n}^{O_n} U_{\mathrm{n}-1}(\Delta_{n-1}) \ldots P_{R_1}^{O_1} U_0(\Delta_0) |\varphi(t_0)\rangle \right\| \tag{15}$$

is the normalization constant. Alternatively,

$$P_{R_m}^{O_m} = P_{S_m}^{H_m} \tag{16}$$

is a projector (7) on an element $\mathcal{H}_{S_m}^{H_m}$ of the Boolean algebra $\mathcal{B}_m$, where

$$S_m = \begin{cases} \bigcap_{k=1}^{n_m} \{E \in \sigma(H_m): \alpha_{mk}(E) = a_{mk}\} \text{ when } n_m > 0 \\ \sigma(H_m) \text{ when } n_m = 0, \end{cases} \tag{17}$$

The projectors $P_{R_m}^{O_m}$ defined by Eq. (16) and (17) are identical with the projectors $P_{R_m}^{O_m}$ defined by Eq. (13) and (14). Equation (17) shows that conditional epistemic state (12) does not depend on the order of eigenvalues and observables in the sequences R_m and O_m.

According to the Born rule, given sequences $O_m = (A_{m1}, \ldots, A_{mn_m})$, $m = 1,2, \ldots, n$, of observables compatible with the respective Hamiltonians H_m, the joint probability that the properties of the system are given by the eigenvalues of these observables in the corresponding sequences $R_m = (a_{m1}, \ldots, a_{mn_m})$ is

$$\mathbb{P}\left(O_n, R_n, t_n; \dots; O_1, R_1, t_1 \middle| \varphi(t_0)\right) = \left\| P_{R_n}^{O_n} U_{\mathrm{n}-1}(\Delta_{n-1}) \dots P_{R_1}^{O_1} U_0(\Delta_0) |\varphi(t_0)\rangle \right\|^2$$

$$= \left\| P_{S_n}^{H_n} U_{\mathrm{n}-1}(\Delta_{n-1}) \dots P_{S_1}^{H_1} U_0(\Delta_0) |\varphi(t_0)\rangle \right\|^2 = \mathbb{P}\left(H_n, S_n, t_n; \dots; H_1, S_1, t_1 \middle| \varphi(t_0)\right). \tag{18}$$

Vectors

$$|O_n, R_n, t_n; \dots; O_1, R_1, t_1 | \varphi(t_0)\rangle = P_{R_n}^{O_n} U_{\mathrm{n}-1}(\Delta_{n-1}) \dots P_{R_1}^{O_1} U_0(\Delta_0) |\varphi(t_0)\rangle$$

$$= P_{S_n}^{H_n} U_{\mathrm{n}-1}(\Delta_{n-1}) \dots P_{S_1}^{H_1} U_0(\Delta_0) |\varphi(t_0)\rangle = |H_n, S_n, t_n; \dots; H_1, S_1, t_1 | \varphi(t_0)\rangle \tag{19}$$

will be called "probability vectors." Normalized non-zero probability vectors are identical with conditional epistemic states (12). However, while conditional epistemic states (12) describe the knowledge about the system represented by known system properties, the probability vectors (19) define probabilities (18) that the system has the specified properties in the respective time intervals, irrespective of whether these properties are registered by external systems interacting with the system or known.

The fact that quantum probabilities (18) are defined by probability vectors (19) is the reason for the most fundamental property of quantum probability calculus. Given $1 \le m \le n$, let $R_m^{(1)} = \left(a_{m1}^{(1)}, \dots, a_{mn_m}^{(1)}\right)$ and $R_m^{(2)} = \left(a_{m1}^{(2)}, \dots, a_{mn_m}^{(2)}\right)$ be two different sequences of properties of the system during the time interval $[t_m, t_{m+1})$, corresponding to a sequence of observables $O_m = \left(A_{m1}, \dots, A_{mn_m}\right)$ compatible with the Hamiltonian H_m, and let $S_m^{(1)}$ and $S_m^{(2)}$ be the corresponding subsets of the spectrum $\sigma(H_m\ )$ defined by Eq. (17). Since $R_m^{(1)}$ and $R_m^{(2)}$ are different, the sets $S_m^{(1)}$ and $S_m^{(2)}$ are disjoint. Therefore, it is straightforward to prove that

$$\left|H_n, S_n, t_n; \dots; H_m, S_m^{(1)} \cup S_m^{(2)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0)\right\rangle$$

$$= \left|H_n, S_n, t_n; \dots; H_m, S_m^{(1)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0)\right\rangle$$

$$+ \left|H_n, S_n, t_n; \dots; H_m, S_m^{(2)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0)\right\rangle \quad (20)$$

Consequently,

$$\left|O_n, R_n, t_n; \dots; O_m, R_m^{(1)} \vee R_m^{(2)}, t_{\mathrm{m}}; \dots; O_1, R_1, t_1 | \varphi(t_0)\right\rangle$$

$$= \left|O_n, R_n, t_n; \dots; O_m, R_m^{(1)}, t_{\mathrm{m}}; \dots; O_1, R_1, t_1 | \varphi(t_0)\right\rangle$$

$$+ \left|O_n, R_n, t_n; \dots; O_m, R_m^{(2)}, t_{\mathrm{m}}; \dots; O_1, R_1, t_1 | \varphi(t_0)\right\rangle \quad (21)$$

since, according to Eq. (17)), $P^{O_m}_{R_m^{(1)} \vee R_m^{(2)}} = P^{H_{\mathrm{m}}}_{S_m^{(1)} \cup S_m^{(2)}}$, where

$$R_m^{(1)} \vee R_m^{(2)} = \left(a_{m1}^{(1)} \vee a_{m1}^{(2)}, \dots, a_{mn_m}^{(1)} \vee a_{mn_m}^{(2)}\right) \quad (22)$$

and, for each $k = 1, \dots, n_m$, $a_{mk}^{(1)} \vee a_{mk}^{(2)}$ means that the property corresponding to A_{mk} is $a_{mk}^{(1)}$ or $a_{mk}^{(2)}$. Equation (20) shows that, unlike in classical probability calculus,

$$\mathbb{P}\left(H_n, S_n, t_n; \dots; H_m, S_m^{(1)} \cup S_m^{(2)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0) \middle| \varphi(t_0)\right)$$

$$\neq \mathbb{P}\left(H_n, S_n, t_n; \dots; H_m, S_m^{(1)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0) \middle| \varphi(t_0)\right)$$

$$+ \mathbb{P}\left(H_n, S_n, t_n; \dots; H_m, S_m^{(2)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0) \middle| \varphi(t_0)\right) \quad (23)$$

unless the probability vectors $\left|H_n, S_n, t_n; \dots; H_m, S_m^{(1)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0)\right\rangle$ and $\left|H_n, S_n, t_n; \dots; H_m, S_m^{(2)}, t_{\mathrm{m}}; \dots; H_1, S_1, t_1 | \varphi(t_0)\right\rangle$ are mutually orthogonal. The latter condition is a very special case and typically is not satisfied.

The property (21) of quantum probability calculus has numerous consequences. One of them is the quantum law of total probability

$$\left|O_n, R_n, t_n; \dots; (A_{m2}, \dots, A_{mn_m}), (a_{m2}, \dots, a_{mn_m}), t_m; \dots; O_1, R_1, t_1 \middle| \varphi(t_0)\right\rangle$$

$$= \left|O_n, R_n, t_n; \dots; (A_{m1}, \dots, A_{mn_m}), \left(a_{m1}^{(1)} \vee \dots \vee a_{m1}^{(d_m)}, a_{m2}, \dots, a_{mn_m}\right), t_m; \dots; O_1, R_1, t_1 \middle| \varphi(t_0)\right\rangle$$

$$= \sum_{a\in\sigma(A_{m1})} \left|O_n, R_n, t_n; \dots; (A_{m1}, \dots, A_{mn_m}), (a, a_{m2}, \dots, a_{mn_m}), t_m; \dots; O_1, R_1, t_1 \middle| \varphi(t_0)\right\rangle \qquad (24)$$

for every observable A_{m1} compatible with H_m. The quantum law of total probability is famously confirmed by the measurements of the distribution of particles diffracted by a plate with two slits although in this case the Hilbert space of the system is infinitely-dimensional.

The joint probability (18) can be also represented using the chain rule:

$$\mathbb{P}\left(O_n, R_n, t_n; \dots; O_1, R_1, t_1 \middle| \varphi(t_0)\right) = \mathbb{P}\left(O_n, R_n, t_n \middle| O_{n-1}, R_{n-1}, t_{n-1}; \dots; O_1, R_1, t_1; \varphi(t_0)\right)$$
$$\times \dots \mathbb{P}\left(O_2, R_2, t_2 \middle| O_1, R_1, t_1; \varphi(t_0)\right)\mathbb{P}\left(O_1, R_1, t_1 \middle| \varphi(t_0)\right), \qquad (25)$$

where the conditional probabilities in Eq. (25) are defined as

$$\mathbb{P}\left(O_m, R_m, t_m \middle| O_{m-1}, R_{m-1}, t_{m-1}; \dots; O_1, R_1, t_1; \varphi(t_0)\right)$$
$$= \begin{cases} 0 \text{ if } \left\| P_{R_{m-1}}^{O_{m-1}} U_{m-2}(\Delta_{m-2}) \dots P_{R_1}^{O_1} U_0(\Delta_0) |\varphi(t_0)\rangle \right\| = 0 \\ \dfrac{\left\| P_{R_m}^{O_m} U_{m-1}(\Delta_{m-1}) \dots P_{R_1}^{O_1} U_0(\Delta_0) |\varphi(t_0)\rangle \right\|^2}{\left\| P_{R_{m-1}}^{O_{m-1}} U_{m-2}(\Delta_{m-2}) \dots P_{R_1}^{O_1} U_0(\Delta_0) |\varphi(t_0)\rangle \right\|} \text{ otherwise.} \end{cases} \qquad (26)$$

The conditional probabilities (26) allow defining a non-stationary quantum Markov chain on the space of epistemic states, i.e., on the unit sphere in the Hilbert space $\mathcal{H}$ associated with the system. The elements of the Markov chain transition probability matrices $\mathbb{P}^{O_m}$ are defined as

$$\mathbb{P}^{O_m}\left(\frac{\psi}{\|\psi\|} \middle| \varphi\right) = \begin{cases} \langle\psi | U_{m-1}(\Delta_{m-1})\varphi\rangle \ \text{ if } \ |\psi\rangle = P_{R_m}^{O_m} U_{m-1}(\Delta_{m-1})|\varphi\rangle \text{ for a sequence } R_m \\ 0 \text{ otherwise.} \end{cases} \qquad (27)$$

The joint probability of a sequence (trajectory) of epistemic states $(\psi_n, \dots, \psi_1)$ is

$$\mathbb{P}^{O_n \dots O_1}(\psi_n; \dots; \psi_1|\varphi_0) = \mathbb{P}^{O_n}(\psi_n|\psi_{n-1}) \dots \mathbb{P}^{O_1}(\psi_1|\varphi_0). \tag{28}$$

For each vector $|\psi_m\rangle$ such that $\mathbb{P}^{O_m}(\psi_m|\psi_{m-1}) > 0$ there exists a sequence R_m and the corresponding conditional epistemic state (12) such that

$$|\psi_m\rangle = \left|\varphi\left(O_m, R_m, t_m; \dots O_1, R_1, t_1\middle|\varphi(t_0)\right)\right\rangle. \tag{29}$$

3.5. Degenerate ontic states

The projector on a one-dimensional eigenspace $\mathcal{H}_E^H$ representing an ontic state can be expressed using an arbitrary normalized eigenvector $|\varphi_E^H\rangle$ in $\mathcal{H}_E^H$ as

$$P_E^H = |\varphi_E^H\rangle\langle\varphi_E^H| \tag{30}$$

and the corresponding ontic component $P_E^H|\varphi\rangle$ of an epistemic state vector $|\varphi\rangle$ is

$$P_E^H|\varphi\rangle = \langle\varphi_E^H|\varphi\rangle|\varphi_E^H\rangle, \tag{31}$$

where $\langle\varphi_E^H|\varphi\rangle$ is a probability amplitude of the ontic component $P_E^H|\varphi\rangle$. Similar expressions for a projector onto, and an ontic component in a multi-dimensional eigenspace $\mathcal{H}_E^H$ representing an ontic state is possible. This can be achieved in the following ways.

First, let $\mathrm{U}(n_E^H)$ denote the group of unitary matrices U on $\mathcal{H}_E^H$. It is shown in Appendix A that the projector P_E^H can be represented as a continuous sum

$$P_E^H = n_E^H \int_{\mathrm{U}(n_E^H)} |U\varphi_E^H\rangle\langle U\varphi_E^H| dU, \tag{32}$$

where $|\varphi_E^H\rangle$ is an arbitrary normalized eigenvector in $\mathcal{H}_E^H$, $|U\varphi_E^H\rangle = U|\varphi_E^H\rangle$, and the integration is carried out over all elements of the unitary group $\mathrm{U}(n_E^H)$ with respect to the normalized Haar measure dU. Hence, the ontic component $P_E^H|\varphi\rangle$ of an epistemic state vector $|\varphi\rangle$ can be represented as a continuous sum of vectors $\langle U\varphi_E^H|\varphi\rangle|U\varphi_E^H\rangle$, i.e., as a continuous superposition of eigenvectors $|U\varphi_E^H\rangle$ in $\mathcal{H}_E^H$:

$$P_E^H|\varphi\rangle = n_E^H \int_{\mathrm{U}(n_E^H)} \langle U\varphi_E^H|\varphi\rangle |U\varphi_E^H\rangle dU. \qquad (33)$$

Second, it is further shown in Appendix B that the integration over the unitary group $\mathrm{U}(n_E^H)$ can be replaced by the integration over the unit sphere $S^{2n_E^H-1}$ in $\mathbb{R}^{2n_E^H}$. The integration is carried out with respect to the measure $d\mu_E^H(x)$ on $S^{2n_E^H-1}$ specified in Appendix B, as follows:

$$P_E^H = \frac{(n_E^H)!}{2\pi^{n_E^H}} \int_{S^{2n_E^H-1}} |\varphi_E^H(x)\rangle\langle\varphi_E^H(x)| d\mu_E^H(x), \qquad (34)$$

where the normalized eigenvectors $|\varphi_E^H(x)\rangle$ in $\mathcal{H}_E^H$ and the measure $d\mu_E^H(x)$ are parametrized using coordinates x of points on the unit sphere $S^{2n_E^H-1}$, for example, Cartesian or spherical coordinates. Hence, the ontic component $P_E^H|\varphi\rangle$ of an epistemic state vector $|\varphi\rangle$ can be represented as a continuous sum of vectors $\langle\varphi_E^H(x)|\varphi\rangle|\varphi_E^H(x)\rangle$, i.e., as a continuous superposition of eigenvectors $|\varphi_E^H(x)\rangle$ in $\mathcal{H}_E^H$:

$$P_E^H|\varphi\rangle = \frac{(n_E^H)!}{2\pi^{n_E^H}} \int_{S^{2n_E^H-1}} \langle\varphi_E^H(x)|\varphi\rangle|\varphi_E^H(x)\rangle d\mu_E^H(x). \qquad (35)$$

Since the projectors $|U\varphi_E^H\rangle\langle U\varphi_E^H|$ in (32) and $|\varphi_E^H(x)\rangle\langle\varphi_E^H(x)|$ in (34) as well as the vectors $\langle U\varphi_E^H|\varphi\rangle|U\varphi_E^H\rangle$ in (33) and $\langle\varphi_E^H(x)|\varphi\rangle|\varphi_E^H(x)\rangle$ in (35) are independent of the phase factor of the respective eigenvectors $|U\varphi_E^H\rangle$ and $|\varphi_E^H(x)\rangle$, the integration in Eq. (32) – (35) can be considered an integration over one-dimensional subspaces of $\mathcal{H}_E^H$, hereinafter called "rays." These rays may be considered one-dimensional ontic states of the system. Thus, $\langle U\varphi_E^H|\varphi\rangle|U\varphi_E^H\rangle$ in (33) and $\langle\varphi_E^H(x)|\varphi\rangle|\varphi_E^H(x)\rangle$ in (35) can be considered ontic components of the epistemic state $|\varphi\rangle$ having probability amplitudes, respectively, $\langle U\varphi_E^H|\varphi\rangle$ and $\langle\varphi_E^H(x)|\varphi\rangle$.

One can represent an epistemic state $|\varphi\rangle$ in $\mathcal{H}$ as a sum (where each integral is considered a continuous sum) of its one-dimensional ontic components,

$$|\varphi\rangle = \sum_{E\in\sigma(H_m)} n_E^H \int_{\mathrm{U}(n_E^H)} \langle U\varphi_E^H|\varphi\rangle |U\varphi_E^H\rangle dU \qquad (36)$$

or

$$|\varphi\rangle = \sum_{E\in\sigma(H_m)} \frac{(n_E^H)!}{2\pi^{n_E^H}} \int_{S^{2n_E^H-1}} \langle\varphi_E^H(x)|\varphi\rangle |\varphi_E^H(x)\rangle d\mu_E^H(x). \qquad (37)$$

as Eq. (32) – (35) are also satisfied for $n_E^H = 1$. Such representation has the advantage that it is unique in the sense that it does not depend on the choice of basis vectors in multi-dimensional eigenspaces $\mathcal{H}_E^H$. However, when the system occupies a degenerate state $\mathcal{H}_E^H$, it is not possible to determine by measurements of observables compatible with the system Hamiltonian (or any other observables), which ontic state corresponding to a ray in the degenerate eigenspace $\mathcal{H}_E^H$ is occupied. These rays are not elements of the Boolean algebra $\mathcal{B}_m$ of events defined in section 3.1. The properties of each ray in $\mathcal{H}_E^H$ are identical with the properties of $\mathcal{H}_E^H$.

Similar representation can be derived for probability vectors (19). They are useful for substantiating the principle of local causality as shown in section 4.5.

4 Applications

4.1. Schrödinger's cat paradox

Schrödinger's cat is a thought experiment devised by Schrödinger to illustrate a problem with the interpretation of quantum superposition. A cat is penned up in a chamber along with a flask of poison which is released when a Geiger counter detects radioactivity due to a decay of a radioactive atom. Thus, according to Schrödinger, after a while the state of the cat is a superposition of two states corresponding to a living and dead cat.

To provide an explanation of this apparent paradox, the first task is to rephrase the problem in terms of the proposed ontological and epistemological interpretation of quantum mechanics. Here is a possible interpretation. The quantum system comprises one "quantum object" called a cat, which at t_0 can occupy one of the two non-degenerate ontic states defined by the eigenvectors $|alive\rangle$ and $|dead\rangle$. Thus, the Hamiltonian of the system may have the form

$$H_0 = |alive\rangle\langle alive| - |dead\rangle\langle dead|. \tag{38}$$

During the time interval $[t_0, t_1)$, when no atom decays, the Hamiltonian does not change, and the cat stays in its initial ontic state $|alive\rangle$ or $|dead\rangle$ with a probability determined by the initial epistemic state

$$|\varphi(t)\rangle = \alpha_{alive}(t)|alive\rangle + \alpha_{dead}(t)|dead\rangle. \tag{39}$$

The probability that the cat is alive is $|\alpha_{alive}|^2$ and the probability that the cat is dead is $|\alpha_{dead}|^2$. The state of the cat can be measured by a measurement apparatus which is compatible with the Hamiltonian, for example, an apparatus for measuring the observable (projector) $|alive\rangle\langle alive|$ or $|dead\rangle\langle dead|$.

When an atom decays and the released poison interacts with the cat at t_1, this must have an effect on the system Hamiltonian. Suppose that

$$H_1 = |u\rangle\langle u| - |v\rangle\langle v| = |alive\rangle\langle dead| + |dead\rangle\langle alive|, \tag{40}$$

where

$$\begin{aligned} |u\rangle &= \frac{1}{\sqrt{2}}(|alive\rangle + |dead\rangle) \\ |v\rangle &= \frac{1}{\sqrt{2}}(|alive\rangle - |dead\rangle) \end{aligned} \tag{41}$$

are the eigenvectors of H_1 defining the new ontic states of the system. Hence, at t_1 an ontic state collapse takes place, and the cat must collapse from an initially occupied ontic state defined by the eigenvector $|alive\rangle$ or $|dead\rangle$ of H_0 to a new ontic state defined by the eigenvector $|u\rangle$ or $|v\rangle$ of H_1. Although the epistemic state $|\varphi(t)\rangle$ can be formally represented as a linear combination of vectors $|alive\rangle$ and $|dead\rangle$ also for $t \in [t_1, t_2)$, it is incorrect to base the interpretation of the system on such representation as these vectors are no longer eigenvectors of H_1 and thus, they do not define ontic states of the system. At any time $t \in [t_1, t_2)$, the epistemic state $|\varphi(t)\rangle$ can be represented as a sum of its ontic components $\langle u|\varphi(t)\rangle|u\rangle$ and $\langle v|\varphi(t)\rangle|v\rangle$

$$|\varphi(t)\rangle = \alpha_u(t)|u\rangle + \alpha_v(t)|v\rangle, \tag{42}$$

and should be interpreted based on the new ontic states defined by the eigenvectors $|u\rangle$ and $|v\rangle$.

4.2. More examples

While it is difficult to imagine what the new ontic states defined by the eigenvectors $|u\rangle$ and $|v\rangle$ mean in the thought-experiment proposed by Schrödinger (perhaps $|good\_ghost\rangle$ and $|bad\_ghost\rangle$), here is a similar example, where such interpretation is obvious. Consider a spin-1/2 particle which interacts with a magnetic field $\boldsymbol{B}$ along the z-axis of a coordinate system $Oxyz$ during the time interval $[t_0, t_1)$. The Hamiltonian of the particle is

$$H_0 = \varepsilon I - \frac{1}{2}\gamma\hbar|\boldsymbol{B}|\sigma_z, \tag{43}$$

where ε is the energy of the particle when the magnetic field is null, I is the identity operator on the Hilbert space $\mathcal{H}$ associated with the spin degrees of freedom of the particle, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices in the coordinate system $Oxyz$, and γ is the particle's gyromagnetic ratio. The ontic states of the system are defined by the non-degenerate eigenvectors of H_0: $|\boldsymbol{z}1\rangle$ and $|\boldsymbol{z}\bar{1}\rangle$. The quantum number $m = \pm 1$ in $|\boldsymbol{z}m\rangle$ is a real property of the particle, an eigenvalue of the observable σ_z compatible with H_0, defining the particle spin component in the direction $\boldsymbol{z}$.

Assume that at t_1, the magnetic field $\boldsymbol{B}$ becomes instantaneously reoriented along a unit vector $\boldsymbol{n}(\theta, \phi)$ having spherical coordinates θ, ϕ. After such reorientation, the Hamiltonian of the particle is

$$H_1 = \varepsilon I - \frac{1}{2}\gamma\hbar|\boldsymbol{B}|\boldsymbol{n}\cdot\boldsymbol{\sigma}. \tag{44}$$

The new ontic states of the system are defined by the non-degenerate eigenvectors of H_1:

$$\begin{aligned} |\boldsymbol{n}1\rangle &= \cos\frac{\theta}{2}|\boldsymbol{z}1\rangle + \ e^{i\phi}\sin\frac{\theta}{2}|\boldsymbol{z}\bar{1}\rangle \\ |\boldsymbol{n}\bar{1}\rangle &= -e^{-i\phi}\sin\frac{\theta}{2}|\boldsymbol{z}1\rangle + \ \cos\frac{\theta}{2}|\boldsymbol{z}\bar{1}\rangle. \end{aligned} \tag{45}$$

The quantum number $m = \pm 1$ in $|\boldsymbol{n}m\rangle$ is a real property of the particle, an eigenvalue of the observable $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ compatible with H_1, defining the particle spin component in the direction $\boldsymbol{n}$. During the time interval $[t_1, t_2)$, one should interpret the epistemic state $|\varphi(t)\rangle$ of the system based on its ontic components shown below:

$$|\varphi(t)\rangle = c_1^1(t)|\boldsymbol{n}1\rangle + c_{\bar{1}}^1(t)|\boldsymbol{n}\bar{1}\rangle. \tag{46}$$

The system occupies the ontic state defined by $|\boldsymbol{n}1\rangle$ with probability $|c_1^1(t_1)|^2$ or the ontic state defined by $|\boldsymbol{n}\bar{1}\rangle$ with probability $\left|c_{\bar{1}}^1(t_1)\right|^2$. However, using Eq. (45) one can express the epistemic state (46) as a superposition of vectors $|\boldsymbol{z}1\rangle$ and $|\boldsymbol{z}\bar{1}\rangle$:

$$|\varphi(t)\rangle = c_1^0(t)|\boldsymbol{z}1\rangle + c_{\bar{1}}^0(t)|\boldsymbol{z}\bar{1}\rangle. \tag{47}$$

It is incorrect to interpret the epistemic state during the time interval $[t_1, t_2)$, based on Eq. (47). One cannot say that the system occupies the ontic state defined by $|\boldsymbol{z}1\rangle$ with probability $|c_1^0(t)|^2$ or the ontic state defined by $|\boldsymbol{z}\bar{1}\rangle$ with probability $\left|c_{\bar{1}}^0(t)\right|^2$ because $|\boldsymbol{z}1\rangle$ and $|\boldsymbol{z}\bar{1}\rangle$ are not eigenvectors of H_1 and do not define ontic states of the particle.

Consider another example, a system comprising two identical spin-1/2 particles. The Hilbert space of this system is the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, each describing the spin states of one particle. The Hamiltonian of the system is

$$H_0 = (\varepsilon_1 I_1) \otimes I_2 + I_1 \otimes (\varepsilon_2 I_2) + \varepsilon_{12} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = (\varepsilon_1 + \varepsilon_2) I_1 \otimes I_2 + \varepsilon_{12} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \tag{48}$$

where, for each $k = 1, 2$, ε_k is the energy of the respective particle when it does not interact with the other particle, I_k is the identity operator on $\mathcal{H}_k$, $\boldsymbol{\sigma}_k = (\sigma_{kx}, \sigma_{ky}, \sigma_{kz})$ is the vector of Pauli matrices in a coordinate system $Oxyz$, the scalar product

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = \sigma_{1x} \otimes \sigma_{2x} + \sigma_{1y} \otimes \sigma_{2y} + \sigma_{1z} \otimes \sigma_{2z}, \tag{49}$$

defines the spin-spin interaction between the two particles, and ε_{12} is the spin-spin interaction energy constant. The spectral decomposition of the Hamiltonian is

$$H_0 = E_S P_S + E_T P_T. \tag{50}$$

Here

$$E_S = \varepsilon_1 + \varepsilon_2 - 3\varepsilon_{12} \tag{51}$$

is the energy of the singlet (non-degenerate) ontic state $\mathcal{H}_S$ defined by the eigenvector

$$|\boldsymbol{z}00\rangle = \frac{1}{\sqrt{2}} (|\boldsymbol{z}1\boldsymbol{z}\bar{1}\rangle - |\boldsymbol{z}\bar{1}\boldsymbol{z}1\rangle) \tag{52}$$

of H_0, and

$$P_S = |\boldsymbol{z}00\rangle\langle\boldsymbol{z}00| = \frac{1}{4} (I_1 \otimes I_2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \tag{53}$$

is the projector from $\mathcal{H}_1 \otimes \mathcal{H}_2$ onto $\mathcal{H}_S$. Next,

$$E_T = \varepsilon_1 + \varepsilon_2 + \varepsilon_{12} \tag{54}$$

is the energy of the triplet (threefold degenerate) ontic state $\mathcal{H}_T$ defined as the linear span of three linearly independent eigenvectors of H_0 in the eigenspace associated with the eigenvalue E_T, for example, as the linear span of vectors

$$
\begin{aligned}
|\boldsymbol{z}11\rangle &= |\boldsymbol{z}1\boldsymbol{z}1\rangle \\
|\boldsymbol{z}10\rangle &= \frac{1}{\sqrt{2}}(|\boldsymbol{z}1\boldsymbol{z}\bar{1}\rangle + |\boldsymbol{z}\bar{1}\boldsymbol{z}1\rangle) \\
|\boldsymbol{z}1\bar{1}\rangle &= |\boldsymbol{z}\bar{1}\boldsymbol{z}\bar{1}\rangle,
\end{aligned}
\tag{55}
$$

and

$$
P_T = I_1 \otimes I_2 - P_S = \frac{1}{4}(3I_1 \otimes I_2 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \tag{56}
$$

is the projector from $\mathcal{H}_1 \otimes \mathcal{H}_2$ onto $\mathcal{H}_T$. In the above formulas $|\boldsymbol{z}m_1\boldsymbol{z}m_2\rangle$ represents the tensor product $|\boldsymbol{z}m_1\rangle \otimes |\boldsymbol{z}m_2\rangle$ and $|\boldsymbol{z}m_k\rangle$, $k = 1, 2$, denotes the eigenvector of σ_{kz} associated with the eigenvalue $m_k = \pm 1$. During the time interval $[t_0, t_1)$, the system occupies one of the ontic states, $\mathcal{H}_S$ or $\mathcal{H}_T$.

It is worth pointing out that since the Hamiltonian (48) is invariant under rotations of the coordinate system, vector $\boldsymbol{z}$ in Eq. (52), (53) and (55) can be replaced by an arbitrary unit vector $\boldsymbol{n}$. In particular, it is straightforward to prove using Eq. (45) and (52) that $|\boldsymbol{z}00\rangle = |\boldsymbol{n}00\rangle$. For this reason, it is often said that the singlet state corresponds to a state in which the spins of the two particles are antiparallel. However, such interpretation is incorrect since $|\boldsymbol{n}1\boldsymbol{n}\bar{1}\rangle$ and $|\boldsymbol{n}\bar{1}\boldsymbol{n}1\rangle$ are not eigenvectors of H_0 and thus, they do not represent ontic states of the system. In fact, the singlet state can be represented as a linear combination of vectors $|\boldsymbol{a}m_1\boldsymbol{b}m_2\rangle$, where the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ define two arbitrary directions in space:

$$
|\boldsymbol{z}00\rangle = \sum_{m_1, m_2} \langle \boldsymbol{a}m_1\boldsymbol{b}m_2 | \boldsymbol{z}00\rangle |\boldsymbol{a}m_1\boldsymbol{b}m_2\rangle. \tag{57}
$$

The total spin number S defined by the eigenvalues $S(S+1)$ of the observable

$$
\boldsymbol{S}^2 = \frac{1}{4}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^2 = \frac{1}{2}(3I_1 \otimes I_2 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) = 2P_T \tag{58}
$$

compatible with H_0, is a property of the system: $S = 0$ in the singlet ontic state and $S = 1$ in the triplet ontic state. The quantum number S defines the value $\hbar S$ of the total spin of the system.

Further, it is worth pointing out that the observable

$$S_z = \frac{1}{2}(\sigma_{1z} \otimes I_2 + I_1 \otimes \sigma_{2z}) \tag{59}$$

is incompatible with H_0 as it cannot be represented as a linear combination of projectors P_S and P_T even though S_z commutes with H_0. In particular, S_z does not define a property of the system in the triplet ontic state. However, assume that at t_1, the system starts interacting with a magnetic field $\boldsymbol{B}$ along the z-axis of the coordinate system $Oxyz$. Based on Eq. (43), (48) and (50), the Hamiltonian of this system is

$$H_1 = H_0 - \frac{1}{2}\gamma\hbar|\boldsymbol{B}|(\sigma_{1z} \otimes I_2 + I_1 \otimes \sigma_{2z}) = E_S P_S + E_{T1} P_{T1} + E_{T0} P_{T0} + E_{T\bar{1}} P_{T\bar{1}}. \tag{60}$$

where

$$E_{TM} = E_T - \frac{1}{2}\gamma\hbar BM, \tag{61}$$

$$P_{TM} = |\boldsymbol{z}1M\rangle\langle\boldsymbol{z}1M|$$

$$= \begin{cases} \frac{1}{4}(I_1 + M\sigma_{1z}) \otimes (I_2 + M\sigma_{2z}) = \frac{1}{4}(I_1 \otimes I_2 + 2MS_z + \sigma_{1z} \otimes \sigma_{2z}) \text{ for } M \neq 0 \\ P_T - P_{T1} - P_{T\bar{1}} = \frac{1}{4}(I_1 \otimes I_2 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - 2\sigma_{1z} \otimes \sigma_{2z}) \text{ for } M = 0. \end{cases} \tag{62}$$

Hence, at t_1 the triplet state is replaced by three non-degenerate (excluding the case of an accidental degeneracy of $|\boldsymbol{z}00\rangle$ and $|\boldsymbol{z}1M\rangle$ for $M = 1$ or $\bar{1}$) ontic states defined by the eigenvectors $|\boldsymbol{z}1M\rangle$ given in Eq. (55). Observable S_z is compatible with H_1 and can be represented as a linear combination of projectors P_S, P_{T1}, P_{T0}, and $P_{T\bar{1}}$:

$$S_z = P_{T1} - P_{T\bar{1}}. \tag{63}$$

The eigenvalue $M = 1, 0, \bar{1}$ of S_z defines the value $\hbar M$ of the component of the total spin in the direction $\boldsymbol{z}$. The quantum numbers S and M are used to label the eigenvectors $|\boldsymbol{z}SM\rangle$ of H_1.

4.3. Quantum entanglement

Quantum entanglement is a physical phenomenon which occurs when the states of subsystems of a combined system cannot be described independently of each other. Consider a combined system comprising two subsystems. Suppose that the two subsystems interact with each other during the initial time interval $[t_0, t_1)$. The Hamiltonian of the combined system is

$$H_0 = h_1 \otimes I_2 + I_1 \otimes h_2 + h_{12}, \tag{64}$$

where, for each $k = 1, 2$,

$$h_k = \sum_{\varepsilon_k \in \sigma(h_k)} \varepsilon_k P_{\varepsilon_k}^{h_k} \tag{65}$$

is the Hamiltonian of the subsystem when it does not interact with the other subsystem, $\sigma(h_k)$ is the spectrum of h_k, $P_{\varepsilon_k}^{h_k}$ is the projector from the Hilbert space $\mathcal{H}_k$ associated with the respective subsystem onto the eigenspace $\mathcal{H}_{\varepsilon_k}^{h_k}$ of h_k corresponding to the eigenvalue ε_k, I_k is the identity operator on $\mathcal{H}_k$, and h_{12} is an operator on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ associated with the combined system, resulting from the interaction between the subsystems.

Assume that the subsystems cease to interact at t_1. Then the Hamiltonian of the combined system during the time interval $[t_1, t_2)$ is

$$H_1 = H_0 - h_{12} = h_1 \otimes I_2 + I_1 \otimes h_2, \tag{66}$$

and the ontic states of the combined system are tensor products $\mathcal{H}_{\varepsilon_1}^{h_1} \otimes \mathcal{H}_{\varepsilon_2}^{h_2}$ of eigenspaces $\mathcal{H}_{\varepsilon_1}^{h_1}$ and $\mathcal{H}_{\varepsilon_2}^{h_2}$ corresponding to eigenvalues ε_1 and ε_2 of the respective Hamiltonians h_1 and h_2. Eigenspaces $\mathcal{H}_{\varepsilon_1}^{h_1}$ and $\mathcal{H}_{\varepsilon_2}^{h_2}$ are ontic states of the respective subsystems. The energy of the ontic state $\mathcal{H}_{\varepsilon_1}^{h_1} \otimes \mathcal{H}_{\varepsilon_2}^{h_2}$ is $\varepsilon_1 + \varepsilon_2$. However, when $(\varepsilon_1, \varepsilon_2)$ and $(\varepsilon_1', \varepsilon_2')$ are energies of subsystems such that $\varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2'$, the subspaces $\mathcal{H}_{\varepsilon_1}^{h_1} \otimes \mathcal{H}_{\varepsilon_2}^{h_2}$ and $\mathcal{H}_{\varepsilon_1'}^{h_1} \otimes \mathcal{H}_{\varepsilon_2'}^{h_2}$ are considered two different ontic states.

An ontic state of a combined system is considered non-entangled if and only if it can be represented as a tensor product $\mathcal{H}_{\varepsilon_1}^{h_1} \otimes \mathcal{H}_{\varepsilon_2}^{h_2}$ of ontic states of its subsystems and the energy of the system in this state is equal to the sum $\varepsilon_1 + \varepsilon_2$ of subsystem energies. Hence, every ontic state of a combined system comprising mutually non-interacting subsystems is non-entangled.

When the interaction term h_{12} in H_0 is compatible with the H_1, or when it is incompatible but commutes with H_1, or when it does not commute with H_1, there exist entangled ontic states of the system, although some ontic states may be non-entangled. The entanglement of ontic states of such combined systems may also depend on other system characteristics such as distinguishability of particles comprised in the system.

Assume that the ontic state occupied by a combined system during the time interval $[t_0, t_1)$ is entangled. Then, according to the ontic state collapse postulate (section 3.2), when the subsystems of the combined system cease to interact with each other at t_1, the entangled ontic state occupied by the combined system collapses to a non-entangled ontic state $\mathcal{H}_{\varepsilon_1}^{h_1} \otimes \mathcal{H}_{\varepsilon_2}^{h_2}$.

Assume now that at t_2 the first subsystem begins interacting with an external system, for example, with a measurement apparatus for measuring an observable A incompatible with h_1. It is further assumed that the measurement apparatus does not interact with the second subsystem. Thus, the Hamiltonian of the combined system is

$$H_2 = h_1^A \otimes I_2 + I_1 \otimes h_2, \tag{67}$$

where h_1^A is the Hamiltonian of the first subsystem accounting for its interaction with the measurement apparatus. The Hamiltonian of the second subsystem is not affected by the measurement. Consequently, ontic states of the combined system one of which is occupied during the time interval $[t_2, t_3)$ are $\mathcal{H}_{\varepsilon_1^A}^{h_1^A} \otimes \mathcal{H}_{\varepsilon_2}^{h_2}$, where $\mathcal{H}_{\varepsilon_1^A}^{h_1^A}$ denotes an eigenspace of the Hamiltonian h_1^A corresponding to an eigenvalue ε_1^A. The ontic state collapse may take place only in the first subsystem. The ontic state of the second subsystem is not affected by the measurement performed on the first subsystem.

Similarly, when the second subsystem begins interacting with an external system and the external system does not interact with the first subsystem, the ontic state collapse may take place only in the second subsystem and the Hamiltonian as well as the ontic state of the first subsystem is not affected by this interaction.

An epistemic state of a combined system is considered a non-entangled epistemic state when it can be represented as a tensor product of epistemic states of subsystems. An entangled epistemic state can be represented only as a linear combination of two or more tensor products of epistemic states of subsystems.

While ontic states of a combined system are always non-entangled at t_1, when the subsystems cease to interact with each other, the ontic state collapse does not lead to a disentanglement of the epistemic state. If the initial epistemic state $|\varphi(t_0)\rangle$ of the combined system is entangled, then the epistemic state $|\varphi(t_1)\rangle = U_0(\Delta_0)|\varphi(t_0)\rangle$ is entangled and the epistemic state $|\varphi(t)\rangle = U_1(t - t_1)|\varphi(t_1)\rangle$ remains entangled also during the time interval $[t_1, t_2)$. Only information obtained during subsequent measurements of properties of subsystems may lead to a disentanglement of an entangled epistemic state of the combined system due to a collapse of the entangled epistemic state. The entanglement of an epistemic state means that the properties of ontic states of the first and second subsystem may be correlated. It appears that the lack of a clear distinction between ontic and epistemic states is the root-cause of the EPR paradox.

As mentioned in section 2 and 3.3, it may happen that quantum states of some external systems, possibly measurement apparatuses, must be taken into account when they interact with the system of interest. This situation occurs when ontic states of the system of interest and of external systems become entangled during their interaction. In this case, all these external systems must be included in the system together with the system of interest. Let $\mathcal{H}_1, \dots, \mathcal{H}_l$, where $1 \leq l \leq n$, be the Hilbert spaces associated with these external systems when they are isolated. The Hilbert space $\mathcal{H}$ of the system is now the tensor product $\mathcal{H}_0 \otimes \mathcal{H}_{ext}$ of the Hilbert space $\mathcal{H}_0$ of the system of interest and the Hilbert space $\mathcal{H}_{ext} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_l$ of the external systems. During the time intervals $[t_m, t_{m+1})$, when none of these external systems interacts with the system of interest, the Hamiltonian of the system is

$$H_m = H_{0,m} \otimes I_{ext} + I_0 \otimes H_{ext,m}, \tag{68}$$

where $H_{0,m}$ is the Hamiltonian of the system of interest, an operator on $\mathcal{H}_0$, $H_{ext,m}$ is the Hamiltonian of the external systems, an operator on $\mathcal{H}_{ext}$, I_0 is the identity operator on $\mathcal{H}_0$, and I_{ext} is the identity operator on $\mathcal{H}_{ext}$. Thus, the ontic states of the system are $\mathcal{H}_{E_{0,m}}^{H_{0,m}} \otimes \mathcal{H}_{E_{ext,m}}^{H_{ext,m}}$, where $\mathcal{H}_{E_{0,m}}^{H_{0,m}}$ is an ontic state of the system of interest, an eigenspace of $H_{0,m}$, and $\mathcal{H}_{E_{ext,m}}^{H_{ext,m}}$ is an ontic state of the external systems, an eigenspace of $H_{ext,m}$. Consequently, ontic states of the combined system comprising the system of interest and the external systems are non-entangled. Moreover, when a measurement is performed on the system of interest and one has no interest in the properties of the external systems, each observable compatible with H_m may be represented as tensor product $A \otimes I_{ext}$, where A is an operator on $\mathcal{H}_0$ representing an observable compatible with $H_{0,m}$. Nevertheless, one needs to remember that the epistemic state of the system may still be entangled.

During the remaining time intervals, when an external system interacts with the system of interest, the behavior of the system depends on many factors such as the number of external systems interacting with the system of interest and the nature of these interactions. The author believes that the results presented in this section are useful to deal with specific cases.

4.4. EPR paradox

To illustrate entanglement of ontic and epistemic quantum states, consider the Bohm's variant of the EPR thought-experiment. Assume that the combined system comprises two identical spin-1/2 particles "1" and "2" moving apart. Initially, the particles interact with each other via the spin-spin interaction. Such system is analyzed in section 4.2. The Hamiltonian H_0 of the system during the time interval $[t_0, t_1)$ is given in Eq. (48). However, the spin-spin interaction energy $\varepsilon_{12}(t)$ is now a function of time, and the absolute value of $\varepsilon_{12}(t)$ decreases as the distance between the particles increases. This has no effect on the ontic states of the combined system. Assume that $|\varphi(t_0)\rangle = |\boldsymbol{z}00\rangle$ (Eq. (52)) is the initial epistemic state of the combined system. Thus, during the time interval $[t_0, t_1)$, the ontic state of this system is known to be the singlet state $\mathcal{H}_S$ and the epistemic as well as the ontic state is entangled. The initial epistemic state evolves according to the equation $|\varphi(t)\rangle = U_0(t - t_0)|\boldsymbol{z}00\rangle$. Since the effect of $U_0(t - t_0)$ on $|\boldsymbol{z}00\rangle$ is multiplying $|\boldsymbol{z}00\rangle$ by the phase factor $e^{-i\int_{t_0}^{t_1}(\varepsilon_1+\varepsilon_2-3\varepsilon_{12}(t'))dt'/\hbar}$, it will be ignored as it has no influence on the information included in the epistemic state of the two particles.

Assume that the particles are so far apart from each other that they essentially cease to interact at t_1: $\varepsilon_{12}(t) = 0$ for $t \geq t_1$. The Hamiltonian of the two particles during the time interval $[t_1, t_2)$ is

$$H_1 = (\varepsilon_1 + \varepsilon_2) I_1 \otimes I_2. \tag{69}$$

Hence, at t_1 the combined system collapses from the singlet ontic state to the fourfold degenerate ontic state $\mathcal{H}_1 \otimes \mathcal{H}_2$ corresponding to the eigenvalue $\varepsilon_1 + \varepsilon_2$. The ontic state $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not entangled. The maximal set of linearly independent observables compatible with H_1 includes only one observable, e.g., the identity operator $I_1 \otimes I_2$. Thus, the system has one trivial property, the eigenvalue 1 of $I_1 \otimes I_2$. Meanwhile, the epistemic state $|\varphi(t)\rangle = U_1(t-t_1)|\boldsymbol{z}00\rangle$ is still entangled during the time interval $[t_1, t_2)$. Since the effect of $U_1(t-t_1)$ on $|\boldsymbol{z}00\rangle$ is multiplying $|\boldsymbol{z}00\rangle$ by the phase factor $e^{-i(\varepsilon_1+\varepsilon_2)(t-t_2)/\hbar}$, it will be ignored as it has no influence on the information included in the epistemic state of the two particles.

Consider the spin-measurement in a direction $\boldsymbol{a}$ performed on particle "1", which begins at t_2. Such measurement is performed by applying a magnetic field $\boldsymbol{B}$ in the direction $\boldsymbol{a}$ to particle "1". The Hamiltonian of particle "1" during the measurement is (see Eq. (44))

$$h_1^A = \varepsilon_1 I_1 - \frac{1}{2}\gamma\hbar|\boldsymbol{B}|\boldsymbol{a}\cdot\boldsymbol{\sigma}_1, \tag{70}$$

and the Hamiltonian of the two particles is

$$H_2 = h_1^A \otimes I_2 + \varepsilon_2 I_1 \otimes I_2. \tag{71}$$

The spectral decomposition of h_1^A is

$$h_1^A = \varepsilon_1(1) P_{\varepsilon_1(1)}^{h_1^A} + \varepsilon_1(\bar{1}) P_{\varepsilon_1(\bar{1})}^{h_1^A} \tag{72}$$

where

$$P_{\varepsilon_1(m_1)}^{h_1^A} = |\boldsymbol{a}m_1\rangle\langle\boldsymbol{a}m_1| \tag{73}$$

is the projector from $\mathcal{H}_1$ onto the eigenspace $\mathcal{H}_{\varepsilon_1(m_1)}^{h_1^A}$ defined by the eigenvector $|\boldsymbol{a}m_1\rangle$ of h_1^A corresponding to the eigenvalue

$$\varepsilon_1(m_1) = \varepsilon_1 - \frac{1}{2}\gamma_1\hbar B m_1, \tag{74}$$

and the spectral decomposition of H_2 is

$$H_2 = (\varepsilon_1(1) + \varepsilon_2) P_{\varepsilon_1(1)}^{h_1^A} \otimes I_2 + (\varepsilon_1(\bar{1}) + \varepsilon_2) P_{\varepsilon_1(\bar{1})}^{h_1^A} \otimes I_2 \tag{75}$$

As a result of the interaction with the measurement apparatus, the ontic state $\mathcal{H}_1$ occupied by particle "1" prior to the measurement collapses at t_2 to one of the two ontic states $\mathcal{H}_{\varepsilon_1(m_1)}^{h_1^A}$ corresponding to $m_1 = \pm 1$. The Hamiltonian h_2 and the ontic state $\mathcal{H}_2$ of particle "2" is not affected by the measurement. The ontic states of the combined system are $\mathcal{H}_{\varepsilon_1(m_1)}^{h_1^A} \otimes \mathcal{H}_2$.

The observable corresponding to particle "1" spin measurement is $A \otimes I_2$, where $A = \boldsymbol{a} \cdot \boldsymbol{\sigma}_1$. Observables A and $A \otimes I_2$ are compatible with the respective Hamiltonians (70) and (71) and

$$A = P_{\varepsilon_1(1)}^{h_1^A} - P_{\varepsilon_1(\bar{1})}^{h_1^A} \tag{76}$$

During the time interval $[t_2, t_3)$, the eigenvalue m_1 of A is a property of particle "1" (as well as of the combined system) and defines the particle "1" spin component in the direction $\boldsymbol{a}$. If the property m_1 registered by the measurement apparatus is known, the epistemic state of the two particles collapses from the entangled epistemic state $|\varphi(t_2)\rangle = |\boldsymbol{z}00\rangle = |\boldsymbol{a}00\rangle$ to a non-entangled conditional epistemic state

$$|\varphi(A, m_1, t_2|\boldsymbol{z}00)\rangle = \frac{P_{m_1}^A \otimes I_2 |\boldsymbol{a}00\rangle}{\left\| P_{m_1}^A \otimes I_2 |\boldsymbol{a}00\rangle \right\|} = |\boldsymbol{a} m_1 \boldsymbol{a} \bar{m}_1\rangle. \tag{77}$$

The probability of each outcome $m_1 = \pm 1$ of the measurement of A is

$$\mathbb{P}(A, m_1, t_2|\boldsymbol{z}00) = \left\| P_{m_1}^A \otimes I_2 |\boldsymbol{a}00\rangle \right\|^2 = \frac{1}{2}. \tag{78}$$

The conditional epistemic state $|\varphi(A, m_1, t_2|\boldsymbol{z}00)\rangle = |\boldsymbol{a} m_1 \boldsymbol{a} \bar{m}_1\rangle$ is a tensor product of epistemic states $|\boldsymbol{a} m_1\rangle$ and $|\boldsymbol{a} \bar{m}_1\rangle$ of the two particles. The epistemic state of particle "2" depends on the outcome m_1 of the measurement of A because the initial epistemic state $|\boldsymbol{z}00\rangle$ is entangled. Therefore, the outcome of a subsequent spin-measurement performed on particle "2" at t_3 may be correlated with the outcome of the measurement of the spin of particle "1" at t_2.

During the time interval $[t_2, t_3)$, the evolution of the conditional epistemic state $|\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle$ is given by $|\varphi(A, m_1, t|\boldsymbol{z}00)\rangle = U_2(t - t_2)|\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle$. Since the effect of $U_2(t - t_2)$ on $|\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle$ is multiplying $|\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle$ by the phase factor $e^{-i(\varepsilon_1(m_1)+\varepsilon_2)(t-t_2)/\hbar}$, it will be ignored as it has no influence on the information included in the epistemic state of the two particles.

Finally, consider the spin-measurement performed on particle "2" during the time interval $[t_3, t_4)$ by applying a magnetic field $\boldsymbol{B}$ in a direction $\boldsymbol{b}$ to particle "2". The Hamiltonian of particle "2" during the measurement is (see Eq. (44))

$$h_2^B = \varepsilon_2 I_2 - \frac{1}{2}\gamma\hbar|\boldsymbol{B}|\boldsymbol{b}\cdot\boldsymbol{\sigma}_2 \tag{79}$$

and the Hamiltonian of the two particles is

$$H_3 = h_1^A \otimes I_2 + I_1 \otimes h_2^B. \tag{80}$$

The spectral decomposition of h_1^B is

$$h_2^B = \varepsilon_2(1)P_{\varepsilon_2(1)}^{h_2^B} + \varepsilon_2(\bar{1})P_{\varepsilon_2(\bar{1})}^{h_2^B} \tag{81}$$

where

$$P_{\varepsilon_2(m_2)}^{h_2^B} = |\boldsymbol{b}m_2\rangle\langle\boldsymbol{b}m_2| \tag{82}$$

is the projector from $\mathcal{H}_2$ onto the eigenspace $\mathcal{H}_{\varepsilon_2(m_2)}^{h_2^B}$ defined by the eigenvector $|\boldsymbol{b}m_2\rangle$ of h_2^B corresponding to the eigenvalue

$$\varepsilon_2(m_2) = \varepsilon_2 - \frac{1}{2}\gamma\hbar|\boldsymbol{B}|m_2, \tag{83}$$

and the spectral decomposition of H_3 is

$$\begin{aligned} H_3 = &\left(\varepsilon_1(1) + \varepsilon_2(1)\right)P_{\varepsilon_1(1)}^{h_1^A} \otimes P_{\varepsilon_2(1)}^{h_2^B} + \left(\varepsilon_1(1) + \varepsilon_2(\bar{1})\right)P_{\varepsilon_1(1)}^{h_1^A} \otimes P_{\varepsilon_2(\bar{1})}^{h_2^B} \\ &+ \left(\varepsilon_1(\bar{1}) + \varepsilon_2(1)\right)P_{\varepsilon_1(\bar{1})}^{h_1^A} \otimes P_{\varepsilon_2(1)}^{h_2^B} + \left(\varepsilon_1(\bar{1}) + \varepsilon_2(\bar{1})\right)P_{\varepsilon_1(\bar{1})}^{h_1^A} \otimes P_{\varepsilon_2(\bar{1})}^{h_2^B}. \end{aligned} \tag{84}$$

As a result of the interaction with the measurement apparatus, the ontic state $\mathcal{H}_2$ of particle "2" occupied prior to the measurement collapses at t_3 to one of the two ontic states $\mathcal{H}_{\varepsilon_2(m_2)}^{h_2^B}$ corresponding to $m_2 = \pm 1$. The Hamiltonian h_1^A and the ontic state $\mathcal{H}_{\varepsilon_1(m_1)}^{h_1^A}$ of particle "1" is not affected by the measurement. The ontic states of the combined system are $\mathcal{H}_{\varepsilon_1(m_1)}^{h_1^A} \otimes \mathcal{H}_{\varepsilon_2(m_2)}^{h_2^B}$. The observable corresponding to particle "2" spin measurement is $I_1 \otimes B$, where $B = \boldsymbol{b} \cdot \boldsymbol{\sigma}_2$. Observables B and $I_1 \otimes B$ are compatible with the respective Hamiltonians (79) and (80) and

$$B = P_{\varepsilon_2(1)}^{h_2^B} - P_{\varepsilon_2(\bar{1})}^{h_2^B} \tag{85}$$

During the time interval $[t_3, t_4)$, the eigenvalue m_2 of B is a property of particle "2" (as well as of the combined system) and defines the particle "2" spin component in the direction $\boldsymbol{b}$. If the property m_2 registered by the measurement apparatus is known, the epistemic state of the two particles collapses from the non-entangled conditional epistemic state $|\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle$ to a non-entangled conditional epistemic state

$$|\varphi(B, m_2, t_3; A, m_1, t_2|\boldsymbol{z}00)\rangle = \frac{I_1 \otimes P_{m_2}^B |\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle}{\left\| I_1 \otimes P_{m_2}^B |\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle \right\|} = \frac{\langle \boldsymbol{b}m_2|\boldsymbol{a}\bar{m}_1\rangle}{\| \langle \boldsymbol{b}m_2|\boldsymbol{a}\bar{m}_1\rangle \|} |\boldsymbol{a}m_1\boldsymbol{b}m_2\rangle. \tag{86}$$

The conditional probability of the outcome m_2 of the measurement of B is

$$\mathbb{P}(B, m_2, t_3|A, m_1, t_2; \boldsymbol{z}00) = \left\| I_1 \otimes P_{m_2}^B |\boldsymbol{a}m_1\boldsymbol{a}\bar{m}_1\rangle \right\|^2 = |\langle \boldsymbol{b}m_2|\boldsymbol{a}\bar{m}_1\rangle|^2 = \frac{1}{2}(1 - m_1 m_2 \boldsymbol{a} \cdot \boldsymbol{b}) \tag{87}$$

and the joint probability of the outcomes m_1 at t_2 and m_2 at t_3 is

$$\begin{aligned} \mathbb{P}(B, m_2, t_3; A, m_1, t_2|\boldsymbol{z}00) &= \mathbb{P}(B, m_2, t_3|A, m_1, t_2; \boldsymbol{z}00)\mathbb{P}(A, m_1, t_2|\boldsymbol{z}00) \\ &= \left\| P_{m_1}^A \otimes P_{m_2}^B |\boldsymbol{z}00\rangle \right\|^2 = |\langle \boldsymbol{a}m_1\boldsymbol{b}m_2|\boldsymbol{z}00\rangle|^2 = \frac{1}{4}(1 - m_1 m_2 \boldsymbol{a} \cdot \boldsymbol{b}), \end{aligned} \tag{88}$$

as is well known from the literature. Equation (88) shows that the outcomes of the two measurements are correlated unless $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Eq. (87), (88) and, up to a phase factor, (86) also hold when particle “1” ceases to interact with the magnetic field after the first and before the second measurement. Then, during this time-period, the Hamiltonian of the two particles is given by Eq. (69). As a result, the ontic state $\mathcal{H}_{\varepsilon_1(m_1)}^{h_1^A}$ of particle “1” collapses to the ontic state $\mathcal{H}_1$. The ontic state of particle “2” remains $\mathcal{H}_2$, and the epistemic state of the combined system remains, up to a phase factor, $|\boldsymbol{a}m_1\boldsymbol{a}\overline{m}_1\rangle$.

The initial epistemic state $|\varphi(t_0)\rangle = |\boldsymbol{z}00\rangle$ can be considered a simple probability vector (19). It is a quantum analogue of an initial probability distribution of ontic states of a classical system. Like in classical systems, the correlation between the outcomes of two measurements performed independently on subsystems of a combined quantum system results from the entanglement of the initial state. Only the probability calculus used in quantum physics is different from the probability calculus used in classical physics.

4.5. Local causality principle

Consider the conditional epistemic state $|\varphi(A, m_1, t_2|\boldsymbol{z}00)\rangle = |\boldsymbol{a}m_1\boldsymbol{a}\overline{m}_1\rangle$ of the system discussed in the preceding section. One can argue that the information about the outcome m_1 of the measurement of particle “1” spin defining its new epistemic state $|\boldsymbol{a}m_1\rangle$ is transferred at t_2 to particle “2” because the epistemic state of particle “2” after the measurement is $|\boldsymbol{a}\overline{m}_1\rangle$. This might be considered the reason for the outcome m_2 of the subsequent measurement of particle “2” spin to be correlated with the outcome of the measurement of particle “1” spin. Such argument is incorrect. The correlation between the outcomes of these two measurements results from the entanglement of the initial epistemic state. No information is transferred between particles when they cease to interact. This can be seen using the argument presented below.

During the time interval $[t_1, t_2)$, for each $k = 1,2$, vectors $|\boldsymbol{n}_k m_k\rangle = |\boldsymbol{n}(\theta_{\boldsymbol{k}}, \phi_k) m_k\rangle$, $m_k = \pm 1$, defined by Eq. (45), are eigenvectors of the Hamiltonian $h_k = \varepsilon_k I_k$, and each vector describes the spin state of particle *"k"*. They represent points on the Bloch sphere, i.e., rays in $\mathcal{H}_k$ which are elements of the projective space $P(\mathcal{H}_k)$. It is straightforward to verify that the singlet epistemic state can be represented as a continuous sum of its ontic components defined by eigenvectors $|\boldsymbol{n}(\theta_1, \phi_1) m_1' \; \boldsymbol{n}(\theta_2, \phi_2) m_2'\rangle = |\boldsymbol{n}(\theta_1, \phi_1) m_1'\rangle \otimes |\boldsymbol{n}(\theta_2, \phi_2) m_2'\rangle$ of H_1 (Eq. 71)):

$$|\boldsymbol{z}00\rangle \; = \frac{1}{4\pi^2} \int_0^{\pi} \sin(\theta_1)\, d\theta_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} \sin(\theta_2)\, d\theta_2 \int_0^{2\pi} d\phi_2 \; |\boldsymbol{n}(\theta_1, \phi_1) m_1'\rangle \otimes |\boldsymbol{n}(\theta_2, \phi_2) m_2'\rangle$$
$$\times \langle \boldsymbol{n}(\theta_1, \phi_1) m_1' \; \boldsymbol{n}(\theta_2, \phi_2) m_2' | \boldsymbol{z}00\rangle \,, \qquad (89)$$

Equation (89) is satisfied for an arbitrary pair of numbers $m_1', m_2' = \pm 1$ since the integration in (89) runs over all points of $P(\mathcal{H}_1) \times P(\mathcal{H}_2)$.

Eq. (89) can be interpreted as follows. During the time interval $[t_1, t_2)$ the particles can occupy any non-entangled, one-dimensional ontic state $|\boldsymbol{n}(\theta_1, \varphi_1) m_1'\rangle \otimes |\boldsymbol{n}(\theta_2, \varphi_2) m_2'\rangle$ of the system. However, it is impossible to determine, which ontic state of the system is occupied after the collapse at t_1. Therefore, it results from the quantum law of total probability (Eq. (24) modified to represent a continuous sum of ontic states) that the probability vector (up to a phase factor) that the measurement of particle "1" spin at t_2 gives value m_1 is

$$|\mathrm{A}, m_1, t_2 | \boldsymbol{z}00\rangle = P_{\varepsilon_1(m_1)}^{h_1^A} \otimes I_2 |\boldsymbol{z}00\rangle$$

$$= \frac{1}{4\pi^2} \int_0^{\pi} \sin(\theta_1)\, d\theta_1 \int_0^{2\pi} d\varphi_1 \int_0^{\pi} \sin(\theta_2)\, d\theta_2 \int_0^{2\pi} d\varphi_2 \; \langle \boldsymbol{n}(\theta_1, \varphi_1) m_1' \; \boldsymbol{n}(\theta_2, \varphi_2) m_2' | \boldsymbol{z}00\rangle$$
$$\times P_{\varepsilon_1(m_1)}^{h_1^A} |\boldsymbol{n}(\theta_1, \varphi_1) m_1'\rangle \otimes |\boldsymbol{n}(\theta_2, \varphi_2) m_2'\rangle, \qquad (90)$$

and the joint probability vector (up to a phase factor) that the measurement of particle "1" spin at t_2 gives value m_1 and the measurement of particle "2" spin at t_3 gives value m_2 is

$$|B,m_2,t_3;\mathrm{A},m_1,t_2|\boldsymbol{z}00\rangle = I_2 \otimes P^{h_2^B}_{\varepsilon_2(m_2)}|\mathrm{A},m_1,t_2|\boldsymbol{z}00\rangle = P^{h_1^A}_{\varepsilon_1(m_1)} \otimes P^{h_2^B}_{\varepsilon_2(m_2)}|\boldsymbol{z}00\rangle$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\sin(\theta_1)\,d\theta_1\int_0^{2\pi}d\varphi_1\int_0^{\pi}\sin(\theta_2)\,d\theta_2\int_0^{2\pi}d\varphi_2\,\langle \boldsymbol{n}(\theta_1,\varphi_1)m_1'\,\boldsymbol{n}(\theta_2,\varphi_2)m_2'|\boldsymbol{z}00\rangle$$

$$\times P^{h_1^A}_{\varepsilon_1(m_1)}|\boldsymbol{n}(\theta_1,\varphi_1)m_1'\rangle \otimes P^{h_2^B}_{\varepsilon_2(m_2)}|\boldsymbol{n}(\theta_2,\varphi_2)m_2'\rangle. \tag{91}$$

Equation (91) shows that the outcome of each measurement and its probability is determined exclusively by the one-dimensional ontic state occupied by the system on which the measurement is performed. The correlation between the outcomes of the measurements is determined by the probability amplitudes $\langle \boldsymbol{n}(\theta_1,\varphi_1)m_1'\,\boldsymbol{n}(\theta_2,\varphi_2)m_2'|\boldsymbol{z}00\rangle$ and results from the entanglement of the initial state of the system.

In general, using the methods described in Appendix A or B, one can show that every epistemic state $|\varphi\rangle$ of a combined system comprising two non-interacting subsystems, the Hamiltonian of which is given in Eq. (66), can be written as a continuous sum of ontic components associated with normalized eigenvectors $\left|U_1\varphi^{h_1}_{\varepsilon_1}\right\rangle\otimes\left|U_2\varphi^{h_2}_{\varepsilon_2}\right\rangle$ or $\left|\varphi^{h_1}_{\varepsilon_1}\right\rangle\otimes\left|\varphi^{h_2}_{\varepsilon_2}\right\rangle$ defining one-dimensional ontic states of the combined system:

$$|\varphi\rangle = \sum_{\varepsilon_1\in\sigma(h_1)}\sum_{\varepsilon_2\in\sigma(h_2)} n^{h_1}_{\varepsilon_1}n^{h_2}_{\varepsilon_2}\int_{\mathrm{U}(n^{h_1}_{\varepsilon_1})}\int_{\mathrm{U}(n^{h_2}_{\varepsilon_2})}\left\langle U_1\varphi^{h_1}_{\varepsilon_1}\otimes U_2\varphi^{h_2}_{\varepsilon_2}\middle|\varphi\right\rangle\left|U_1\varphi^{h_1}_{\varepsilon_1}\right\rangle\otimes\left|U_2\varphi^{h_2}_{\varepsilon_2}\right\rangle dU_1 dU_2, \tag{92}$$

where $\left|\varphi^{h_1}_{\varepsilon_1}\right\rangle$ and $\left|\varphi^{h_2}_{\varepsilon_2}\right\rangle$ are arbitrary eigenvectors in $\mathcal{H}^{h_1}_{\varepsilon_1}$ and $\mathcal{H}^{h_2}_{\varepsilon_2}$, the integration variables U_1 and U_2 are matrices of the respective unitary groups $\mathrm{U}(n^{h_1}_{\varepsilon_1})$ and $\mathrm{U}(n^{h_2}_{\varepsilon_2})$ acting on eigenspaces $\mathcal{H}^{h_1}_{\varepsilon_1}$ and $\mathcal{H}^{h_2}_{\varepsilon_2}$, and the integration is with respect to the normalized Haar measures dU_1 and dU_2 on these groups, or

$$|\varphi\rangle = \sum_{\varepsilon_1\in\sigma(h_1)} \sum_{\varepsilon_2\in\sigma(h_2)} \frac{\left(n_{\varepsilon_1}^{h_1}\right)!\left(n_{\varepsilon_2}^{h_2}\right)!}{4\pi^{\left(n_{\varepsilon_1}^{h_1}+n_{\varepsilon_1}^{h_1}\right)}} \int_{S^{2n_{\varepsilon_1}^{h_1}-1}} \int_{S^{2n_{\varepsilon_1}^{h_1}-1}} \langle \varphi_{\varepsilon_1}^{h_1}(x_1) \otimes \varphi_{\varepsilon_2}^{h_2}(x_2) | \varphi \rangle$$

$$\times \left|\varphi_{\varepsilon_1}^{h_1}(x_1)\right\rangle \otimes \left|\varphi_{\varepsilon_2}^{h_2}(x_2)\right\rangle d\mu_{\varepsilon_1}^{h_1}(x_1)\, d\mu_{\varepsilon_2}^{h_2}(x_2) \tag{93}$$

where the eigenvectors $\left|\varphi_{\varepsilon_1}^{h_1}(x_1)\right\rangle$ in $\mathcal{H}_{\varepsilon_1}^{h_1}$ and $\left|\varphi_{\varepsilon_2}^{h_2}(x_2)\right\rangle$ in $\mathcal{H}_{\varepsilon_2}^{h_2}$ as well as the measures $d\mu_{\varepsilon_1}^{h_1}(x_1)$ and $d\mu_{\varepsilon_2}^{h_2}(x_2)$ are parametrized using coordinates x_1 and x_2 of points on the respective unit spheres $S^{2n_{\varepsilon_1}^{h_1}-1}$ and $S^{2n_{\varepsilon_1}^{h_1}-1}$, over which they are integrated. Therefore, outcomes of measurements performed on subsystems and their probabilities depend only on the one-dimensional ontic states occupied by the subsystem on which the measurements are performed, and the correlation between measurements performed on these subsystems is determined by the respective probability amplitudes $\left\langle U_1\varphi_{\varepsilon_1}^{h_1} \otimes U_2\varphi_{\varepsilon_2}^{h_2} \middle| \varphi) \right\rangle$ or $\left\langle \varphi_{\varepsilon_1}^{h_1} \otimes \varphi_{\varepsilon_2}^{h_2} \middle| \varphi \right\rangle$. However, since it is not known which ontic states in each degenerate eigenspace $\mathcal{H}_{\varepsilon_1}^{h_1}$ and $\mathcal{H}_{\varepsilon_2}^{h_2}$ are occupied, one must use the continuous analogue of the quantum law of total probability (24) to calculate probabilities of these outcomes. Similar arguments for substantiating the principle of local causality are presented in [20] although without a reference to ontic and epistemic states of the system.

5 Comparison with other interpretations of quantum mechanics

While the author will not attempt to explain how the proposed ontological and epistemological interpretation of quantum mechanics, hereinafter referred to as the "present approach," aligns with or differs from every other interpretation, at least due to the vast number of such interpretations, he will make a few observations he deems particularly useful or relevant.

In the quantum histories approach proposed by Griffiths [12] and further co-developed by Omnes [16], and Gell-Mann and Hartle [10], the fundamental concept is a "quantum history" of a physical system defined as "*a sequence of quantum events at successive times, where a quantum event at a particular time can be any quantum property of the system in question. Thus given a set of times $t_1 < t_2 < \cdots < t_f$, a quantum history is specified by a collection of projectors $(F_1, F_2, \dots F_f)$, one projector for each time.*" (section 8.3 in [12]). The chain operator corresponding to a quantum history is constructed by inserting unitary time-development operators between every pair of consecutive projectors. It is used to calculate weights (probabilities) of quantum histories (section 10.1 in [12]).

In the present approach, each probability vector $|O_n, R_n, t_n; \dots; O_1, R_1, t_1|\varphi(t_0)\rangle$ (Eq. (19)) allows defining a quantum history $P_{R_1}^{O_1}, \dots, P_{R_n}^{O_n}, |\varphi(t_0)\rangle\langle\varphi(t_0)|$, as well as the corresponding chain operator. However, quantum histories introduced in this way are limited to those defined by observables compatible with the system Hamiltonians $H_1, \dots, H_n$ and by eigenvalues of these observables. These eigenvalues, the system properties, are defined for the entire time intervals $[t_m, t_{m+1})$. Based on Eq. (16) and (17), each projector $P_{R_m}^{O_m} = P_{S_m}^{H_m}$ corresponds to a subspace $\mathcal{H}_{S_m}^{H_m}$ of the Hilbert space $\mathcal{H}$, i.e., to an event of the Boolean algebra $\mathcal{B}_m$ defined in section 3.1. Thus, it is possible to define a Boolean algebra on the set of quantum histories $P_{R_1}^{O_1}, \dots, P_{R_n}^{O_n}, |\varphi(t_0)\rangle\langle\varphi(t_0)|$ defined by observables compatible with the system Hamiltonians. This is not the case in the space of quantum histories considered by Griffith, where it is not always possible to define conjunction or disjunction of two quantum histories from this space (section 8.4 in [12]). Further, the consistency condition defined in section 10.2 in [12] is generally not satisfied, even by histories corresponding to different sequences R_m of eigenvalues of given sequences O_m of observables compatible with H_m, as such consistency condition requires that the probability vectors (19), which define these different quantum histories, are mutually orthogonal. Therefore, it is not possible to define a probability measure on the Boolean algebra of quantum histories, as shown in section 3.4.

While the present approach is based on a rigorous definition of ontic states of a quantum system as eigenspaces of the system Hamiltonians, and the properties of the system are associated with these ontic states, it is difficult to conclude what an ontic state in the quantum history approach might be. Perhaps it is "[a] *physical property of a quantum system* [which] *is associated with a subspace* $\mathcal{P}$ *of the quantum Hilbert space* $\mathcal{H}$ *(...) and the projector P onto* $\mathcal{P}$" (section 4.1 in [12]) or the space of quantum histories defined by sequences of these properties. But then, there seems to be too many ontic states.

In the quantum information approach such as QBism, an abbreviation for Quantum Bayesianism, originally developed by Fuchs, Schack and Mermin [7, 8], the notion of an ontic state is typically not used. In this approach, only the information (knowledge) about the system is analyzed. In the present approach, this information is represented by epistemic states. QBists use epistemic states to describe how quantum systems evolve in time and to calculate probabilities of measurement outcomes which comply with observations and experiments. One may say that the main object of study in QBism are the quantum Markov chains on the space of epistemic states in the Hilbert space associated with the system, defined by the Markov chain transition probability matrices (22). Such processes are investigated in [5]. While, in the opinion of the author, the description of quantum systems in QBism approach is correct, it misses ontological aspects of quantum mechanics, which are useful for a more complete understanding of quantum phenomena. This deficiency is sometimes addressed by considering quantum Markov chains with hidden states [6], where such hidden states might play the role of ontic states. However, this seems impossible because such approach leads to a violation of the key properties of quantum probability derived in section 4.1.

The present approach may be also considered a starting point for describing random interactions of a quantum system with its environment. In this case, the terms V_m of the Hamiltonians H_m in Eq. (1) cause the quantum system to undergo random, spontaneous collapses from time to time. The outcomes of measurements performed on such quantum systems are the ensemble or time averages of outcomes of measurements performed on individual quantum systems. Such averages explain macroscopic observations. Nevertheless, the evolution of each quantum system is still unitary, albeit stochastic. Such approach has been used, for example, to describe spin-lattice interactions in magnetic resonances and magnetic relaxation processes [9].

Another spontaneous collapse theory developed by Pearle and others [12, 17] proposes to model a classical behavior of an individual quantum system by including nonlinear, stochastic terms in the Schrödinger equation governing the system evolution. Such theories, however, are outside the scope of the present approach.

The author believes the reader will recognize the similarities of the present approach to, as well as the differences between the present approach and other interpretations of quantum mechanics, including the Copenhagen interpretation and quantum logic interpretation.

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Appendix A

Let $|\varphi_1\rangle, \ldots, \left|\varphi_{n_E^H}\right\rangle$ be basis vectors of $\mathcal{H}_E^H$ chosen in such a way such that $|\varphi_1\rangle = |\varphi_E^H\rangle$, where $|\varphi_E^H\rangle$ is an eigenvector of the Hamiltonian H corresponding to an eigenvalue E. For every pair of normalized vectors $|\varphi\rangle$ and $|\psi\rangle$ in $\mathcal{H}_E^H$

$$n_E^H \int_{\mathrm{U}(n_E^H)} \langle\psi|U\varphi_E^H\rangle\langle U\varphi_E^H|\varphi\rangle dU = n_E^H \sum_{k,l} \langle\psi|\varphi_l\rangle\langle\varphi_k|\varphi\rangle \int_{\mathrm{U}(n_E^H)} \langle\varphi_\mathrm{l}|U\varphi_1\rangle\langle U\varphi_1|\varphi_k\rangle dU$$

$$= \sum_k \langle\psi|\varphi_k\rangle\langle\varphi_k|\varphi\rangle = \langle\psi|\varphi\rangle \tag{A1}$$

since using the invariant integration method described in [1][1] one can see that

$$\int_{\mathrm{U}(n_E^H)} \langle\varphi_\mathrm{l}|U\varphi_1\rangle\langle U\varphi_1|\varphi_k\rangle dU = \frac{\delta_{kl}}{n_E^H}\,. \tag{A2}$$

Equation (A1) proves Eq. (32) and (33).

Appendix B

A finite-dimensional Hilbert space $\mathcal{H}_E^H$ is identical with $\mathbb{C}^n$ for $n = n_E^H$. Let $\omega: z \to \omega(z)$ be a mapping from $\mathbb{C}^n$ into $\mathbb{R}^{2n}$ defined as

$$\omega(z) = \omega(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n), \tag{B1}$$

where $z_k = x_k + iy_k$ for $k = 1, \ldots, n$. ω establishes a one-to-one correspondence between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. Let $(x_1, y_1, \ldots, x_n|x_1', y_1', \ldots, x_n', y_n')$ denote the inner product in $\mathbb{R}^{2n}$ defined as

$$(x_1, y_1, \dots, x_n | x_1', y_1', \dots, x_n', y_n') = \sum_{k=1}^{n} (x_k x_k' + y_k y_k'). \tag{B2}$$

The inner product $\langle z | z' \rangle$ in $\mathbb{C}^n$ is

$$\langle z | z' \rangle = \sum_{k=1}^{n} \big(x_k x_k' + y_k y_k' + i(x_k y_k' - y_k x_k')\big). \tag{B3}$$

It is easy to verify that

$$\langle z | z' \rangle = (\omega(z) | \omega(z')) + i(\omega(z) | \omega(iz')) = (\omega(z) | \omega(z')) - i\big(\omega(iz) \big| \omega(z')\big) \tag{B4}$$

Thus, the integral of the product $\langle z' | z \rangle \langle z | z'' \rangle$ over the unit sphere $S(\mathbb{R}^{2n})$ in $\mathbb{R}^{2n}$ can be calculated in the following way

$$\int_{S(\mathbb{R}^{2n})} \langle z' | z \rangle \langle z | z'' \rangle d\mu$$

$$= \int_{S(\mathbb{R}^{2n})} \big[(\omega(z') | \omega(z))(\omega(z) | \omega(z'')) - i\big(\omega(iz') \big| \omega(z)\big)\big(\omega(z), \omega(z'')\big)$$

$$+ i\big(\omega(z') \big| \omega(z)\big)\big(\omega(z) \big| \omega(iz'')\big) + (\omega(iz') | \omega(z))(\omega(z) | \omega(iz''))\big] d\mu$$

$$= \frac{\pi^n}{n!} \big[(\omega(z') | \omega(z'')) - i\big(\omega(iz') \big| \omega(z'')\big) + i\big(\omega(z') \big| \omega(iz'')\big) + (\omega(iz') | \omega(iz''))\big], \tag{B5}$$

where $d\mu$ denotes the standard surface measure on the unit sphere $S(\mathbb{R}^{2n})$ in $\mathbb{R}^{2n}$ normalized as

$$\int_{S(\mathbb{R}^{2n})} d\mu = \frac{2\pi^n}{(n-1)!}, \tag{B6}$$

and where the formula

$$\int_{S(\mathbb{R}^{2n})} (a | \mu)(\mu | b) d\mu = \frac{\pi^n}{n!} (a | b) \tag{B7}$$

for integrating products of inner products $(a | \mu)(\mu | b)$ of unit vectors a, b, μ in $\mathbb{R}^{2n}$ over a unit sphere $S(\mathbb{R}^{2n})$ was used [14].

Further, since

$$(\omega(iz')|\omega(iz'')) = (\omega(z')|\omega(z'')). \tag{B8}$$

one obtains using (B4):

$$(\omega(z')|\omega(z'')) - i\big(\omega(iz')\big|\omega(z'')\big) + i\big(\omega(z')\big|\omega(iz'')\big) + \big(\omega(iz')\big|\omega(iz'')\big) = 2\langle z|z'\rangle$$

Hence,

$$\int_{S(\mathbb{R}^{2n})} \langle z'|z\rangle\langle z|z''\rangle d\mu = \frac{2\pi^n}{n!}\langle z'|z''\rangle, \tag{B9}$$

which proves Eq. (34) and (35).

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