The Series of Reciprocals of The Primes Diverges

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1 Introduction

The divergence of the harmonic series was first proven in the 14th century by Nicole[2]. Then people started to think about what would happen if we replaced the natural numbers with primes. Intuitively, the set of prime numbers is a subset of natural numbers. Will the series converge in that case? The answer is that the series of the reciprocals of the primes diverges. This was first proved by Leonhard Euler in 1737[1], so the theorem is also called *Euler's Theorem*. The proof that I'm referring to is provided by Prof. Ruben and Peter[3]. They proved it by contradiction, and the key idea of their proof is to use the Fundamental Theorem of Arithmetic[1].

Fundamental Theorem of Arithmetic:

> Every positive integer (except the number 1) can be represented in exactly one way apart from rearrangement as a product of one or more primes [5]

In this paper, I will provide more details of their proofs and some basic definitions of series.

[1] The proof actually does not require uniqueness of FTA. It only requires that every integer greater than 1 is either prime or a product of primes[3].

2 Definitions

Definition 2.1. A *series* is an operation of adding a_i one after another. Usually, it writes as

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

We call it the n^{th} partial sum of the series.

If we take the limit of the sum as n goes to infinity, then it becomes an *infinity series.*

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

We have seen a lot of examples of infinity series, such as geometric series, harmonic series, and the series of reciprocals of the primes[2].

Definition 2.2. A series *converges* to a real number L, if the sequence of its partial sums converges to L[3]. Usually writes:

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = L$$

If the series does not converge, we say it *diverges* to $\pm\infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \pm\infty$$

3 Theorems and Proofs

Theorem 3.1. *If p_j denotes the j^{th} prime number, then the series $\sum_{j=1}^{\infty} \frac{1}{p_j}$ diverges.*

Proof. Suppose the series converges. There would exist an M such that

$$\sum_{j=M}^{\infty} \frac{1}{p_j} < \frac{1}{2}$$

[2] According to Euclid's theorem, there are infinitely many prime numbers

[3] A sequence (a_n) of real numbers is said to converge to the real number L provided that for each $\epsilon > 0$, there exists a number N such that $n > N$ implies $|a_n - L| < \epsilon$ [4]

This is because a series converges if and only if it satisfies the Cauchy criterion[4]. Cauchy criterion implies the tails of that series also converges. So we know there must exist a such M.

Now we consider the following series.

$$\sum_{k=0}^{\infty}\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^k \tag{1}$$

Since $\sum_{j=M}^{\infty}\frac{1}{p_j}$ converges and it's less than a half, then series (1) will be a convergent geometric series[5].

Fix the j, then for each $j < M$, the series

$$\sum_{k=0}^{\infty}\left(\frac{1}{p_j}\right)^k \tag{2}$$

is also a convergent geometric series because $\frac{1}{p_j} \leq \frac{1}{2}$ for all j[6].

For each $j < M$, let's multiply the series (2) together with the series (1). Then we have the following expression.

$$\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)\cdots\left(\sum_{k=0}^{\infty}\frac{1}{p_{M-1}^k}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^k\right) \tag{3}$$

For now, the only thing we know about the expression (3) is that each term is a convergent geometric series. Hence, after expanding the whole expression, it equals a real number.

Now suppose the set $\{p_{n_1}, p_{n_2}, \ldots, p_{n_s}\}$ is a collection of primes with $p_{n_i} \geq p_M$ for all i. Let the set $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ be a subset of $\mathbb{N}$. Then I claim that

$$\frac{1}{p_{n_1}^{\alpha_1}\cdot p_{n_2}^{\alpha_2}\cdots p_{n_s}^{\alpha_s}} \tag{4}$$

is a term that occurs in the expansion of

$$\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_s} \tag{5}$$

[4]The Cauchy criterion is satisfied when, for all $\epsilon > 0$, there is a fixed number N such that $|a_j - a_i| < \epsilon$ for all $i, j > N$ [4]

[5]For $|r| < 1$, the geometric series converges to $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$[4]

[6]2 is the smallest prime number

To see this, we can expand the expression (5).

$$\begin{aligned}
\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_s} &= \left(\frac{1}{p_M}+\frac{1}{p_{M+1}}+\cdots\right)^{\alpha_1}\cdot\left(\frac{1}{p_M}+\frac{1}{p_{M+1}}+\cdots\right)^{\alpha_2}\cdots \\
&= \left(\frac{1}{p_M^{\alpha_1}}+\frac{1}{p_{M+1}^{\alpha_1}}+\cdots\right)\cdot\left(\frac{1}{p_M^{\alpha_2}}+\frac{1}{p_{M+1}^{\alpha_2}}+\cdots\right)\cdots \\
&= \frac{1}{p_{n_1}^{\alpha_1}\cdot p_{n_2}^{\alpha_2}\cdots p_{n_s}^{\alpha_s}}+\cdots
\end{aligned}$$

The last equality holds because we have $p_{n_i} \geq p_M$ for all i and for each $1/p_{n_i}$, it will multiply itself α_i times. Hence, after expanding the bracket, one of the term will be the expression (4)

Let M be our cutting line that separates the prime numbers into two parts: $p_{m_i} < p_M$, and $p_{n_i} \geq p_M$. Let any n be a natural number that is greater than 1 and the set $\{\beta_1, \beta_2, \ldots, \beta_t\}$ be another subset of $\mathbb{N}$. By the *Fundamental Theorem of Arithmetic*, we can write n as

$$n = p_{m_1}^{\beta_1}\cdot p_{m_2}^{\beta_2}\cdots p_{m_t}^{\beta_t}\cdot p_{n_1}^{\alpha_1}\cdot p_{n_2}^{\alpha_2}\cdots p_{n_s}^{\alpha_s}$$

Note that there is nothing special about p_{m_i} and p_{n_i}. They are all prime factors of n. The whole expression is just saying n has a prime factorization.

Then we can write $1/n$ as the following expression.

$$\frac{1}{n} = \frac{1}{p_{m_1}^{\beta_1}\cdot p_{m_2}^{\beta_2}\cdots p_{m_t}^{\beta_t}}\cdot\left(\frac{1}{p_{n_1}^{\alpha_1}\cdot p_{n_2}^{\alpha_2}\cdots p_{n_s}^{\alpha_s}}\right)$$

We just showed that the part in the bracket occurs in the expansion of equation (5). So, we can say $1/n$ occurs as a term in the product of

$$\frac{1}{p_{m_1}^{\beta_1}\cdot p_{m_2}^{\beta_2}\cdots p_{m_t}^{\beta_t}}\cdot\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_s} \tag{6}$$

Now I claim that the equation (6) is in the expansion of equation (3) which is the product of geometric series.

$$\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)\cdots\left(\sum_{k=0}^{\infty}\frac{1}{p_{M-1}^k}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{j=M}^{\infty}\frac{1}{p_j}\right)^k\right) \quad (3)$$

This is because $p_{m_i} < p_M$ and we have all the p_{m_i} in (3). For each p_{m_i}, it multiplies itself k times and since $k \in [0, \infty)$, we must have the term, $1/(p_{m_1}^{\beta_1} \cdot p_{m_2}^{\beta_2} \cdots p_{m_t}^{\beta_t})$ in the expansion of the product of the first p_{M-1} geometric series. Similarly, for the double sum part, since we the power k is from 0 to ∞, there must exist a number where $k = \alpha_1 + \alpha_2 + \cdots + \alpha_s$. Hence, if we expand the products (3), we will see the expression (6). Moreover, we can conclude that the product (3) contains $1/n$ for every natural number[7].

However, as we showed before, expression (3) is the product of a bunch of convergent geometric series and that means $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ should converge to real number which contradicts the fact that the harmonic series diverges[8].

$\square$

4 Conclusion

Looking back at the proof, we start by constructing a product of convergent geometric series. Then by the *fundamental theorem of arithmetic*, we express n in terms of the product of p_j. And we found that every term of the harmonic series $(1/n)$ for all $n \geq 1$ occurs in the expansion of that product of convergent geometric series. Then we reach the contradiction. Since each geometric series converges to a real number, the product of them is equal to a real number as well.

We see that prime factorization plays an important role in the proof. It builds a bridge between the primes and the natural numbers, and because we know the harmonic series diverges, it allows us to obtain a contradiction.

[7] Note that we initially assume our n is any natural number that is greater than 1, but it is not hard to see when $k = 0$, the expression (3) is equal to 1.

[8] See the appendix for the proof

5 Appendix

Theorem 5.1. *Harmonic series diverges.*

Proof. First, we write down the expression of harmonic series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

We replace the denominator with the next-largest power of 2 and get the following expression.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots$$

Since we replaced the denominator by a larger number, the equality will become inequality. Then we have

$$\begin{aligned}
&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&\geq 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \sum_{n=1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

□

References

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