Analyzing several equations concerning various aspects of Quantum Mechanics, some Ramanujan parameters and the developments of the MRB Constant. New possible mathematical connections with some parameters of Number Theory

Michele Nardelli¹, Antonio Nardelli

Abstract

In this paper, we analyze several equations concerning various aspects of Quantum Mechanics, some Ramanujan parameters and the developments of the MRB Constant. We describe new possible mathematical connections with some parameters of Number Theory.

¹ M.Nardelli studied at Dipartimento di Scienze della Terra Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10 - 80138 Napoli, Dipartimento di Matematica ed Applicazioni “R. Caccioppoli” - Università degli Studi di Napoli “Federico II” – Polo delle Scienze e delle Tecnologie Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

A. Nardelli studies at the Università degli Studi di Napoli Federico II - Dipartimento di Studi Umanistici – Sezione Filosofia - scholar of Theoretical Philosophy
We have:

\[
\frac{G_N E^2}{\hbar c^5} \Rightarrow \frac{G_N E^2}{\hbar c^5} \times \left(\frac{\hbar c}{E}\right)^2 \leq \ell_s
\]

\[\Delta x \Delta p \geq \hbar\]
\[(G\cdot E^2)/(h\cdot c^5) \cdot (((h\cdot c)/E)\cdot 1/l)^2\]

**Input**

\[
\frac{G \cdot e^2}{h \cdot c^5} \left( \frac{h \cdot c}{e} \times \frac{1}{l} \right)^2
\]

**Result**

\[
\frac{G \cdot h}{l^2 \cdot c^3}
\]

**Roots**

\[
G = 0, \quad cl \neq 0
\]

\[
h = 0, \quad cl \neq 0
\]

**Property as a function**

Parity

even

**Derivative**

\[
\frac{\partial}{\partial c} \left( \frac{(G \cdot e^2) \left( \frac{h \cdot c}{e \cdot l} \right)^2}{h \cdot c^5} \right) = -\frac{3 \cdot G \cdot h}{c^3 \cdot l^2}
\]
Indefinite integral

\[ \int \frac{Gh}{c^3 l^2} \, dc = -\frac{Gh}{2c^2 l^2} + \text{constant} \]

Limit

\[ \lim_{l \to \pm \infty} \frac{Gh}{c^3 l^2} = 0 \]

Alternative representation

\[ \left( \frac{\hbar}{\epsilon l} \right)^2 \left( G e \right)^2 \left( \frac{Gh}{\exp(z) l} \right)^2 \left( \frac{\exp(z)}{\hbar c^5} \right) \quad \text{for } z = 1 \]

Series representations

\[ \frac{Gh}{c^3 l^2} = \sum_{n=0}^{\infty} \frac{(-1 + l)^n (-1)^n (Gh (1 + n))}{c^3} \quad \text{for } |\pm 1 + l| < 1 \]

\[ \frac{Gh}{c^3 l^2} = \sum_{n=0}^{\infty} \frac{(-1 + c)^n ((-1)^n Gh (1 + n) (2 + n))}{2 l^2} \quad \text{for } |\pm 1 + c| < 1 \]

\[ \frac{Gh}{c^3 l^2} = \sum_{n=-\infty}^{\infty} \left( \begin{array}{c} \frac{Gh}{c^2} \quad n = -3 \\ 0 \quad \text{otherwise} \end{array} \right) c^n \]

|z| is the absolute value of z.
Definite integral over a hypersphere of radius R

\[ \iiint_{\rho^2 + C^2 + h^2 + l^2 < R^2} \frac{G \hbar}{c^3 \ell^2} \, d\rho \, dG \, dh \, dl = 0 \]

We have:

\[ \alpha = \frac{(G \cdot E^2)}{(h \cdot c^5)} \]

For:

\[ \alpha = \frac{(G \cdot E^2)}{(h \cdot c^5)} \]

we obtain:

\[ \left( \frac{G \cdot E^2}{h \cdot c^5} \right) \left[ \frac{(h \cdot c^2)}{E} \right] \left[ \frac{1}{l^2} \right] \]

\[ \left( \frac{G \cdot E^2}{(G \cdot E^2) / \alpha} \right) \left[ \frac{(h \cdot c^2)}{E} \right] \left[ \frac{1}{l^2} \right] \]
Input

\[ \frac{G e^2}{\alpha} \left( \frac{\hbar c}{e} \times \frac{1}{l} \right)^2 \]

Exact result

\[ \frac{\alpha c^2 h^2}{\epsilon^2 l^2} \]

Roots

\[ c = 0, \ l \neq 0 \]

\[ h = 0, \ l \neq 0 \]

\[ \alpha = 0, \ l \neq 0 \]

Property as a function

Parity

even

Derivative

\[ \frac{\partial}{\partial c} \left( \frac{(G e^2) \left( \frac{\hbar c}{\epsilon l} \right)^2}{\frac{G e^2}{\alpha}} \right) = \frac{2 \alpha c h^2}{\epsilon^2 l^2} \]
Indefinite integral

\[
\int \frac{c^2 h^2 a}{e^2 l^2} \, dc = \frac{α c^3 h^2}{3 e^2 l^2} + \text{constant}
\]

From:

\[
\frac{G h}{l^2 c^3}
\]

and:

\[
\frac{α c^2 h^2}{e^2 l^2}
\]

we obtain:

\[
\frac{(G*h)/(l^2*c^3)}{(G*h)/(l^2*c^3) * 1/(((α*c^2*h^2)/(E^2*l^2)))}
\]

Input

\[
\frac{G h}{l^2 c^3} \times \frac{1}{α c^2 h^2} \frac{e^2 l^2}{c^2 l^2}
\]

Result

\[
\frac{G e^2}{α h c^5}
\]
Root
\( \alpha \neq 0, \quad G = 0 \)

Property as a function
Parity
odd

Derivative

\[
\frac{\partial}{\partial c} \left( \frac{G h}{(c^2 + \alpha^2 h^2)} \right) = -\frac{5 e^2 G}{\alpha c^6 h}
\]

Indefinite integral

\[
\int \frac{e^2 G}{c^5 h} \, dc = -\frac{e^2 G}{4 \alpha c^4 h} + \text{constant}
\]

Limit

\[
\lim_{\alpha \to \infty} \frac{e^2 G}{c^5 h} = 0
\]

Alternative representation

\[
\frac{G h}{(\alpha c^2 h^2 + t^2 \alpha^2)} = \frac{G h}{\exp^2(\pi t^2)} \quad \text{for} \quad z = 1
\]
Series representations

\[
\frac{G \hbar}{\left(\alpha \frac{c^2 \hbar^2}{e^2 t^2}\right)} = \frac{G \sum_{k=0}^{\infty} \frac{2^k}{k!}}{c^5 \hbar \alpha}
\]

\[
\frac{G \hbar}{\left(\alpha \frac{c^2 \hbar^2}{e^2 t^2}\right)} = \frac{G \left(\sum_{k=0}^{\infty} \frac{1}{k!}\right)^2}{c^5 \hbar \alpha}
\]

\[
\frac{G \hbar}{\left(\alpha \frac{c^2 \hbar^2}{e^2 t^2}\right)} = \frac{G}{c^5 \hbar \alpha \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\right)^2}
\]

\(n!\) is the factorial function

We have

\[
\frac{G e^2}{\alpha \hbar c^5}
\]

\((6.67430\times 10^{-11}\times 1.054571817\times 10^{-34}\times 299792458^5)\)

for \(E = X\)

\((6.67430\times 10^{-11}\times (X)^2)/(1/137\times 1.054571817\times 10^{-34}\times 299792458^5)\)

Input interpretation

\[
\frac{6.67430 \times 10^{-11} \ X^2}{\frac{1}{137} \times 1.054571817 \times 10^{-34} \times 299792458^5}
\]
Result
$3.58052 \times 10^{-17} X^2$

Plot (figure that can be related to an open string)

Geometric figure
line

Alternate form assuming $X$ is real
$3.58052 \times 10^{-17} X^2 + 0$

Root
$X = 0$

Polynomial discriminant
$\Delta = 0$

Property as a function
Parity
even
Derivative

\[
\frac{d}{dX} (3.58052 \times 10^{-17} X^2) = 7.16105 \times 10^{-17} X
\]

Indefinite integral

\[
\int 3.58052 \times 10^{-17} X^2 \, dX = 1.19351 \times 10^{-17} X^3 + \text{constant}
\]

Global minimum

\[
\min\{3.58052 \times 10^{-17} X^2\} = 0 \text{ at } X = 0
\]

Definite integral after subtraction of diverging parts

\[
\int_0^\infty \left(3.58052 \times 10^{-17} X^2 - 3.58052 \times 10^{-17} X^2\right) \, dX = 0
\]

We consider:

\[
E = 0.510998995 \times 299792458^2
\]

from

\[
\frac{6.67430 \times 10^{-11} X^2}{\frac{1}{137} \times 1.054571817 \times 10^{-34} \times 299792458^5}
\]

for \(X = E\), we obtain:
\[
\frac{(6.67430 \times 10^{-11} \times (0.510998995 \times 299792458^2)^2)}{(1/137 \times 1.054571817 \times 10^{-34} \times 299792458^5)}
\]

**Input interpretation**

\[
\frac{6.67430 \times 10^{-11} \times (0.510998995 \times 299792458^2)^2}{\frac{1}{137} \times 1.054571817 \times 10^{-34} \times 299792458^5}
\]

**Result**

\[
7.5521310764142600971918261897058768763123868174133044187367... \times 10^{16}
\]

\[7.552131076... \times 10^{16}\]

From which, performing the ln and after some calculations, we obtain:

\[
\frac{55 \times 1}{\ln((6.67430 \times 10^{-11} \times (0.510998995 \times 299792458^2)^2)/(1/137 \times 1.054571817 \times 10^{-34} \times 299792458^5))} - 5 + \frac{1}{3\sqrt{\pi}} C_{MRB}
\]

**Input interpretation**

\[
55 \times \frac{1}{\log \left( \frac{6.67430 \times 10^{-11} \times (0.510998995 \times 299792458^2)^2}{\frac{1}{137} \times 1.054571817 \times 10^{-34} \times 299792458^5} \right)} - 5 + \frac{1}{3\sqrt{\pi}} C_{MRB}
\]

\(\log(x)\) is the natural logarithm

\(C_{MRB}\) is the MRB constant

**Result**

\[1.6180535308545234399701233352668694422910281160258154673299935606\]

\[...
1.61805353...\] result that is a very good approximation to the value of the golden ratio \[1.618033988749...\]
Possible closed forms

\[ \Phi + 1 \approx 1.61803398 \]

\[ \sqrt{\frac{3 \mathcal{K}_1}{2}} \approx 1.6180570118 \]

\[ 93 (\bar{V}) \approx 1.618036249 \]

\[ \frac{61}{12 \pi} \approx 1.618075254 \]

\[ \frac{\log^2(3)}{e^2 \log^6(2)} \approx 1.618048844 \]

\[ \frac{3 \sqrt{2}}{L} \approx 1.6180578035 \]

\[ \frac{\log(6)}{-1 + \sqrt{2} + \log(2)} \approx 1.618044960 \]

\( \Phi \) is the golden ratio conjugate
\( \mathcal{K}_1 \) is the Khinchin harmonic mean
\( \bar{V} \) is the mean tetrahedron-in-tetrahedron volume
\( \log(x) \) is the natural logarithm
\( L \) is the lemniscate constant
We analyze some equations concerning various aspects of Quantum Mechanics

\[ \Delta = 2\hbar K e^{-\frac{1}{\hbar} S_0}, \]

and

\[
K_E(\eta, -\eta; T) = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{\omega T}{2}} \sum_{n \text{ odd}} \frac{\left(K T e^{-\frac{1}{\hbar} S_0}\right)^n}{n!}
\]

\[ = \frac{1}{2} \sqrt{\frac{m\omega}{\pi\hbar}} \left[ e^{-\frac{T}{\hbar} \left( \frac{b\omega}{2} - \frac{1}{2} \right)} - e^{-\frac{T}{\hbar} \left( \frac{b\omega}{2} + \frac{1}{2} \right)} \right], \]

(3.349)

For: \( m = 9.109 \times 10^{-31} \quad T = 300 \quad \omega = 7.81 \times 10^{20} \quad h = 6.582119569 \times 10^{-16} \)

\( \Delta = -3.7 \quad S = 5.46296 \times 10^{-30}, \) we obtain:
\[ \exp\left(\frac{1}{2}(6.582119569 \times 10^{-16} \times 7.81 \times 10^{20} - x)\right) - \exp\left(\frac{1}{2}(6.582119569 \times 10^{-16} \times 7.81 \times 10^{20} + x)\right) \]

**Input interpretation**

\[ \exp\left(\frac{1}{2}(6.582119569 \times 10^{-16} \times 7.81 \times 10^{20} - x)\right) - \exp\left(\frac{1}{2}(6.582119569 \times 10^{-16} \times 7.81 \times 10^{20} + x)\right) \]

**Result**

\[ e^{1/2(514064. - x)} = e^{(x+514064.)/2} \]

**Plots**

(figures that can be related to the open strings)

![Graph 1](x from -6.4 to -3.7)

![Graph 2](x from -10 to 0)

**Alternate forms**

\[ -3.0131733835 \times 10^{111627} e^{-x/2} (e^{x} - 1) \]
Alternate form assuming $x$ is real

$$3.0131733835 \times 10^{111627} e^{-x/2} - 3.0131733835 \times 10^{111627} e^{x/2}$$

Real root

$x = 0.10^{-11} \approx 0$

Roots

$x \approx 12.56637061435917 i \ n, \ n \in \mathbb{Z}$

$x \approx 2 i (6.283185307179586 n + 3.141592653589793), \ n \in \mathbb{Z}$

$\mathbb{Z}$ is the set of integers

Integer root

$x = 0$

Properties as a real function

Domain

$\mathbb{R}$ (all real numbers)

Range

$\mathbb{R}$ (all real numbers)
Bijectivity

bijective from its domain to $\mathbb{R}$

Parity

odd

$R$ is the set of real numbers

Series expansion at $x=0$

$$-3.0131733835 \times 10^{111627} x - 1.255489908 \times 10^{111626} x^3 - 1.5693611373 \times 10^{111624} x^5 + O(x^6)$$

(Taylor series)

Indefinite integral

$$\int \left( e^{1/2(514064. - x)} - e^{1/2(514064. + x)} \right) dx =$$

$$-6.026346767 \times 10^{111627} \left( 2.71828^x + 1 \right) e^{-0.5 x} + \text{constant}$$

$$(-300/6.582119569*10^{-16}) \left( (e^{(1/2 (514064. + 3.7)))} - e^{(1/2 (514064. - 3.7)))} \right)$$

Input interpretation

$$- \frac{300}{6.582119569 \times 10^{-16}} \left( e^{1/2(514064. + 3.7)} - e^{1/2(514064. - 3.7)} \right)$$

Result

$$-1.07300... \times 10^{111646}$$

$$-1.07300... \times 10^{111646}$$
\[
\frac{1}{2}\sqrt{\left(\frac{9.109 \times 10^{-31} \times 7.81 \times 10^{20}}{\pi \times 6.582119569 \times 10^{-16}}\right) \left(-\frac{300}{6.582119569 \times 10^{-16}}\right)} \left(e^{\frac{1}{2} (514064. +3.7)} - e^{\frac{1}{2} (514064. -3.7)}\right)
\]

**Input interpretation**

\[
\frac{1}{2}\sqrt{\frac{9.109 \times 10^{-31} \times 7.81 \times 10^{20}}{\pi \times 6.582119569 \times 10^{-16}}} \left(-\frac{300}{6.582119569 \times 10^{-16}}\right) \left(e^{\frac{1}{2} (514064. +3.7)} - e^{\frac{1}{2} (514064. -3.7)}\right)
\]

**Result**

-3.14683... × 10^{111648}

-3.14683...×10^{111648}

From:

\[
K_E(-\eta, -\eta; T) = \frac{\sqrt{m\omega}}{\pi \hbar} e^{-\frac{\omega T}{2}} \sum_{n \text{ even}} \frac{(KT e^{-\frac{1}{\hbar} S_0})^n}{n!}
\]

\[
= \frac{1}{2} \sqrt{\frac{m\omega}{\pi \hbar}} \left[ e^{-\frac{T}{\hbar} \left(\frac{\hbar \omega}{2} - \frac{\Delta}{2}\right)} + e^{-\frac{T}{\hbar} \left(\frac{\hbar \omega}{2} + \frac{\Delta}{2}\right)} \right],
\]

we obtain:

\[
\frac{1}{2}\sqrt{\left(\frac{9.109 \times 10^{-31} \times 7.81 \times 10^{20}}{\pi \times 6.582119569 \times 10^{-16}}\right) \left(-\frac{300}{6.582119569 \times 10^{-16}}\right)} \left(e^{\frac{1}{2} (514064. +3.7)} + e^{\frac{1}{2} (514064. -3.7)}\right)
\]
Dividing the two above expressions, after some calculations, we obtain:

\[((-3.30638 \times 10^{111648} \times 1/((1/2 \sqrt{(9.109 \times 10^{-31} \times 7.81 \times 10^{20})/(\pi \times 6.582119569 \times 10^{-16})} (-300/6.582119569 \times 10^{-16}) ((e^{1/2 (514064. +3.7)} - e^{1/2 (514064. -3.7)}))))))^{10} + 4(\text{MRB constant})^{1-(1/(4\pi)+\pi)}\]

Input interpretation

\[
\left(-\frac{1}{2} \sqrt{\frac{9.109 \times 10^{-31} \times 7.81 \times 10^{20}}{\pi \times 6.582119569 \times 10^{-16}}} \left(-\frac{300}{6.582119569 \times 10^{-16}}\right) \left(e^{1/2 (514064. +3.7)} - e^{1/2 (514064. -3.7)}\right)\right)^{10} + 4 C_{\text{MRB}}^{1-(1/(4\pi)+\pi)}
\]

\(C_{\text{MRB}}\) is the MRB constant
Result
1.644934 (trace of the instanton shape)

Considering:

\[ m = 9.109 \times 10^{-31} ; \quad T = 300 ; \quad \omega = 495.672 ; \quad h = 6.582119569 \times 10^{-16} \]

\[ \Delta = -3.7 ; \quad S = 5.46296 \ldots \times 10^{-30} ; \quad \eta = 1 \quad \text{and} \quad t = 1 , \text{we obtain:} \]

where:

<table>
<thead>
<tr>
<th>#</th>
<th>( \text{Phi}^\lambda(n/7) )</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000000</td>
<td>306.342</td>
</tr>
<tr>
<td>2</td>
<td>1.0711625</td>
<td>328.142</td>
</tr>
<tr>
<td>3</td>
<td>1.1473892</td>
<td>351.494</td>
</tr>
<tr>
<td>4</td>
<td>1.2290403</td>
<td>376.508</td>
</tr>
<tr>
<td>5</td>
<td>1.3165020</td>
<td>403.300</td>
</tr>
<tr>
<td>6</td>
<td>1.4101876</td>
<td>432.000</td>
</tr>
<tr>
<td>7</td>
<td>1.5105401</td>
<td>462.742</td>
</tr>
<tr>
<td>8</td>
<td>1.6180340</td>
<td>495.672</td>
</tr>
<tr>
<td>9</td>
<td>1.7331774</td>
<td>530.945</td>
</tr>
<tr>
<td>10</td>
<td>1.8565147</td>
<td>568.729</td>
</tr>
<tr>
<td>11</td>
<td>1.9886290</td>
<td>609.201</td>
</tr>
</tbody>
</table>

Table 46: \( \text{Phi}^\lambda(n/7) \) scale (octave = 4)

Note. Author’s calculation with data
From Lange, Nardelli, & Bini (2013, p.3). ©

Table of Frequency System based on Phi
From:

$$\xi_1 = \sqrt{\frac{m}{S_0}} \frac{\eta \omega}{2} \frac{1}{\cosh^2 \left( \frac{\omega t}{2} \right)},$$

(3.355)

we obtain:

$$\sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times 1/2 \times 495.672 \times \frac{1}{\cosh^2 \left( \frac{1/2 \times 495.672}{2} \right)}$$

**Input interpretation**

\[
\sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times \frac{1}{2} \times 495.672 \times \frac{1}{\cosh^2 \left( \frac{1}{2} \times 495.672 \right)}
\]

\(\cosh(x)\) is the hyperbolic cosine function

**Result**

\(2.18591\ldots \times 10^{-213}\)

\(2.18591\ldots \times 10^{-213}\)

We have that

$$\alpha = \sqrt{\frac{m}{S_0}} 2 \eta \omega.$$

$$\sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times 2 \times 495.672$$

**Input interpretation**
\[ \sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times 2 \times 495.672 \]

**Result**

404.805...

404.805....

From

\[ \xi_2 \sim_{t \to \pm \infty} \pm a e^{\omega |t|}, \]

we obtain:

\[ ((\sqrt{(9.109 \times 10^{-31})/(5.46296 \times 10^{-30})} \times 2 \times 495.672)) \times e^{495.672} \]

**Input interpretation**

\[ \left( \sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times 2 \times 495.672 \right) e^{495.672} \]

**Result**

7.49653... \times 10^{217}

7.49653... \times 10^{217}

Multiplying the two above expressions, we obtain:

\[
[[(((\sqrt{(9.109 \times 10^{-31})/(5.46296 \times 10^{-30})} \times 2 \times 495.672)) \times e^{495.672})] \times
[[((\sqrt{(9.109 \times 10^{-31})/(5.46296 \times 10^{-30})})^{1/2} \times (495.672) \times 1/(\cosh^2(1/2(495.672))))]]
\]
Input interpretation

\[
\left( \left( \frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}} \times 2 \times 495.672 \right) e^{495.672} \right)
\left( \left( \frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}} \times \frac{1}{2} \times 495.672 \times \frac{1}{\cosh^2 \left( \frac{1}{2} \times 495.672 \right)} \right) \right)^{1/24 - 4(MRB \text{ const})^{1-1/(4\pi)+\pi}}
\]

\(\cosh(x)\) is the hyperbolic cosine function

Result

1.63867... \times 10^5

1.63867...*10^5

From which:

\[
(((\sqrt{(9.109\times10^{-31})/(5.46296\times10^{-30})} \times 2 \times 495.672) \times e^{495.672}) \times

[(((\sqrt{(9.109\times10^{-31})/(5.46296\times10^{-30})} \times \frac{1}{2} \times 495.672) \times \frac{1}{\cosh^2 (\frac{1}{2} \times 495.672)})]^{1/24 - 4(MRB \text{ const})^{1-1/(4\pi)+\pi}}
\]

Input interpretation

\[
\left( \left( \frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}} \times 2 \times 495.672 \right) e^{495.672} \right)
\left( \left( \frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}} \times \frac{1}{2} \times 495.672 \times \frac{1}{\cosh^2 \left( \frac{1}{2} \times 495.672 \right)} \right) \right)^{1/24 - 4 \times \text{C}_\text{MRB}^{1-1/(4\pi)+\pi}}
\]

\(\cosh(x)\) is the hyperbolic cosine function

\(C_{\text{MRB}}\) is the MRB constant
Result
$1.6446979988900433964519279316486328825720236337909906838369154055$
...
$1.64469799889... \approx \zeta(2) = \pi^2/6 = 1.644934$ (trace of the instanton shape)

Now, we have

$$\mathcal{K} = \sqrt{\frac{S_0}{2\pi \hbar}} \sqrt{2 \omega} \alpha = 2 \sqrt{\frac{\eta^2 \omega^3 m}{\pi \hbar}}. \quad (3.372)$$

$$\Delta = \frac{\hbar \omega}{\pi} e^{-\frac{1}{\hbar} S_0} \sqrt{\frac{2 \pi \omega^3 m^2}{\hbar \lambda}}, \quad (3.374)$$

From (3.372), we obtain:

For: $m = 9.109 \times 10^{-31}$; $T = 300$; $\omega = 495.672$; $\hbar = 6.582119569 \times 10^{-16}$

$\Delta = -3.7$; $S = 5.46296... \times 10^{-30}$; $\eta = 1$

$$\mathcal{K} = \sqrt{\frac{S_0}{2\pi \hbar}} \sqrt{2 \omega} \alpha = 2 \sqrt{\frac{\eta^2 \omega^3 m}{\pi \hbar}}. \quad (3.372)$$
\[ 2\sqrt{\frac{495.672^3 \times 9.109 \times 10^{-31}}{\pi \times 6.582119569 \times 10^{-16}}} \]

**Input interpretation**

\[ 2 \sqrt{\frac{495.672^3 \times 9.109 \times 10^{-31}}{\pi \times 6.582119569 \times 10^{-16}}} \]

**Result**

0.000463233...

0.000463233...

We have:

\[ \lambda \approx 2 \omega e^{-\omega T} = 4 \omega \alpha^2 e^{-\omega T} \]

\[ 4 \times 495.672 \times (\sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times 2 \times 495.672)^2 \exp(-495.672 \times 300) \]

**Input interpretation**

\[ 4 \times 495.672 \left( \sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}}} \times 2 \times 495.672 \right)^2 \exp(-495.672 \times 300) \]

**Result**

1.68817... \times 10^{-64572}

1.68817... \times 10^{-64572}
From:

\[
\Delta = \frac{\hbar \omega}{\pi} \sqrt{\frac{2 \pi}{\hbar \omega}} \frac{e^{-\frac{i}{\hbar} \delta_0}}{\sqrt{\left(2 \pi \omega^3 \frac{m^2}{\hbar \lambda}\right)}},
\]


Input interpretation

\[
\frac{1}{\pi} \left(6.582119569 \times 10^{-16} \times 495.672\right) \exp\left(-\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}}\right) \sqrt{\frac{2 \pi \times 495.672^3 \left(9.109 \times 10^{-31}\right)^2}{6.582119569 \times 10^{-16} \times \frac{1.68817}{10^{64.572}}}\}
\]

Result

\[\infty\]

\[
\infty \text{ is complex infinity}
\]

Decimal approximation

7.85003... \times 10^{32254}

7.85003... \times 10^{32254}

From the two previous expressions,

\[
4 \times 495.672 \left(\sqrt{\frac{9.109 \times 10^{-31}}{5.46296 \times 10^{-30}} \times 2 \times 495.672}\right)^2 \exp(-495.672 \times 300)
\]

1.68817... \times 10^{-64572}
and

\[
\frac{1}{\pi} \left( \frac{6.582119569 \times 10^{-16} \times 495.672}{5.46296 \times 10^{-30}} \right) \exp \left( -\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}} \right) \sqrt{\frac{2 \pi \times 495.672^3 \left(9.109 \times 10^{-31}\right)^2}{6.582119569 \times 10^{-16} \times \frac{1.68817}{10^{64.572}}} }
\]

\[
7.85003 \ldots \times 10^{3254}
\]

after some calculations, we obtain:

\[
(7-9/e)^{11161} \sqrt{1/(1.68817e-64572)}\times1/(((1/Pi*(6.582119569e-16*495.672)*e^(-5.46296e-30/6.582119569e-16)*
\sqrt{(((2*Pi*495.672^3*(9.109e-31)^2)/((6.582119569e-16 *(1.68817e-64572))))))))^2)
\]

**Input interpretation**

\[
\left(7 - \frac{9}{e}\right) \times 11161 \sqrt{\frac{1}{1.68817} \times \frac{6.582119569 \times 10^{-16} \times 495.672}{5.46296 \times 10^{-30}} \exp \left( -\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}} \right) \sqrt{\frac{2 \pi \times 495.672^3 \left(9.109 \times 10^{-31}\right)^2}{6.582119569 \times 10^{-16} \times \frac{1.68817}{10^{64.572}}} }
\]

**Result**

\[
4.03685 \times 10^{35}
\]

\[
4.03685 \times 10^{35} \sim 4.036978 \times 10^{35} \quad \text{(Planck mass flow)}
\]
We observe that 11161 is given by the following Ramanujan taxicab number:

\[ 11161 = (14258^3 - 11468^3 + 1)^{1/3} \]

Furthermore, from

\[ \frac{1}{\pi} \left( \frac{6.582119569 \times 10^{-16} \times 495.672}{5.46296 \times 10^{-50}} \right) \exp \left( -\frac{2 \pi \times 495.672^3 (9.109 \times 10^{-31})^2}{6.582119569 \times 10^{-16} \times \frac{1.68817}{10^{64.572}} \times 10^{32254}} \right) \]

we obtain:

\[ \frac{1}{43} \ln(7.85003 \times 10^{32254}) + 2 - C_{\text{MRB}} \]

**Input interpretation**

\[ \frac{1}{43} \log(7.85003 \times 10^{32254}) + 2 - C_{\text{MRB}} \]

\( \log(x) \) is the natural logarithm

\( C_{\text{MRB}} \) is the MRB constant

**Result**

1729.0130731...

1729.0130731...
This result is very near to the mass of candidate glueball \( f_0(1710) \) scalar meson. Furthermore, 1728 occurs in the algebraic formula for the \( j \)-invariant of an elliptic curve. \((1728 = 8^2 \times 3^3)\) The number 1728 is one less than the Hardy–Ramanujan number \( 1729 \) (taxicab number)

and again:

\[
(\frac{1}{43}\ln(7.85003 \times 10^{32254})+2-\text{MRB const})^{1/15}+(\text{MRB const})^{(1-1/(4\pi)+\pi)}
\]

**Input interpretation**

\[
\sqrt[15]{\frac{1}{43} \log(7.85003 \times 10^{32254}) + 2 - C_{\text{MRB}} + C_{\text{MRB}}^{1-1/(4\pi)+\pi}}
\]

\( \log(x) \) is the natural logarithm  
\( C_{\text{MRB}} \) is the MRB constant  

**Result**  
1.64493885273...  
1.64493885273... \( \approx \zeta(2) = \pi^2/6 = 1.644934 \) (trace of the instanton shape)

\[
(\frac{1}{27}((\frac{1}{43}\ln(7.85003 \times 10^{32254})+2-\text{MRB const})-1))^2
\]

**Input interpretation**

\[
\left(\frac{1}{27} \left(\frac{1}{43} \log(7.85003 \times 10^{32254}) + 2 - C_{\text{MRB}} \right) - 1\right)^2
\]

\( \log(x) \) is the natural logarithm  
\( C_{\text{MRB}} \) is the MRB constant  

**Result**  
4096.0619763...  
4096.0619763... \( \approx 4096 = 64^2 \)
Now, we have:

\[ E(\theta) = \frac{\hbar \omega}{2} - \frac{\hbar \omega}{\pi} e^{-\frac{1}{\hbar} S_0} \sqrt{\frac{2 \pi \omega^3 m^2}{\hbar \lambda}} \cos \theta. \]  

(3.379)

For: \( m = 9.109 \times 10^{-31} \); \( T = 300 \); \( \omega = 495.672 \); \( \hbar = 6.582119569 \times 10^{-16} \)

\( \Delta = -3.7 \); \( S = 5.46296... \times 10^{-30} \); \( \eta = 1 \); \( \lambda = 1.68817... \times 10^{-64572} \)

\[(1/2 \times 6.582119569e-16 \times 495.672)-(1/\pi \times 6.582119569e-16 \times 495.672)\times \exp(-5.46296e-30/6.582119569e-16) \times \sqrt{((1/(6.582119569e-16 \times 1.68817e-64572)) \times (2\pi \times 495.672^3 \times (9.109 \times 10^{-31})^2))} \times \cos(\pi/6)\]

Input interpretation

\[ \frac{1}{2} \times 6.582119569 \times 10^{-16} \times 495.672 - \]

\[ \left( \frac{1}{\pi} \times 6.582119569 \times 10^{-16} \times 495.672 \right) \exp\left( -\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}} \right) \]

\[ \sqrt{\frac{1}{6.582119569 \times 10^{-16} \times 1.68817 \times 10^{-64572}} \times (2\pi \times 495.672^3 \times (9.109 \times 10^{-31})^2) \times \cos(\pi/6)} \]

Result

\( \infty \)

\( \infty \) is complex infinity

Decimal approximation

\(-6.79833... \times 10^{32254} \)

\(-6.79833... \times 10^{32254} \)
Dividing the previous expression

\[
\frac{1}{\pi} \left( 6.582119569 \times 10^{-16} \times 495.672 \right) \\
\exp \left( -\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}} \right) \sqrt{\frac{2 \pi \times 495.672^3 \left( 9.109 \times 10^{-31} \right)^2}{6.582119569 \times 10^{-16} \times 1.68817 \times 10^{64.572}}} 
\]

by

\[
\frac{1}{2} \times 6.582119569 \times 10^{-16} \times 495.672 - \left( \frac{1}{\pi} \times 6.582119569 \times 10^{-16} \times 495.672 \right) \exp \left( -\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}} \right) \sqrt{\frac{1}{6.582119569 \times 10^{-16} \times 1.68817 \times 10^{64.572}}} \left( 2 \pi \times 495.672^3 \left( 9.109 \times 10^{-31} \right)^2 \right) \cos \left( \frac{\pi}{6} \right) 
\]

that is equal to

\[-6.79833 \ldots \times 10^{32254}\]

we obtain:

\[-1/(-6.79833*10^{32254})*(1/\pi*(6.582119569*10^{-16}*495.672)*e^(-5.46296*10^{-30}/6.582119569*10^{-16})*
\sqrt{(((2*Pi*495.672^3*(9.109*10^{-31})^2)/(6.582119569*10^{-16} *(1.68817*10^{-64572})))})
\]

Input interpretation

\[
\left\{ -\frac{1}{\pi} \left( 6.582119569 \times 10^{-16} \times 495.672 \right) \exp \left( -\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}} \right) \sqrt{\frac{2 \pi \times 495.672^3 \left( 9.109 \times 10^{-31} \right)^2}{6.582119569 \times 10^{-16} \times 1.68817 \times 10^{64.572}}} \right\} / (-(6.79833 \times 10^{32254})) 
\]
From which, after some calculations, we obtain:

\[
\begin{align*}
(-1/(-6.79833e+32254)*(1/Pi*(6.582119569e-16*495.672)*e^{-5.46296e-30/6.582119569e-16})* \sqrt{((2*Pi*495.672^3*(9.109e-31)^2)/((6.582119569e-16*(1.68817e-64572))))})^{7/2} - 9 (\text{MRB const})^{1-1/(4\pi)+\pi}
\end{align*}
\]

\[
\binom{\frac{1}{\pi} \left(6.582119569 \times 10^{-16} \times 495.672\right) \exp\left(-\frac{5.46296 \times 10^{-30}}{6.582119569 \times 10^{-16}}\right)}{\left(\sqrt{\frac{2\pi \times 495.672^3 \times (9.109 \times 10^{-31})^2}{6.582119569 \times 10^{-16} \times 1.68817^{10^{-54572}}}}\right)}/\left(-6.79833 \times 10^{32254}\right)^{7/2} - 9 \ C_{\text{MRB}}^{1-1/(4\pi)+\pi}
\]

\( C_{\text{MRB}} \) is the MRB constant
We have the following equation:

\[
\frac{1}{\sqrt{\pi}} \int da_1 = \frac{1}{\sqrt{\pi}} \int dz \left\| \frac{dq_z}{dz} \right\| = \frac{1}{\sqrt{\pi}} \left[ \int \left( \frac{dq_z}{dz} \right)^2 dx \right]^{1/2} \int dz
\]

\[
= \frac{(2\varepsilon)^{3/4}}{(-3\alpha)^{1/2}} \frac{L}{\sqrt{\pi}} \tag{29.15}
\]

\[
(2\varepsilon)^{3/4} / (-3\alpha)^{1/2} \times L / (\sqrt{\pi})
\]

**Input**

\[
\frac{(2\varepsilon)^{3/4}}{\sqrt{-3\alpha}} \times \frac{L}{\sqrt{\pi}}
\]

**Exact result**

\[
\frac{2^{3/4} \varepsilon^{3/4} L}{\sqrt{3\pi} \sqrt{-\alpha}}
\]
Alternate form assuming $L$, $\alpha$, and $\varepsilon$ are positive

$$-i \, 2^{3/4} \varepsilon^{3/4} L \sqrt{3/\pi} \sqrt{-\alpha}$$

Real roots

$L < 0, \quad \alpha < 0, \quad \varepsilon = 0$

$L = 0, \quad \alpha < 0, \quad \varepsilon \geq 0$

$L > 0, \quad \alpha < 0, \quad \varepsilon = 0$

Root for the variable $\varepsilon$

$\varepsilon = 0$

Derivative

$$\frac{\partial}{\partial L} \left( \frac{(2 \varepsilon)^{3/4} L}{\sqrt{-3 \alpha} \sqrt{\pi}} \right) = \frac{2^{3/4} \varepsilon^{3/4}}{\sqrt{3/\pi} \sqrt{-\alpha}}$$

Indefinite integral

$$\int \frac{2^{3/4} L \varepsilon^{3/4}}{\sqrt{3/\pi} \sqrt{-\alpha}} \, dL = \frac{\varepsilon^{3/4} L^2}{4 \sqrt{2} \sqrt{3/\pi} \sqrt{-\alpha}} + \text{constant}$$
Series representations

\[
\frac{L(2\varepsilon)^{3/4}}{\sqrt{\pi} \sqrt{-3\alpha}} = -\frac{2^{3/4} L \sqrt{-\alpha} \varepsilon^{3/4}}{\sqrt{3} \alpha \sqrt{-1 + \pi} \sum_{k=0}^{\infty} (-1 + \pi)^{-k} \left(\frac{1}{2}\right)_k} \\
\frac{L(2\varepsilon)^{3/4}}{\sqrt{\pi} \sqrt{-3\alpha}} = -\frac{2^{3/4} L \sqrt{-\alpha} \varepsilon^{3/4}}{\sqrt{3} \alpha \sqrt{-1 + \pi} \sum_{k=0}^{\infty} (-1)^k(-1 + \pi)^{-k}\left(\frac{1}{2}\right)_k} \\
\frac{L(2\varepsilon)^{3/4}}{\sqrt{\pi} \sqrt{-3\alpha}} = -\frac{2 \times 2^{3/4} L \sqrt{-\alpha} \varepsilon^{3/4} \sqrt{\pi}}{\sqrt{3} \alpha \sum_{j=0}^{\infty} \text{Res}_{s=\frac{1}{2}+j} (-1 + \pi)^{-s} \Gamma\left(-\frac{1}{2} - s\right) \Gamma(s)}
\]

\(\left(\frac{n}{m}\right)\) is the binomial coefficient
\(n!\) is the factorial function
\((a)_n\) is the Pochhammer symbol (rising factorial)
\(\Gamma(x)\) is the gamma function
\(\text{Res}\ f\) is a complex residue
\(s=0\)

From the alternate form assuming \(L, \alpha,\) and \(\varepsilon\) are positive

\[- \frac{i \ 2^{3/4} \varepsilon^{3/4} L}{\sqrt{3} \pi \sqrt{\alpha}}\]

for \(L = 4, \ \alpha = 8\) and \(\varepsilon = 16\), we obtain:

\[-(i \ 2^{(3/4)} \ 4 \ 16^{(3/4)})/(\text{sqrt}(3 \pi) \text{ sqrt}(8))\]

Input

\[- \frac{i \times 2^{3/4} \times 4 \times 16^{3/4}}{\sqrt{3} \pi \sqrt{8}}\]
Exact result

\[- \frac{16 \, i \frac{\sqrt{2}}{\sqrt{3} \pi}}{\pi} \]

Decimal approximation

\[-6.19786222467364352062929705825519740442215067769748628397145550... \, i \]

\[-6.19786222467... \, i \]

Property

\[- \frac{16 \, i \frac{\sqrt{2}}{\sqrt{3} \pi}}{\pi} \text{ is a transcendental number} \]

Alternate complex forms

\[16 \frac{\frac{\sqrt{2}}{\sqrt{3} \pi} \left( \cos\left( -\frac{\pi}{2} \right) + i \sin\left( -\frac{\pi}{2} \right) \right)}{\pi} \]

\[16 \frac{\frac{\sqrt{2}}{\sqrt{3} \pi} \, e^{-i \pi/2}}{\sqrt{3} \pi} \]

Polar coordinates

\[r = \frac{16 \sqrt{2}}{\sqrt{3} \pi} \text{ (radius), } \theta = -\frac{\pi}{2} \text{ (angle)} \]
Series representations

\[ i 2^{3/4} \times 4 \times 16^{3/4} \left/ \sqrt{3 \pi} \sqrt{8} \right. = - \frac{32 \times 2^{3/4} i}{\sqrt{7} \sqrt{-1 + 3 \pi} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\left( \frac{1}{7} \right)^k}{k!} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(-1 + 3 \pi)^k}{k!} \left( \frac{1}{2} \right)^k} \]

\[ i 2^{3/4} \times 4 \times 16^{3/4} \left/ \sqrt{3 \pi} \sqrt{8} \right. = - \frac{32 \times 2^{3/4} i}{\sqrt{7} \sqrt{-1 + 3 \pi} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{1}{7} \right)^k \frac{\left( \frac{1}{2} \right)^k}{k!} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(-1 + 3 \pi)^k}{k!} \left( \frac{1}{2} \right)^k} \]

\[ i 2^{3/4} \times 4 \times 16^{3/4} \left/ \sqrt{3 \pi} \sqrt{8} \right. = \frac{32 \times 2^{3/4} i}{\sqrt{z_0^2 \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\left( \frac{1}{2} \right)^k}{k!} \left( 8 - z_0^k \right)^k}{z_0^k} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\left( \frac{1}{2} \right)^k}{k!} \left( 3 \pi - z_0^k \right)^k}{z_0^k} \right.} \]

for (not \( z_0 \in \mathbb{R} \) and \( -\infty < z_0 \leq 0 \))

\[ \binom{n}{m} \] is the binomial coefficient

\( n! \) is the factorial function

\( (\alpha)_n \) is the Pochhammer symbol (rising factorial)

\( \mathbb{R} \) is the set of real numbers

From the exact result

\[ - \frac{16 i \sqrt{2}}{\sqrt{3 \pi}} \]

after some calculations, we obtain:
\((-16i\sqrt[4]{2}/\sqrt[4]{3\pi})^4 + 233 + 21 - 3C_{\text{MRB}}\) const

**Input**

\[
\left( -\frac{16i\sqrt[4]{2}}{\sqrt[4]{3\pi}} \right)^4 + 233 + 21 - 3C_{\text{MRB}}
\]

\(i\) is the imaginary unit

\(C_{\text{MRB}}\) is the MRB constant

**Exact result**

\[-3C_{\text{MRB}} + 254 + \frac{131072}{9\pi^2}\]

**Decimal approximation**

1729.033108…

This result is very near to the mass of candidate glueball \(f_0(1710)\) scalar meson. Furthermore, 1728 occurs in the algebraic formula for the \(j\)-invariant of an elliptic curve. \((1728 = 8^3 \times 3^3)\) The number 1728 is one less than the Hardy–Ramanujan number 1729 (taxicab number)

**Alternate forms**

\[-\frac{27\pi^2 C_{\text{MRB}} - 131072 - 2286\pi^2}{9\pi^2}\]

\[-\frac{27\pi^2 C_{\text{MRB}} - 2(65536 + 1143\pi^2)}{9\pi^2}\]
\((-16 \, i \, 2^{(1/4)})/\sqrt{3 \, \pi})^4 + 233 + 21 - 3 \text{MRB const})^{1/15} + (\text{MRB const})^{(1-1/(4\pi)+\pi)}

**Input**

\[
\sqrt[15]{\left( -\frac{16 \, i \, \sqrt{2}}{\sqrt{3 \, \pi}} \right)^4 + 233 + 21 - 3 \, C_{\text{MRB}} + C_{\text{MRB}}^{1-1/(4\pi)+\pi}}
\]

\(i\) is the imaginary unit

\(C_{\text{MRB}}\) is the MRB constant

**Exact result**

\[
C_{\text{MRB}}^{1-1/(4\pi)+\pi} + \sqrt[15]{-3 \, C_{\text{MRB}} + 254 + \frac{131 \, 072}{9 \, \pi^2}}
\]

**Decimal approximation**

1.644940122566\ldots \approx \zeta(2) = \frac{\pi^2}{6} = 1.644934 \text{ (trace of the instanton shape)}

**Alternate forms**

\[
C_{\text{MRB}}^{1-1/(4\pi)+\pi} + \sqrt[15]{-27 \, \pi^2 \, C_{\text{MRB}} + 131 \, 072 + 2286 \, \pi^2}
\]

\[
C_{\text{MRB}}^{1-1/(4\pi)+\pi} + \sqrt[15]{2 \left(65 \, 536 + 1143 \, \pi^2\right) - 27 \, \pi^2 \, C_{\text{MRB}}}
\]

\[
\frac{1}{3 \, \pi^{2/15}} C_{\text{MRB}}^{-1/(4\pi)} \left(3 \, \pi^{2/15} \, C_{\text{MRB}}^{1+\pi} + 3^{13/15} \, 4 \, \pi \, C_{\text{MRB}}^{15} \, -27 \, \pi^2 \, C_{\text{MRB}} + 131 \, 072 + 2286 \, \pi^2 \right)
\]
\[
\frac{1}{27}((-16 i 2^{1/4})/\sqrt{3 \pi})^4+233+21-3 \text{MRB const}-1)^2 - \text{MRB const}
\]

**Input**

\[
\left(\frac{1}{27} \left( \left(- \frac{16 i \sqrt{2}}{\sqrt{3 \pi}} \right)^4 + 233 + 21 - 3 \text{CMRB} \right) - 1 \right)^2 - \text{CMRB}
\]

\(i\) is the imaginary unit
\(\text{CMRB}\) is the MRB constant

**Exact result**

\[
\frac{1}{729} \left(-3 \text{CMRB} + 253 + \frac{131072}{9 \pi^2} \right)^2 - \text{CMRB}
\]

**Decimal approximation**

4095.969098317202848556222328253322264584135410019010777619211607

\(\ldots\)

4095.9690983172\ldots \approx 4096 = 64^2

**Alternate forms**

\[
\frac{1}{729} \left(-2247 \text{CMRB} + 9 \text{CMRB}^2 + 64009\right) - \frac{262144 \left(3 \text{CMRB} - 253\right)}{6561 \pi^2} + \frac{17179869184}{59049 \pi^4}
\]

\[
\frac{1}{59049 \pi^4} \left(-7077888 \pi^2 \text{CMRB} - 182007 \pi^4 \text{CMRB} + 729 \pi^4 \text{CMRB}^2 + 17179869184 + 596901888 \pi^2 + 5184729 \pi^4\right)
\]

\[
729 \pi^4 \text{CMRB}^2 - 27 \pi^2 \left(262144 + 6741 \pi^2\right) \text{CMRB} + \left(131072 + 2277 \pi^2\right)^2
\]

\[
\frac{59049 \pi^4}
\]
Expanded form

\[- \frac{749 C_{\text{MRB}}}{243} + \frac{C_{\text{MRB}}^2}{81} - \frac{262144 C_{\text{MRB}}}{2187 \pi^2} + \frac{64009}{729} + \frac{17179869184}{59049 \pi^4} + \frac{66322432}{6561 \pi^2} \]

We have:

Our main interest throughout has been the imaginary part of \( \psi \) which, because \( Z_0 \) is real, has the form

\[
\text{Im } \psi(\alpha) = -\frac{1}{L} \lim_{L \to \infty} \text{Im} \left( \frac{Z_1}{Z_0} \right)
\]

\[
= \pm \frac{1}{2} \sqrt{\frac{1}{3e\pi}} \left[ \frac{2(2\varepsilon)^{3/4}}{(-3\alpha)^{1/2}} \right] \exp \left[ \frac{-(2\varepsilon)^{3/2}}{(-3\alpha)} \right] \frac{\Pi'\lambda^{-1/2}}{\Pi\lambda^{(0)^{-1/2}}}
\]

(29.24)

Bringing our expression for the ratio of the product of continuum states from the appendix we have

\[
\text{Im } \psi(\alpha) = \pm \frac{2^{7/4}e^{5/4}}{(-\pi\alpha)^{1/2}} \exp \left[ -\frac{(2\varepsilon)^{3/2}}{(-3\alpha)} \right]
\]

(29.25)
From the right-hand side:

\[
\text{Im } \psi(\alpha) = \pm \frac{2^{7/4} \varepsilon^{5/4}}{(-\pi \alpha)^{1/2}} \exp \left[ -\frac{(2\varepsilon)^{3/2}}{(-3\alpha)} \right]
\]

we obtain:

\[
(1/(-\pi\alpha)^{(1/2)})(2)^{(7/4)}\varepsilon^{(5/4)}\exp((-2\varepsilon)^{(3/2)/(-3\alpha)})
\]

**Input**

\[
\frac{1}{\sqrt{-\pi \alpha}} \times 2^{7/4} \varepsilon^{5/4} \exp \left( \frac{-(2 \varepsilon)^{3/2}}{-3 \alpha} \right)
\]

**Exact result**

\[
2 \times 2^{3/4} \varepsilon^{5/4} e^{(2\sqrt{2} \varepsilon^{3/2}/(3\alpha))} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{-\alpha}}
\]
3D plots
Real part
(figures that can be related to the D-branes/Instantons)

Imaginary part
Contour plots
Real part
Imaginary part

Alternate form assuming $\alpha$ and $\varepsilon$ are positive

\[ \frac{-2i \cdot 2^{3/4} \varepsilon^{5/4} e^{(2\sqrt{2} \varepsilon^{3/2})/(3\alpha)}}{\sqrt{\pi} \sqrt{\alpha}} \]

Roots

$\alpha = 0$

$\varepsilon = 0$
Series expansion at $\alpha=0$

\[
\begin{cases}
\left(e^{2\sqrt{2} \epsilon^{3/2}/(3\alpha)} \left(\frac{2 \cdot 3^{3/4} \epsilon^{5/4}}{\sqrt{\pi} \sqrt{-\alpha}} + O(\epsilon^{7/9})\right)\right) & \text{Im}(\alpha) \leq 0 \\
\left(e^{2\sqrt{2} \epsilon^{3/2}/(3\alpha)} \left(\frac{2 \cdot 3^{3/4} \epsilon^{5/4}}{\sqrt{\pi} \sqrt{-\alpha}}\right)^* + O(\epsilon^{7/9})\right) & \text{otherwise}
\end{cases}
\]

(Im($z$) is the imaginary part of $z$ 
$z^*$ is the complex conjugate of $z$)

Series expansion at $\alpha=\infty$

\[
\frac{2 \cdot 2^{3/4} \sqrt{\alpha} \sqrt{\frac{1}{\alpha}}} {32 \sqrt{2} \sqrt{\alpha} \cdot \frac{1}{\alpha}^{7/2} \epsilon^{23/4}} + \frac{8 \sqrt{2} \sqrt{\alpha} \left(\frac{1}{\alpha}\right)^{3/2} \epsilon^{11/4}} {8 \sqrt{\pi} \sqrt{-\alpha}} + \frac{8 \cdot 2^{3/4} \sqrt{\alpha} \left(\frac{1}{\alpha}\right)^{5/2} \epsilon^{17/4}} {9 \sqrt{\pi} \sqrt{-\alpha}} + \frac{3 \sqrt{\pi} \sqrt{-\alpha}} {243 \sqrt{\pi} \sqrt{-\alpha}} + O\left(\frac{1}{\alpha}\right)^5
\]

(generalized Puiseux series)

Derivative

\[
\frac{\partial}{\partial \alpha} \left(2^{7/4} \epsilon^{-5/4} \exp\left(\frac{-2 \sqrt{2} \epsilon^{3/2}}{-3\alpha}\right)\right) = -\frac{4 \sqrt{2} \epsilon^{7/4} \left(2 \sqrt{2} \epsilon^{3/2}\right)}/(3\alpha) \left(3 \sqrt{2} \alpha + 8 \epsilon^{3/2}\right)
\]

Indefinite integral

\[
\int \frac{2 \cdot 2^{3/4} e^{2\sqrt{2} \epsilon^{3/2}/(3\alpha)} \epsilon^{5/4}} {\sqrt{\pi} \sqrt{-\alpha}} \, d\alpha =
\]

\[-4 \sqrt{\frac{2}{3}} \sqrt{-\alpha} \epsilon^{5/4} \sqrt{-\frac{\epsilon^{3/2}}{\alpha}} \Gamma\left(-\frac{1}{2}, -\frac{2 \sqrt{2} \epsilon^{3/2}}{3\alpha}\right) + \text{constant}
\]

$\Gamma(\alpha, x)$ is the incomplete gamma function
\textbf{Limit}

\[
\lim_{a \to \infty} \frac{2 \cdot 2^{3/4} e^{\left(2 \sqrt{2} \cdot 3^{3/2} / (3 \cdot a)\right) \cdot 5/4}}{\sqrt{\pi} \sqrt{-a}} = 0
\]

From

\[
\frac{2 \cdot 2^{3/4} e^{5/4} \left(2 \sqrt{2} \cdot 3^{3/2} / (3 \cdot a)\right)}{\sqrt{\pi} \sqrt{-a}}
\]

for \(\epsilon = -4\) and \(\alpha = -2\), we obtain:

\[(2 \cdot 2^{(3/4)} e^{((2 \sqrt{2} \cdot (-4)^{(3/2)})/(3 \cdot -2)) \cdot (-4)^{(5/4)})/(\sqrt{\pi} \cdot \sqrt{2})})
\]

\textbf{Input}

\[
2 \cdot 2^{3/4} e^{\left(2 \sqrt{2} \cdot (-4)^{3/2} / (3 \cdot -2)\right) \cdot (-4)^{5/4}}
\]

\textbf{Exact result}

\[
-8 \sqrt{-1} \cdot 2^{3/4} e^{-16i \sqrt{2}}
\]

\textbf{Decimal approximation}

\[
7.506547812785244722804783274829031749770751411880234423982135857\ldots +
\]

\[
1.127822755766103078679078537022844541792814424120995524201372348\ldots i
\]
Alternate complex forms

\[
-8 \sqrt{2} \sin(16 \sqrt{2}) - 8 \sqrt{2} \cos(16 \sqrt{2}) + \frac{i\left(8 \sqrt{2} \sin(16 \sqrt{2}) - 8 \sqrt{2} \cos(16 \sqrt{2})\right)}{\sqrt{\pi}}
\]

\[
8 \cdot 2^{3/4} \left(\cos \left(\frac{\sin(16 \sqrt{2}) - \cos(16 \sqrt{2})}{\sqrt{2}}\right) \right) + \frac{i \sin \left(\frac{\sin(16 \sqrt{2}) - \cos(16 \sqrt{2})}{\sqrt{2}}\right)}{\sqrt{\pi}}
\]

\[
8 \cdot 2^{3/4} \exp \left(i \tan^{-1} \left(\frac{\sin(16 \sqrt{2}) - \cos(16 \sqrt{2})}{\sqrt{2}}\right)\right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \r...
Series representations

\[
2^{3/4} e^{(2 \sqrt{2} \sqrt{-4}^{3/2} (3-2) \sqrt{-4}^{5/4})} = \\
\frac{\sqrt{\pi} \sqrt{2}}{16 \sqrt{-2} \exp \left(-16 i \sqrt{z_0} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-z_0 \sqrt{-4})^k z_0^{-k}}{k!} \right)} \\
- \frac{\sqrt{z_0} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-z_0 \sqrt{-4})^k z_0^{-k}}{k!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi-z_0 \sqrt{-4})^k z_0^{-k}}{k!}}{\sqrt{z_0} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-z_0 \sqrt{-4})^k z_0^{-k}}{k!}}
\]

for \((z_0 \in \mathbb{R} \text{ and } -\infty < z_0 \leq 0)\)

\[
2^{3/4} e^{(2 \sqrt{2} \sqrt{-4}^{3/2} (3-2) \sqrt{-4}^{5/4})} = \\
\frac{\sqrt{\pi} \sqrt{2}}{16 \sqrt{-2} \exp \left(-16 i \exp \left(i \pi \left[\frac{\arg(2-x)}{2 \pi}\right]\right) \sqrt{x} \right) \left[\sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} \left(-\frac{1}{2}\right)_k}{k!}\right]} \left[\sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} \left(-\frac{1}{2}\right)_k}{k!}\right] \\
\frac{\exp \left(i \pi \left[\frac{\arg(\pi-x)}{2 \pi}\right]\right) \sqrt{x} \cdot \left(\sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} \left(-\frac{1}{2}\right)_k}{k!}\right)}{\sum_{k=0}^{\infty} \frac{(-1)^k (\pi-x)^k x^{-k} \left(-\frac{1}{2}\right)_k}{k!}}
\]

for \((x \in \mathbb{R} \text{ and } x < 0)\)
\[
2 \left( \frac{2^{3/4} e^{2\sqrt{2} (-4)^{3/2}/(3-2)} (-4)^{5/4}}{\sqrt{\pi} \sqrt{2}} \right) = \\
-\left( 16 \sqrt{-2} \exp \left( -16 i \left( \frac{1}{z_0} \right)^{1/2 \mid \arg(2-z_0)/(2\pi) \mid} \frac{1}{2+1/2 \mid \arg(2-z_0)/(2\pi) \mid} \sum_{k=0}^{\infty} \frac{(-1)^k \left( -\frac{1}{2} \right)_k (2-z_0)_k z_0^{-k}}{k!} \right) \right) \\
\left( \frac{\left( \frac{1}{z_0} \right)^{-1/2 \mid \arg(2-z_0)/(2\pi) \mid} \frac{1}{2 \mid \arg(\pi-z_0)/(2\pi) \mid} \sum_{k=0}^{\infty} \frac{(-1)^k \left( -\frac{1}{2} \right)_k (\pi-z_0)_k z_0^{-k}}{k!} \right) \right) \\
\eta! \text{ is the factorial function}
\]
\[ (a)_\eta \text{ is the Pochhammer symbol (rising factorial) } \]
\[ \mathbb{R} \text{ is the set of real numbers} \]
\[ \arg(z) \text{ is the complex argument} \]
\[ \lfloor x \rfloor \text{ is the floor function} \]

From which:
\[
4 \left( 2 \left( 2^{3/4} e^{(2 \sqrt{2} (-4)^{3/2}/(3-2)} (-4)^{5/4}}/(\sqrt{\pi} \sqrt{2}) \right)^{3} - 21 - \phi \right)
\]

\[ \text{Input} \]
\[
4 \left( 2 \times 2^{3/4} e^{(2 \sqrt{2} (-4)^{3/2}/(3-2)} (-4)^{5/4})/(\sqrt{\pi} \sqrt{2}) \right)^{3} - 21 - \phi
\]
\[ \phi \text{ is the golden ratio} \]

\[ \text{Exact result} \]
\[
-\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2} e^{-48 i \sqrt{2}}}{\pi^{3/2}}
\]
Decimal approximation

$$1554.727079682360636453971104808202854670254148244754266873044136\ldots +$$
$$756.87190863764878419890589882576778740449265419986667774338393\ldots i$$

Alternate complex forms

$$\frac{-\phi - 21 + \frac{i}{\pi^{3/2}} \left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)}{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})} +$$

$$\left( \cos^{-1} \left( \frac{-4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2})}{\pi^{3/2} \left( -\phi - 21 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}} \right)} \right) + \frac{i}{\pi^{3/2}} \left( \sin^{-1} \left( \frac{-4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2})}{\pi^{3/2} \left( -\phi - 21 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}} \right)} \right) \right) /$$

$$\pi^{3/2} \sqrt{2 \left( 134217728 \sqrt{2} + (927 + 43 \sqrt{5}) \pi^3 - 8192 \times 2^{3/4} (43 + \sqrt{5}) \pi^{3/2} (\cos(48 \sqrt{2}) - \sin(48 \sqrt{2})) \right)}$$

$$\exp \left( \frac{i}{\pi^{3/2}} \left( \frac{-4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2})}{\pi^{3/2} \left( -\phi - 21 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}} \right)} \right) \right)$$

$$\frac{2}{\sqrt{134217728 \sqrt{2} + (927 + 43 \sqrt{5}) \pi^3 - 8192 \times 2^{3/4} (43 + \sqrt{5}) \pi^{3/2} (\cos(48 \sqrt{2}) - \sin(48 \sqrt{2}))}}$$

\(i\) is the imaginary unit

\(\tan^{-1}(x)\) is the inverse tangent function
Polar coordinates
\[ r \approx 1729.2 \text{ (radius)}, \quad \theta \approx 0.45305 \text{ (angle)} \]

1729.2

This result is very near to the mass of candidate glueball \( f_0(1710) \) scalar meson. Furthermore, 1728 occurs in the algebraic formula for the \( j \)-invariant of an elliptic curve. (1728 = \( 8^2 \times 3^3 \)) The number 1728 is one less than the Hardy–Ramanujan number 1729 (taxicab number)

Alternate forms

\[-\phi - 21 - \frac{8192 i \sqrt{-2} e^{-48 i \sqrt{2}}}{\pi^{3/2}}\]

\[\frac{1}{2} (-43 - \sqrt{5}) - \frac{8192 i \sqrt{-2} e^{-48 i \sqrt{2}}}{\pi^{3/2}}\]

\[-\pi^{3/2} (\phi + 21) + 8192 (-1)^{3/4} \sqrt{2} e^{-48 i \sqrt{2}} \frac{\pi^{3/2}}{\pi^{3/2}}\]

\[-\phi - 21 - \frac{8192 \sqrt{2} e^{(3 i \pi)/4 - 48 i \sqrt{2}}}{\pi^{3/2}}\]

Expanded form

\[-\frac{43}{2} \sqrt{5} \frac{1}{2} - \frac{8192 (-1)^{3/4} \sqrt{2} e^{-48 i \sqrt{2}}}{\pi^{3/2}}\]
Series representations

\[
4 \left( \frac{2^{3/4} \sqrt{2} \left(2 \sqrt{2} (-4)^{3/2})/(3-2i) (-4)^{5/4} \right)}{\sqrt{\pi} \sqrt{2}} \right)^3 - 21 - \phi =
\]

\[
- \left( \exp \left( -48 i \sqrt{z_0} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2 - z_0)^k z_0^{-k}}{k!} \right) \right)
\]

\[
16384 (-2)^{3/4} + 21 \exp \left( 48 i \sqrt{z_0} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2 - z_0)^k z_0^{-k}}{k!} \right)
\]

\[
\sqrt{z_0}^6 \left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2 - z_0)^k z_0^{-k}}{k!} \right)^3
\]

\[
\exp \left( 48 i \sqrt{z_0} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2 - z_0)^k z_0^{-k}}{k!} \right) \phi
\]

\[
\sqrt{z_0}^6 \left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2 - z_0)^k z_0^{-k}}{k!} \right)^3
\]

\[
\left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2 - z_0)^k z_0^{-k}}{k!} \right)^3
\]

\[
\left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi - z_0)^k z_0^{-k}}{k!} \right)^3
\]

\[
\left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi - z_0)^k z_0^{-k}}{k!} \right)^3
\]

\[
\left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi - z_0)^k z_0^{-k}}{k!} \right)^3
\]

\[
\left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi - z_0)^k z_0^{-k}}{k!} \right)^3
\]

for (not \(z_0 \in \mathbb{R}\) and \(-\infty < z_0 \leq 0\))
\[ 4 \left( \frac{2^{3/4} e^{(2\sqrt{2} (-4)^{3/2})/(3-2) (-4)^{5/4}}}{\sqrt{\pi} \sqrt{2}} \right)^3 - 21 - \phi = \]
\[ - \left( \exp \left( -48i \exp \left( i \pi \left[ \frac{\text{arg}(2-x)}{2\pi} \right] \right) \sqrt{x} \sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right) \right. \]
\[ \left. \left( 16384 (-2)^{3/4} + 21 \exp \left( 48i \exp \left( i \pi \left[ \frac{\text{arg}(2-x)}{2\pi} \right] \right) \sqrt{x} \sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right) \right) \exp^{3} \left( i \pi \left[ \frac{\text{arg}(2-x)}{2\pi} \right] \right) \exp^{3} \left( i \pi \left[ \frac{\text{arg}(\pi-x)}{2\pi} \right] \right) \exp^{3} \left( i \pi \left[ \frac{\text{arg}(\pi-x)}{2\pi} \right] \right) \sqrt{x} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right)^3 \right) \]
\[ + \left( \sqrt{x} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (\pi-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right)^3 \right) \]
\[ \phi \exp^{3} \left( i \pi \left[ \frac{\text{arg}(2-x)}{2\pi} \right] \right) \exp^{3} \left( i \pi \left[ \frac{\text{arg}(\pi-x)}{2\pi} \right] \right) \sqrt{x} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right)^3 \]
\[ \left( \sum_{k=0}^{\infty} \frac{(-1)^k (\pi-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right) \right)^3 \right) \]
\[ \left( \exp^{3} \left( i \pi \left[ \frac{\text{arg}(2-x)}{2\pi} \right] \right) \exp^{3} \left( i \pi \left[ \frac{\text{arg}(\pi-x)}{2\pi} \right] \right) \sqrt{x} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (2-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right)^3 \]
\[ \left( \sum_{k=0}^{\infty} \frac{(-1)^k (\pi-x)^k x^{-k} (-\frac{1}{2})^k}{k!} \right) \right)^3 \right) \right) \text{ for } (x \in \mathbb{R} \text{ and } x < 0) \]
\[
4 \left( \frac{2^{3/4} e^{(2\sqrt{2} (-4)^{3/2})/(2-\pi)}}{\sqrt{\pi} \sqrt{2}} \right)^3 - 21 - \phi = \\
- \left( 48 \frac{1}{\zeta_0} \right)^{1/2} \arg(2-\zeta_0)/(2\pi) + z_{-2+1/2} ^{1/2} \arg(2-\zeta_0)/(2\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-\zeta_0)^k \zeta_0^{-k}}{k!} \right)
\]

\[
\left( \frac{1}{\zeta_0} \right)^{3/2} \arg(2-\zeta_0)/(2\pi) + 3/2 \arg(\pi-\zeta_0)/(2\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-\zeta_0)^k \zeta_0^{-k}}{k!} \right)
\]

\[
48 \left( \frac{1}{\zeta_0} \right)^{1/2} \arg(2-\zeta_0)/(2\pi) + z_{-2+1/2} ^{1/2} \arg(2-\zeta_0)/(2\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-\zeta_0)^k \zeta_0^{-k}}{k!} \right)
\]

\[
\left( \frac{1}{\zeta_0} \right)^{3/2} \arg(2-\zeta_0)/(2\pi) + 3/2 \arg(\pi-\zeta_0)/(2\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-\zeta_0)^k \zeta_0^{-k}}{k!} \right)
\]

\[
\left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-\zeta_0)^k \zeta_0^{-k}}{k!} \right)^3 \left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi-\zeta_0)^k \zeta_0^{-k}}{k!} \right)^3
\]

\[
\left( \left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (2-\zeta_0)^k \zeta_0^{-k}}{k!} \right)^3 \left( \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\right)_k (\pi-\zeta_0)^k \zeta_0^{-k}}{k!} \right)^3 \right)^{-1}
\]

\[
(\alpha)_n \text{ is the Pochhammer symbol (rising factorial)}
\]

\[
\mathbb{R} \text{ is the set of real numbers}
\]

\[
\text{arg}(z) \text{ is the complex argument}
\]
\[
(4((2\ 2^{(3/4)} e^{((2 \sqrt{2} (-4)^{(3/2)})/(3 -2)) (-4)^{(5/4)}/(\sqrt{\pi} \ sqrt{2}))})^{3-21-\phi})^{1/15}+(\text{MRB const})^{(1-1/(4\pi)+\pi)}
\]

Input

\[
\sqrt[15]{4 \left( \frac{2 \times 2^{3/4} e^{(2 \sqrt{2} (-4)^{3/2})/(3-2) (-4)^{5/4}/\sqrt{\pi} \sqrt{2})} {\sqrt{2} \ \sqrt{\pi}} \right)^3 - 21 - \phi + \text{MRB}^{1-1/(4\pi)+\pi}
\]

\(\phi\) is the golden ratio
\(\text{MRB}\) is the MRB constant

Exact result

\[
\text{MRB}^{1-1/(4\pi)+\pi} + \sqrt[15]{-\phi - 21 - \frac{8192 (-1)^{3/4} 4^{3/2} e^{-48 i \sqrt{2}}}{\pi^{3/2}}}
\]

Decimal approximation

1.6441991249471788605975599334062970061045550543879695550911969... + 0.04964125353032495602765875345720559087787119094655990361824442... i

Alternate complex forms

1.644948324801209948584303505812372965477409736855405361375516 (cos(0.03018258412864432549625657456870221253746361485371547026939·4229) + i sin(0.030182584128644325496256574568702212537463614853715470269·394229))
1.6449483324801209948584330505812372965477409736855405361375516
\[ e \approx \zeta(2) = \frac{\pi^2}{6} = 1.644934 \] (trace of the instanton shape)

**Polar coordinates**

\[ r = 1.6449483324801209948584330505812372965477409736855405361375516 \]
(radius), \[ \theta = 0.030182584128644325496256574568702212537463614853715470269394229 \] (angle)

1.64494833248… \[ \approx \zeta(2) = \frac{\pi^2}{6} = 1.644934 \] (trace of the instanton shape)

**Alternate complex forms**

\[ i \text{ Im} \left( \sqrt[15]{-\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2} e^{-48i \sqrt{2}}}{\pi^{3/2}}} \right) + \]

\[ C_{MRB}^{1-1/(4\pi)+\pi} + \text{Re} \left( \sqrt[15]{-\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2} e^{-48i \sqrt{2}}}{\pi^{3/2}}} \right) \]
\[
\sqrt{\left(\frac{1}{4} C_{\text{MRB}}^{-1/(2\pi)}\right)^2}
\left(2 C_{\text{MRB}}^{1+\pi} + \frac{1}{15} \tan^{-1}\left(\frac{1}{15} \tan^{-1}\left(\frac{8192 \times 2^{3/4} (\sin(48 \sqrt{2}) + \cos(48 \sqrt{2}))}{8192 \times 2^{3/4} (\cos(48 \sqrt{2}) - \sin(48 \sqrt{2})) - (43 + \sqrt{5}) \pi^{3/2}}\right)\right) \right)
\left(\sin^2\left(\frac{1}{15} \tan^{-1}\left(\frac{8192 \times 2^{3/4} (\sin(48 \sqrt{2}) + \cos(48 \sqrt{2}))}{8192 \times 2^{3/4} (\cos(48 \sqrt{2}) - \sin(48 \sqrt{2})) - (43 + \sqrt{5}) \pi^{3/2}}\right)\right)\right)\]
\left(\sqrt{\pi} \left(-\frac{2}{(-134217728 \sqrt{2} - (927 + 43 \sqrt{5}) \pi^3 + 8192 \times 2^{3/4} (43 + \sqrt{5}) \pi^{3/2}}\right)\right)^{(1/15)}
\left(\cos(48 \sqrt{2}) - \sin(48 \sqrt{2})\right)\]
\left(\cos\left(\tan^{-1}\left(\frac{\text{Im}\left(\sqrt{\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2}}{\pi^{3/2}}} e^{-48 i \sqrt{2}}\right)}{C_{\text{MRB}}^{-1/(4\pi) + \pi} + \text{Re}\left(\sqrt{\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2}}{\pi^{3/2}}} e^{-48 i \sqrt{2}}\right)}\right)\right)\right)
\left(\sin\left(\tan^{-1}\left(\frac{\text{Im}\left(\sqrt{\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2}}{\pi^{3/2}}} e^{-48 i \sqrt{2}}\right)}{C_{\text{MRB}}^{-1/(4\pi) + \pi} + \text{Re}\left(\sqrt{\phi - 21 - \frac{8192 (-1)^{3/4} \sqrt{2}}{\pi^{3/2}}} e^{-48 i \sqrt{2}}\right)}\right)\right)\right)
\]
\[
\sqrt{\frac{1}{4} C_{\text{MRB}}^{-1/(2\pi)}} \left( 2C_{\text{MRB}}^{1+\pi} + \frac{1}{10^{\pi \sqrt{2} \pi}} 2^{3/4} \left( 134.217728 \sqrt{2} + (927 + 43 \sqrt{5}) \pi^3 - 8192 \times \right) \left( 43 + \sqrt{5} \right)^{3/2} \left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) \right)^{(1/30) \cos \left( \frac{1}{15} \tan^{-1} \left( \frac{8192 \times 2^{3/4} \left( \sin(48 \sqrt{2}) + \cos(48 \sqrt{2}) \right)}{8192 \times 2^{3/4} \left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) - (43 + \sqrt{5}) \pi^{3/2} \right) \right)^{4\sqrt{C_{\text{MRB}}}} \right)^2 + \\
\sin^2 \left( \frac{1}{15} \tan^{-1} \left( \frac{8192 \times 2^{3/4} \left( \sin(48 \sqrt{2}) + \cos(48 \sqrt{2}) \right)}{8192 \times 2^{3/4} \left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) - (43 + \sqrt{5}) \pi^{3/2} \right) \right)^{\sqrt{\pi \left( -2 \left( -134.217728 \sqrt{2} - (927 + 43 \sqrt{5}) \pi^3 + \right) 8192 \times 2^{3/4} \left( 43 + \sqrt{5} \right)^{3/2} \\
\left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) \right) \right)^{(1/15)}} \\
\exp \left( i \tan^{-1} \left( \frac{\text{Im} \left( \frac{15^{\sqrt{-\phi - 21 - \frac{8192 (-1)^{3/4} \frac{4}{\sqrt{2}} e^{-48 i \sqrt{2}}}{\pi^{3/2}}}}}{C_{\text{MRB}}^{1-1/(4 \pi) + \pi} + \text{Re} \left( \frac{15^{\sqrt{-\phi - 21 - \frac{8192 (-1)^{3/4} \frac{4}{\sqrt{2}} e^{-48 i \sqrt{2}}}{\pi^{3/2}}}} \right) \right) \right) \right) \right)
\]

\text{Im}(z) \text{ is the imaginary part of } z  \\
\text{Re}(z) \text{ is the real part of } z  \\
i \text{ is the imaginary unit}  \\
\tan^{-1}(x) \text{ is the inverse tangent function}
Alternate forms

\[
C_{MRB}^{1-1/(4\pi)+\pi} + 15 \sqrt{\frac{1}{2} \left(-43 - \sqrt{5}\right) - \frac{8192 \sqrt{2} e^{4\sqrt{2} i} \phi}{\pi^{3/2}}} + 15 \sqrt{-\pi^{3/2} \phi} + 21 - 8192 \left(-1\right)^{3/4} e^{4\sqrt{2} i} \phi - 4MRB\text{ const}
\]

Expanded form

\[
C_{MRB}^{1-1/(4\pi)+\pi} + 15 \sqrt{-\phi - 21 - \frac{6192 \sqrt{2} e^{(3i\pi)/4 + 4\sqrt{2} i}}{\pi^{3/2}}} + 15 \sqrt{-\frac{43}{2} - \frac{\sqrt{5}}{2} - \frac{8192 (-1)^{3/4} \sqrt{2} e^{-4\sqrt{2} i}}{\pi^{3/2}}} + (1/27(((4((2 2^{3/4} e^{(2 \sqrt{2} ((-4)^{(3/2)})/(3 - 2)) (-4)^{(5/4)})/(\sqrt{\pi} sqrt(2))))^3-21-\phi))-1))^2-2\Phi-4MRB\text{ const}
\]
\[
\left( \frac{1}{27} \left( 4 \left( 2 \times 2^{3/4} e^{(2 \sqrt{2} (-4)^{3/2})/(3 \sqrt{2})} (-4)^{5/4} \right)^3 - 21 - \phi \right) - 1 \right)^2 - 2 \Phi - 4 C_{\text{MRB}}
\]

\( \phi \) is the golden ratio
\( \Phi \) is the golden ratio conjugate
\( C_{\text{MRB}} \) is the MRB constant

**Exact result**

\[-4 C_{\text{MRB}} - 2 \Phi + \frac{1}{729} \left( -\phi - 22 - \frac{8192 (-1)^{3/4} 2^{3/2} e^{-48 i \sqrt{2}}}{\pi^{3/2}} \right)^2\]

**Decimal approximation**

2523.681563484859445529621508795021364495211915626822662739330821...

+ 3226.261674351683161024392445998532953064773655082397897805763417...

**Alternate complex forms**

\[-4 C_{\text{MRB}} - 2 \Phi + \frac{1}{729 \pi^{3/2}} \left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)
\]

\[-\phi - 22 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}} \right) +
\]

\[-\frac{1}{729} \left( \left( -\phi - 22 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}} \right)^2 - \right.
\]

\[-\frac{4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2})}{\pi^{3}} \right) \]

\]
\[
\frac{1}{1458 \pi^3} \sqrt{\left( \pi^3 (5832 C_{MBB} + 2916 \Phi - 1015 - 45 \sqrt{5}) + 134217728 \sqrt{2} \sin(96 \sqrt{2}) + 8192 \times 2^{3/4} \left( 45 + \sqrt{5} \right) \pi^{3/2} \left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) \right)^2 + 134217728 \sqrt{2} \left( 45 + \sqrt{5} \right) \pi^{3/2} \left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) \right)^2}
\]

\[
\left( \sin(48 \sqrt{2}) + \cos(48 \sqrt{2}) \right)^2
\]

\[
\cos\left( \tan^{-1} \left( \frac{2 \left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)}{- \phi - 22 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}}} \right) \right)
\]

\[
729 \pi^{3/2} \left( -4 C_{MBB} - 2 \Phi + \frac{1}{729} \left( - \phi - 22 + \frac{1}{\pi^{3/2}} (4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2}) \right) \right)^2
\]

\[
\left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)^2
\]

\[
i \sin\left( \tan^{-1} \left( \frac{2 \left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)}{- \phi - 22 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}}} \right) \right)
\]

\[
729 \pi^{3/2} \left( -4 C_{MBB} - 2 \Phi + \frac{1}{729} \left( - \phi - 22 + \frac{1}{\pi^{3/2}} (4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2}) \right) \right)^2
\]

\[
\left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)^2
\]
\[
\frac{1}{1458 \pi^3} \sqrt{\left( \pi^3 (5832 \text{CMRB} + 2916 \Phi - 1015 - 45 \sqrt{5}) + 134217728 \sqrt{2} \sin(96 \sqrt{2}) + 8192 \times 2^{3/4} (45 + \sqrt{5}) \pi^{3/2} \left( \cos(48 \sqrt{2}) - \sin(48 \sqrt{2}) \right) \right)^2 + 134217728 \sqrt{2} \left( (45 + \sqrt{5}) \pi^{3/2} - 8192 \times 2^{3/4} (\cos(48 \sqrt{2}) - \sin(48 \sqrt{2})) \right)^2} \\
\exp\left( i \tan^{-1}\left( \frac{2 (-4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}))}{-\phi - 22 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}}} \right) \right) \\
\left( 729 \pi^{3/2} \left( -4 \text{CMRB} - 2 \Phi + \frac{1}{729} \left( \left( -\phi - 22 + \frac{4096 \times 2^{3/4} \cos(48 \sqrt{2}) - 4096 \times 2^{3/4} \sin(48 \sqrt{2})}{\pi^{3/2}} \right)^2 \right) \right) - \left( -4096 \times 2^{3/4} \sin(48 \sqrt{2}) - 4096 \times 2^{3/4} \cos(48 \sqrt{2}) \right)^2 \right) \\
\right) \text{ is the imaginary unit} \\
\tan^{-1}(x) \text{ is the inverse tangent function}
\]

**Polar coordinates**

\( r \approx 4096.1 \) (radius), \( \theta \approx 0.90698 \) (angle)

\[ 4096.1 \approx 4096 = 64^2 \]

**Alternate forms**

\[-4 \text{CMRB} - 2 \Phi + \frac{1}{729} \left( \phi + 22 - \frac{4096 - 4096 i 2^{3/4} e^{-48 i \sqrt{2}}}{\pi^{3/2}} \right)^2 \]

\[-4 \text{CMRB} - 2 \Phi + \frac{1}{729} \left( \frac{1}{2} (-45 - \sqrt{5}) - \frac{8192 i \sqrt{-2} e^{-48 i \sqrt{2}}}{\pi^{3/2}} \right)^2 \]
\[-4 \text{MRB} - 2 \Phi + \frac{1}{729} \left( -22 + \frac{1}{2} (-1 - \sqrt{5}) - \frac{8192 (-1)^{3/4} \sqrt{2} e^{-48 i \sqrt{2}}}{\pi^{3/2}} \right)^2 \]

\[-4 \text{MRB} - 2 \Phi + \frac{1}{729} \left( -\phi - 22 - \frac{8192 \sqrt{2} e^{(3 i \pi)/4 - 48 i \sqrt{2}}}{\pi^{3/2}} \right)^2 \]

**Expanded form**

\[-4 \text{MRB} - 2 \Phi + \frac{1015}{1458} + \frac{5 \sqrt{5}}{162} \sqrt{2} e^{-48 i \sqrt{2}} - \frac{67108864 i \sqrt{2} e^{-96 i \sqrt{2}}}{729 \pi^{3/2}} + \frac{8192 (-1)^{3/4} \sqrt{2} \sqrt{5} e^{-48 i \sqrt{2}}}{729 \pi^{3/2}} \]

**On the Ramanujan taxicab numbers**

We have:

\[135^3 + 138^3 = 172^3 - 1 \]
\[11161^3 + 11466^3 = 14256^3 + 1 \]
\[791^3 + 810^3 = 1010^3 - 1 \]
\[9^3 + 10^3 = 12^3 + 1 \]
\[6^3 + 8^3 = 9^3 - 1 \]

We observe that:

\[(172^3 - 1)^{1/31} \]

**Input**

\[\frac{31}{\sqrt[31]{172^3 - 1}} \]
Result

\[ 3^{\frac{1}{31}} \sqrt[31]{188461} \]

Decimal approximation

\[ 1.64566510302... \approx \zeta(2) = \frac{\pi^2}{6} = 1.644934 \] (trace of the instanton shape)

Alternate form

\[ \text{root of } x^{31} - 5088447 \text{ near } x = 1.64567 \]

\[ (14258^3 + 1)^{1/58} \]

Input

\[ \sqrt[58]{14258^3 + 1} \]

Result

\[ \sqrt[29]{21} \sqrt[55]{6572600593} \]

Decimal approximation

\[ 1.6400802564534... \approx \zeta(2) = \frac{\pi^2}{6} = 1.644934 \] (trace of the instanton shape)

\[ ((1010)^3 - 1)^{1/42} \]

Input

\[ \sqrt[42]{1010^3 - 1} \]
Decimal approximation

1.639058233867… result very near to the mean between $\zeta(2) = \frac{\pi^2}{6} = 1.644934… \ldots$, the value of golden ratio 1.61803398… and the 14th root of the Ramanujan’s class invariant $Q=(G_{505}/G_{101/5})^3 = 1164.2696 \ldots$ i.e. 1.65578…, i.e. 1.63958266

$((12^3)+1)^{1/15}$

Input

$\frac{15}{\sqrt{12^3 + 1}}$

Result

$\frac{15}{\sqrt{1729}}$

Decimal approximation

1.64381522874872881305800880313247695143292831436999401726452126788

... 1.6438152287… $\approx \zeta(2) = \frac{\pi^2}{6} = 1.644934$ (trace of the instanton shape)

$(9^3-1)^{1/13}$

Input

$\frac{13}{\sqrt{9^3 - 1}}$

Result

$2^{3/13} \frac{13}{\sqrt{91}}$
**Decimal approximation**

1.660213543033589894465409919904892710280291280277020057159063333

... 1.660213543... result very near to the 14th root of the following Ramanujan’s class invariant \( Q = \left( \frac{G_{505}}{G_{101/5}} \right)^3 = 1164.2696 \) i.e. 1.65578...

In conclusion, we obtain from the mean of the all previous expressions:

\[
\frac{1}{5}\left( \sqrt[13]{9^3-1} + \sqrt[15]{(12^3+1)^{1/15}} + \sqrt[58]{(14258^3+1)^{1/58}} + \sqrt[42]{((1010)^3-1)^{1/42}} + \sqrt[31]{((172^3-1)^{1/31}} \right)
\]

**Input**

\[
\frac{1}{5} \left( \sqrt[13]{9^3-1} + \sqrt[15]{12^3+1} + \sqrt[58]{14258^3+1} + \sqrt[42]{1010^3-1} + \sqrt[31]{172^3-1} \right)
\]

**Result**

\[
\frac{1}{5} \left( 2^{3/13} \sqrt[13]{91} + \sqrt[15]{1729} + \right.
\]

\[
\sqrt[22]{21} \sqrt[38]{6572600593} + \sqrt[14]{7} \sqrt[42]{3003793} + \sqrt[33]{31 \sqrt[31]{188461}}
\]

**Decimal approximation**

1.6457664730248446580518515414601019933590480428714293554700105585

... 1.6457664730248... \( \approx \zeta(2) = \pi^2/6 = 1.644934 \) (trace of the instanton shape)
Alternate forms

\[ \frac{1}{5} \sqrt[29]{21} \frac{58}{\sqrt[21]{6572600593}} + \]
\[ \frac{1}{5} \sqrt[31]{7} \frac{42}{\sqrt[13]{13}} \left( 2^{3/13} \times 7^{18/403} \times 13^{29/546} + 7^{16/465} \times 13^{3/70} \frac{15}{\sqrt[19]{19}} + \right. \]
\[ \left. 3^{3/31} \times 13^{11/1302} \frac{31}{\sqrt[2071]{2071}} + 7^{17/434} \frac{42}{\sqrt[231061]{231061}} \right) \]
\[ \frac{1}{5} \sqrt[31]{7} \frac{58}{\sqrt[13]{13}} \left( 2^{3/13} \times 7^{18/403} \times 13^{45/754} + 7^{16/465} \times 13^{43/876} \frac{15}{\sqrt[19]{19}} + 3^{3/31} \times 13^{27/1798} \frac{31}{\sqrt[2071]{2071}} + \right. \]
\[ \left. 7^{17/434} \times 13^{4/609} \frac{42}{\sqrt[231061]{231061}} + \sqrt[3]{7^{2/899} \frac{58}{\sqrt[505584661]{505584661}}} \right) \]

Expanded form

\[ \frac{1}{5} \times 2^{3/13} \frac{13}{\sqrt[91]{91}} + \frac{15}{5} \sqrt[1729]{1729} + \frac{1}{5} \times 3^{3/31} \frac{31}{\sqrt[188461]{188461}} + \]
\[ \frac{1}{5} \sqrt[14]{7} \frac{42}{\sqrt[3003793]{3003793}} + \frac{1}{5} \sqrt[29]{21} \frac{58}{\sqrt[6572600593]{6572600593}} \]

It’s interesting to observe that also with regard these Ramanujan’s taxicab numbers, the mean of the various \( n \)\(^{th} \) roots that we have calculated, is always a result very near to \( \zeta(2) = 1.64493 \).
Acknowledgments

We would like to thank Professor Augusto Sagnotti theoretical physicist at Scuola Normale Superiore (Pisa – Italy) for his very useful explanations and his availability
References

**String Theory, Gravity and Particle Physics** (Prof. Augusto Sagnotti - SNS) - AstronomiAmo 23.04.2020

**Classical and Quantum Statistical Physics - Fundamentals and Advanced Topics** - CARLO HEISSENBERG, AUGUSTO SAGNOTTI - Cambridge University Press, First published 2022

*Schulman, Lawrence S* - **Techniques and applications of path integration** - Copyright (c) 1981 by John Wiley & Sons, Inc.

*The Geometry of the MRB constant by Marvin Ray Burns*

https://www.academia.edu/22271085/The_Geometry_of_the_MRB_constant