Assuming \( c < \text{rad}^2(abc) \), The abc Conjecture is True

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Abstract

In this paper, we consider the abc conjecture. Assuming that \( c < \text{rad}^2(abc) \) is true, we give the proof of the abc conjecture for \( \epsilon \geq 1 \), then for the case \( \epsilon \in ]0,1[ \), we consider that the abc conjecture is false, from the proof, we arrive in a contradiction.

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MSC Classification: 11AXX , 26AXX , 11JXX

To the memory of my Father who taught me arithmetic,
To my Wife, my Daughter and my Son.

1 Introduction and notations

Let a positive integer \( a = \prod a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod a_i^{\alpha_i} \) noted by \( \text{rad}(a) \). Then \( a \) is written as :

\[ a = \prod a_i^{\alpha_i} = \text{rad}(a) \cdot \prod a_i^{\alpha_i-1} \tag{1} \]

We note:

\[ \mu_a = \prod a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a) \tag{2} \]

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Esterlé of Pierre et Marie Curie University
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(Paris 6) [4]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the \( \text{abc} \) conjecture is given below:

**Conjecture 1 (abc Conjecture):** For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)
\]

(3)

where \( K \) is a constant depending only of \( \epsilon \).

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the \( \text{abc} \) conjecture is due to the incomprehensibility how the prime factors are organized in \( c \) giving \( a, b \) with \( c = a + b \). So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) [4]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) [3]. It is the key to resolve the \( \text{abc} \) conjecture. In my paper, I assume that the conjecture \( c < \text{rad}^2(abc) \) holds, I propose an elementary proof of the \( \text{abc} \) conjecture.

\section{The Proof of the abc conjecture}

**Proof** We note \( R = \text{rad}(abc) \) in the case \( c = a + b \) or \( R = \text{rad}(ac) \) in the case \( c = a + 1 \). We assume that \( c < R^2 \) is true.

\subsection{Case : \( \epsilon \geq 1 \)}

Assuming that \( c < R^2 \) is true, we have \( \forall \epsilon \geq 1: \)

\[
c < R^2 \leq R^{1+\epsilon} < K(\epsilon) \cdot R^{1+\epsilon}, \text{ with } K(\epsilon) = e, \epsilon \geq 1
\]

(4)

Then the \( \text{abc} \) conjecture is true.

\subsection{Case: \( \epsilon < 1 \)}

From the statement of the \( \text{abc} \) conjecture 1, we want to give a proof that \( c < K(\epsilon)R^{1+\epsilon} \implies \log K(\epsilon) + (1 + \epsilon)\log R - \log c > 0 \).

For our proof, we proceed by contradiction of the \( \text{abc} \) conjecture. We suppose that the \( \text{abc} \) conjecture is false:

\[
\exists \epsilon_0 \in ]0, 1[, \forall \epsilon > 0, \exists c_0 = a_0 + b_0; \quad a_0, b_0, c_0 \text{ coprime so that } c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \text{ and } \forall \epsilon \in ]0, 1[, \ c_0 > K(\epsilon)R_0^{1+\epsilon}
\]

(5)

We choose the constant \( K(\epsilon) = e^{e^\epsilon} \). Let :

\[
Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)\log R_0 - \log c_0, \epsilon \in ]0, 1[
\]

(6)
From the above explications, if we will obtain $\forall \epsilon \in ]0, 1[, Y_{c_0}(\epsilon) > 0 \implies c_0 < K(\epsilon)R_0^{1+\epsilon} \implies c_0 < K(\epsilon_0)R_0^{1+\epsilon_0}$, then the contradiction with (5).

About the function $Y_{c_0}$, we have:

$$
\lim_{\epsilon \to 1} Y_{c_0}(\epsilon) = 1 + \log(R_0^2/c_0) = \lambda > 0
$$

$$
\lim_{\epsilon \to 0} Y_{c_0}(\epsilon) = +\infty
$$

The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall \epsilon \in ]0, 1[$, we obtain:

$$
Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + \log R_0 = \frac{\epsilon^3 \log R_0 - 2}{\epsilon^3}
$$

(7)

$$
Y'_{c_0}(\epsilon) = 0 \implies \epsilon = \epsilon' = \sqrt[3]{\frac{2}{\log R_0}} \in ]0, 1[ \text{ for } R_0 \geq 8.
$$

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>0</th>
<th>$\epsilon'$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y'(\epsilon)$</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$Y(\epsilon)$</td>
<td>$-\infty$</td>
<td>$Y(\epsilon')$</td>
<td>$\lambda &gt; 0$</td>
</tr>
</tbody>
</table>

Fig. 1 Table of variations

**Discussion from the table (Fig.: 1):**

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall \epsilon \in ]0, 1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \implies c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$ and the supposition that the abc conjecture is false can not hold. Hence the abc conjecture is true for $\epsilon \in ]0, 1[$.

- If $Y_{c_0}(\epsilon') < 0 \implies \exists 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We recall the following definition:

**Definition 2** The number $\xi$ is called algebraic number if there is at least one polynomial:

$$
l(x) = l_0 + l_1 x + \cdots + a_m x^m, \quad a_m \neq 0
$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$
c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \implies \frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0 b_0)} = \frac{1}{\epsilon_1^2} R_0^{\epsilon_1}
$$

(9)

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $\alpha_1 = R_0$ are also algebraic numbers. We obtain:

$$
\frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0 b_0)} = e^{\beta_0}.\alpha_1^{\beta_1}
$$

(10)
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From the theorem (see theorem 3, page 196 in [1]):

**Theorem 3** $e^{\beta_0 \alpha_1^{\beta_1} \ldots \alpha_n^{\beta_n}}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$.

we deduce that the right member $e^{\beta_0 \alpha_1^{\beta_1}}$ of (10) is transcendental, but the term $\frac{\mu_{c_0}}{\text{rad}(a_0 b_0)}$ is an algebraic number, then the contradiction and the case $Y_{c_0}(\epsilon') < 0$ is impossible. It follows $Y_{c_0}(\epsilon') \geq 0$ then the abc conjecture is true.

ii) - We suppose that $\epsilon_1$ is transcendental, then $1/(\epsilon_1^2), e^{1/(\epsilon_1^2)}$ and $R_1^{\epsilon} = e^{\epsilon \log R_0}$ are also transcendental, we obtain that $c_0/R_0$ is transcendental, then the contradiction with $c_0/R_0$ an algebraic number. It follows that $Y_{c_0}(\epsilon') \geq 0$ and the abc conjecture is true.

Then the proof of the abc conjecture is finished. Assuming $c < R^2$ true, we obtain that $\forall \epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with $a, b, c$ positive integers relatively coprime, then:

$$c < K(\epsilon).\text{rad}^{1+\epsilon}(abc)$$

and the constant $K(\epsilon)$ depends only of $\epsilon$.

Q.E.D

Ouf, end of the mystery! □

3 Conclusion

Assuming $c < R^2$ is true, we have given an elementary proof of the abc conjecture. We can announce the important theorem:

**Theorem 4** Assuming the conjecture $c < R^2$ true, the abc conjecture is true:

For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).\text{rad}^{1+\epsilon}(abc)$$

where $K$ is a constant depending of $\epsilon$.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.
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