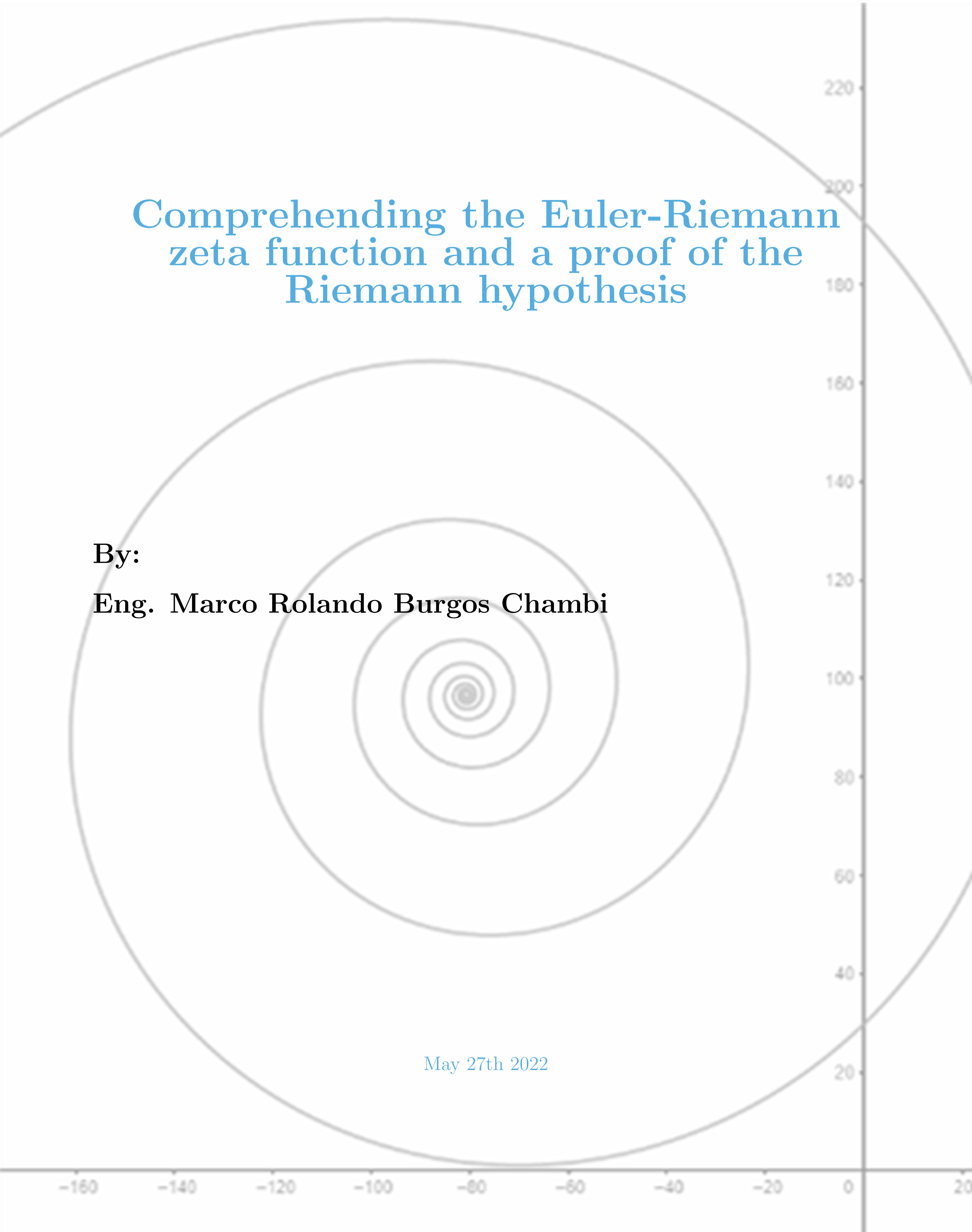
Comprehending the Euler-Riemann zeta function and a proof of the Riemann hypothesis

By:

Eng. Marco Rolando Burgos Chambi

May 27th 2022



Dedication.

This work is dedicated to all my family who supported me at all times and in the most difficult moments of my life, especially my beloved wife Araceli, my two beautiful children Ocram and Arelys, and my parents who never lost faith in me.

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Abstract

On 1859, the german mathematician Georg Friedrich Bernhard Riemann made one of his most famous publications "On the Number of Prime Numbers less than a Given Quantity" when he was developing his explicit formula to give an exact number of primes less than a given number x, in which he conjectured that "all non-trivial zeros of the zeta function have a real part equal to $\frac{1}{2}$". Riemann was sure of his statement, but he could not prove it, remaining as one of the most important hypotheses unproven for 163 years.

This paper will prove that the Riemann Hypothesis is true., based on the following statements:

- The resulting value of the Euler-Riemann zeta function $\zeta(k)$ is the center of a spiral on the complex plane, where $k \in \mathbb{C}$.
- The center of this spiral when $\zeta(k) = 0$, coincides with the origin of coordinates of the complex plane.
- There exists a function related to this spiral $S^*_{-k}(n)$, such that the equality is satisfied:

$$\zeta(k) = \lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^k} - S^*_{-k}(n)\right]$$

And $S^*_{-k}(n)$ is a spiral with center at the origin of coordinates of the complex plane, and $\sum_{n=1}^{n}\frac{1}{n^k}$ is a spiral with center at $\zeta(k)$.

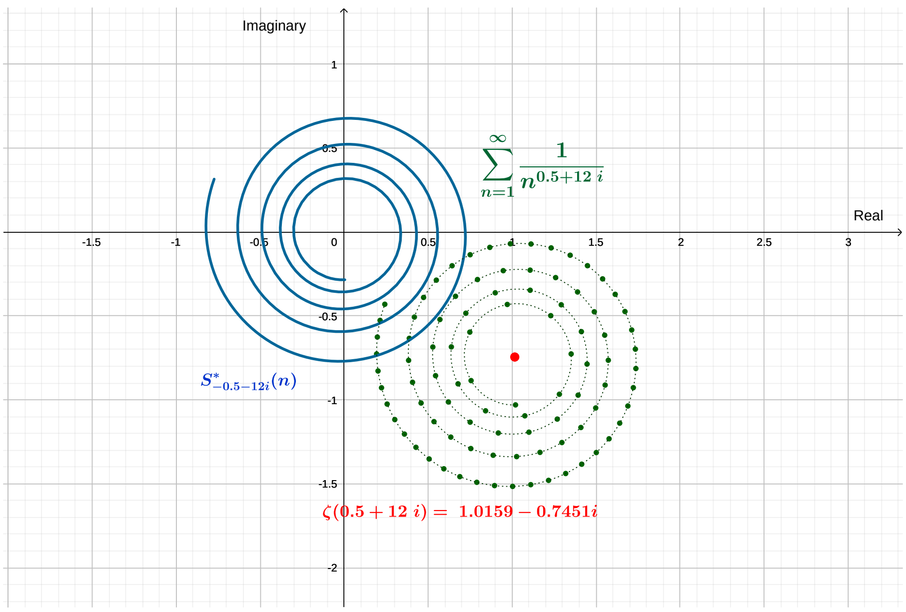

Figure 1: *Spiral graphics in the complex plane given by the function $S^*_{-k}(n)$ (blue), the series $\sum_{n=1}^{n}\frac{1}{n^k}$ (green) and the point $\zeta(k)$ (red).*

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1 Introduction

1.1 Euler product and the sum of inverse powers.

The infinite sum of inverse powers is a series of great interest for mathematics in number theory. Leonhard Euler [2] managed to relate this series to an infinite product that goes through all the prime numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \prod_{p \in \mathbb{P}} \frac{p^k}{p^k - 1}$$

Where $k \in \mathbf{C} \wedge p$ is the n-th prime number.

This series is convergent for values of $Re(k) > 1$, however it is divergent for values of $Re(k) \leq 1$.

Euler was able to find a closed formula for even powers, $2k$ when $k \in \mathbb{N}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!}$$

Where B_{2k} are Bernoulli numbers; $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, etc.

For example for $k = 1$, (the classic Basel problem) [3] is easily solved with this equality:

Example 1

$$\sum_{n=1}^{\infty} \frac{1}{n^{2*1}} = \frac{(-1)^{1-1}(2\pi)^{2*1} B_{2*1}}{2(2*1)!}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(-1)^0(2\pi)^2 B_2}{2(2)!}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1 * 4\pi^2 \frac{1}{6}}{2 * 2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

1.2 The Euler-Riemann zeta function $\zeta(k)$.

Riemann introduced the function $\zeta(k)$ [4], making it equal to the series of the sum of the inverse of k-th power inverses in the convergence range $Re(k) > 1$:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

And it manages to give continuity to the function, in the range of the complex plane, where the series diverges through the functional equation:

$$\zeta(k) = 2^k \pi^{k-1} sin\left(\frac{\pi k}{2}\right) \Gamma(1-k)\zeta(1-k)$$

Where Γ is the gamma function.

If $Re(k) < 0$, then $\zeta(k)$ can be calculated with the functional equation using of the value of the convergence of the series of the inverse of the powers $\sum_{n=1}^{\infty} \frac{1}{n^{1-k}} = \zeta(1-k)$, so for example for $k = -1$:

Example 2

$$\zeta(-1) = 2^{-1}\pi^{-1-1}sin\left(\frac{\pi(-1)}{2}\right)\Gamma(1-(-1))\zeta(1-(-1))$$
$$\zeta(-1) = 2^{-1}\pi^{-2}sin\left(\frac{\pi(-1)}{2}\right)\Gamma(2))\zeta(2)$$
$$\zeta(-1) = 2^{-1}\pi^{-2}(-1)(1)\frac{\pi^2}{6}$$
$$\zeta(-1) = -\frac{1}{12}$$

From the functional equation, we deduce that for even negative values of k the function $\zeta(k) = 0$, at these *"zeros"* Riemann called *"Trivial zeros"*. There also exist values of k that lie within the range $0 < Re(k) < 1$ that makes the function $\zeta(k) = 0$, these values of k are called *"Nontrivial zeros"* of the function $\zeta(k)$ and which Riemann conjectured all lie on the straight line $Re(k) = \frac{1}{2}$.

The conjecture cannot be proved with the Riemann functional equation alone, because the function is redundant for the so-called critical range: $0 < Re(k) < 1$, for example:

Example 3

$$\zeta(0.1) = 2^{0.1}\pi^{-0.9}sin\left(\frac{\pi * 0.1}{2}\right)\Gamma(0.9)\zeta(0.9)$$

y

$$\zeta(0.9) = 2^{0.9}\pi^{-0.1}sin\left(\frac{\pi * 0.9}{2}\right)\Gamma(0.1)\zeta(0.1)$$

Neither $\zeta(0.1)$*, nor* $\zeta(0.9)$ *can be solved. .*

To calculate the values of $\zeta(k)$ in the critical range $0 < Re(k) < 1$, must to be used numerical methods that calculate approximate values of $\zeta(k)$, which do not prove the hypothesis despite the fact that all computationally obtained non-trivial zeros have the value of $Re(k) = \frac{1}{2}$.

2 Complex power of a number and its conjugate.

The complex power of a number is deduced from properties of logarithms and Euler's identity. This will be a basic tool for the study of functions in complex variable, to convert a complex number in polar form to its Cartesian form and vice-versa:

$$z = n^{a+bi} = n^a e^{ibln(n)} = n^a\left[\cos\left(b\ln n\right) + i\sin\left(b\ln n\right)\right]$$

And the conjugate:

$$\overline{z} = n^{a-bi} = n^a e^{-ibln(n)} = n^a\left[\cos\left(b\ln n\right) - i\sin\left(b\ln n\right)\right]$$

3 Bernoulli numbers and the sum of k-th power.

In mathematics, the Bernoulli numbers B_k is a set of successive rational numbers with relevant importance in number theory. They appear in *Combinatorics*, in the expansion of the tangent functions and the hyperbolic tangent by Taylor series. As we have already seen, Euler obtained a closed formula for $\zeta(k)$ when k is a positive even number. If we replace Euler's formula in the Riemann functional equation, we obtain another closed formula for negative integer values of k:

$$\zeta(-k) = \frac{B_{k+1}}{k+1}$$

Where $k \in \mathbb{N}$

They are called Bernoulli numbers because Abraham de Moivre named them that way, in honor of Jakob Bernoulli, the first mathematician who studied them. There are several ways to obtain the values of B_k, but they were obtained for the first time by Jakob Bernoulli, using series of sum of k-th power, so, for example, Jakob managed to deduce that the sum of n consecutive natural numbers can be calculated with the equation:

$$\sum_{n=1}^{n} n^1 = 1+2+3\cdots = S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

Where $k=1$ and $S_k(n)$ is a continuous function.

Similarly Jakob Bernoulli was able to obtain a formula for the sum of square powers $S_2(n)$:

$$\sum_{n=1}^{n} n^2 = 1^2+2^2+3^2\cdots = S_1(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

In general one can obtain the sum of k-th power $S_k(n)$, as a function of: $S_{k-1}(n)$, $S_{k-2}(n)$, $S_{k-3}(n)$,...$S_0(n)$ where $k \in \mathbb{N}$:

$$\sum_{n=1}^{n} n^k = S_k(n) = \frac{1}{k+1}\left[(n+1)^{k+1} - 1 - \sum_{m=0}^{k-1}\binom{k+1}{m}S_m(n)\right]$$

In a posthumous publication by Jakob Bernoulli [1], we can find a listing of the sums of powers up to $k=10$. Because of the relevance of $S_k(n)$ for proving the Riemann hypothesis, in this publication we present the $S_k(n)$ functions up to $k=11$ so that the reader can observe the properties we will state of the $S_k(n)$ functions later:

$$\sum_{n=1}^{n} n^k = S_k(n)$$

$$\sum_{n=1}^{n} n^0 = S_0(n) = n$$

$$\sum_{n=1}^{n} n^1 = S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\sum_{n=1}^{n} n^2 = S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\sum_{n=1}^{n} n^3 = S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$\sum_{n=1}^{n} n^4 = S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$\sum_{n=1}^{n} n^5 = S_5(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$\sum_{n=1}^{n} n^6 = S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$\sum_{n=1}^{n} n^7 = S_7(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$\sum_{n=1}^{n} n^8 = S_8(n) = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

$$\sum_{n=1}^{n} n^9 = S_9(n) = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2$$

$$\sum_{n=1}^{n} n^{10} = S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

$$\sum_{n=1}^{n} n^{11} = S_{11}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2$$

Note 1 *Up to this point, everything presented is public knowledge and available in the bibliography, so it was not necessary to list the equations presented. From now on, when new concepts are presented, all new equations presented will be listed.*

3.1 Obtaining the Bernoulli numbers by the sum of powers.

It was known that to obtain the Bernoulli numbers it was necessary to derive $S_k(n)$ and evaluate it at zero, however this concept is not entirely correct since for B_1 when applying this concept it is not possible to obtain the value of $B_1 = -1/2$, value obtained by other methods. The correct way to obtain the Bernoulli numbers by the sum of k-th power is with the equation:

$$B_k = (-1)^k S_k'(0) \tag{1}$$

The equation (1) *will be proofed later on*

For example, to obtain B_2:

Example 4

$$\begin{aligned} B_2 &= (-1)^2 S_2'(0) \\ B_2 &= (1)\left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right]'_{n=0} \\ B_2 &= \left[n^2 + n + \frac{1}{6}\right]_{n=0} \\ B_2 &= \frac{1}{6} \end{aligned}$$

Similarly, from the function $S_k(n)$ all Bernoulli numbers are obtained: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, B_9 = 0, B_{10} = 5/66, B_{11} = 0$.

It is noted that:

$$B_k = 0 \ / \ k = 2m + 1, m \in \mathbb{N}$$

3.2 Simplified formula to find $S_k(n)$.

Another way to write the formula for the sum of k-th power is as follows:

$$S_k(n) = \sum_{p=1}^{k+1} C_p(k) n^p$$

Where

$$C_p(k) = \frac{(-1)^{k+1-p}}{k+1}\binom{k+1}{p} B_{k+1-p}$$

And B_k is obtained by:

$$B_k = -\frac{1}{k+1}\sum_{m=0}^{k-1}\binom{k+1}{m} B_m$$

Note 2 *This way of finding $S_k(n)$, will be very useful for the purpose of this work.*

4 Properties of the function $S_k(n)$, and its coefficients $C_p(k)$.

4.1 Properties of the function $S_k(n)$.

In the following, we will enumerate some properties of the function $S_k(n)$ that are easy to verify, when $k \in \mathbb{N}$:

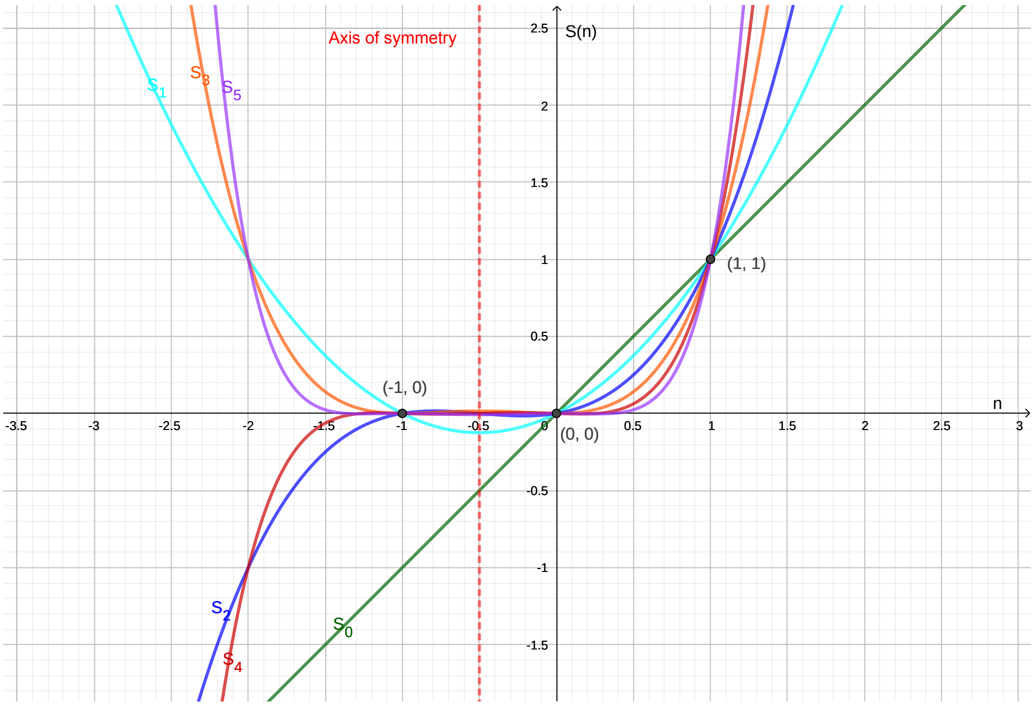

Figure 2: *Graph of the function $S_k(n)$ for values of k from 0 to 5. Common intersection points and the symmetry axis.*

1. In any function $S_k(n)$, or sum of k-th power, the ordered pairs: $(1,1); (0,0); (-1,0) \in S_k(n)$ (view Figure 2).

2. Any function $S_k(n)$ is *"even symmetric"* when k is odd and *"odd symmetric"* when k is even, always with respect to the symmetry axis $n = -\frac{1}{2}$. (view Figure 2).

3. If we integrate $S_k(n)$ between $[-1, 0]$ or, equivalently, evaluate the integral at $n = -1$, we obtain $\zeta(-k)$.

$$\zeta(-k) = -\int_{-1}^{0} S_k(n)\mathrm{d}n \tag{2}$$

   The equation (2) will be proofed later on.

4. In the coefficients of the functions $S_k(n)$ it is observed that there is a pattern betwenn the coefficients of the función $S_k(n)$ and an intimate relationship with the function $\zeta(k)$, and they are function of k in order of ascending degree starting at:$k^{-1}, k^0, k^1, k^2, k^3, ...k^n$. For example the first 6 coefficients satisfy the following functions Cp(k) where p is the position or order of the coefficients of Sk(n):

$$C_1(k) = \frac{1}{k+1} \tag{3}$$

$$C_2(k) = \frac{1}{2} \tag{4}$$

$$C_3(k) = \frac{1}{12}k \tag{5}$$

$$C_4(k) = 0 \tag{6}$$

$$C_5(k) = \frac{k(k+1)(k+2)}{720} \tag{7}$$

$$C_6(k) = 0 \tag{8}$$

4.2 The coefficients $C_p(k)$ of $S_k(n)$.

The coefficients $C_p(k)$ can be calculated by solving polynomial regressions or systems of linear equations using the values of the coefficients of the function $S_k(n)$. But it is easier to obtain the coefficients by working a little bit the known formula:

$$S_k(n) = \sum_{p=1}^{k+1} \frac{(-1)^{k+1-p}}{k+1} \binom{k+1}{p} B_{k+1-p} n^p = \sum_{p=1}^{k+1} C_p(k) n^{k+2-p}$$

To obtain:

$$S_k(n) = \sum_{p=1}^{k+1} C_p(k) n^{k+2-p} \tag{9}$$

Where:

$$C_p(k) = \frac{(-1)^{p-1}}{(k+1)(p-1)!} \prod_{m=1}^{p-1} (k+2-m) B_{p-1} \tag{10}$$

Proof of equation (10):

Factorizing $k+1$ and developing the summation and binomial coefficient of $S_k(n)$:

$$S_k(n) = \frac{1}{1+k} \left[\frac{(-1)^k (k+1)!}{1!k!} B_k n + \frac{(-1)^{k-1}(k+1)!}{2!(k-1)!} B_{k-1} n^2 + \frac{(-1)^{k-2}(k+1)!}{3!(k-2)!} B_{k-2} n^3 + \right.$$
$$\left. \frac{(-1)^{k-3}(k+1)!}{4!(k-3)!} B_{k-3} n^4 + \cdots + \frac{(-1)^1 (k+1)!}{k!1!} B_1 n^k + \frac{(-1)^0 (k+1)!}{(k+1)!(0)!} B_0 n^{k+1} \right]$$

Rearranging terms and accommodating the factorials in order to simplify:

$$S_k(n) = \frac{1}{1+k} \left[\frac{(-1)^0 (k+1)!}{(k+1)!0!} B_0 n^{k+1} + \frac{(-1)^1 (k+1)!}{k!1!} B_1 n^k + \frac{(-1)^2 (k+1)!}{(k-1)!2!} B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k (k+1)!}{1!k!} B_k n \right]$$

$$S_k(n) = \frac{1}{1+k} \left[\frac{(-1)^0 (k+1)!}{(k+1)!0!} B_0 n^{k+1} + \frac{(-1)^1 k!(k+1)}{k!1!} B_1 n^k + \frac{(-1)^2 (k-1)!k(k+1)}{(k-1)!2!} B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k (k+1)!}{1!k!} B_k n \right]$$

$$S_k(n) = \frac{1}{1+k} \left[\frac{(-1)^0}{0!} B_0 n^{k+1} + \frac{(-1)^1 (k+1)}{1!} B_1 n^k + \frac{(-1)^2 k(k+1)}{2!} B_2 n^{k-1} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^k (k+1)!}{1!k!} B_k n \right]$$

Rewriting as a summation of a product of factors:

$$S_k(n) = \sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{(k+1)(p-1)!} \prod_{m=1}^{p-1} (k+2-m) B_{p-1} n^{k+2-p}$$

Where the coefficients $C_p(k)$ are:

$$C_p(k) = \frac{(-1)^{p-1}}{(k+1)(p-1)!} \prod_{m=1}^{p-1} (k+2-m) B_{p-1}$$

The ecuations (9) and (10) has been proofed.

With the equation (4.2) it is possible to prove equation (1)

Proof of equation (1):

Deriving the equation (4.2) and evaluating at zero we obtain:

$$S'_k(0) = \sum_{p=1}^{k+1} (k+2-p)\frac{(-1)^{p-1}}{(k+1)(p-1)!}\prod_{m=1}^{p-1}(k+2-m)B_{p-1}(0)^{k+1-p}$$

Where the only term different from zero is when $p = k+1$, so the expression reduces to:

$$S'_k(0) = \frac{(-1)^k(k+1)!}{(k+1)k!}B_k$$

$$S'_k(0) = \frac{(-1)^k(k+1)!}{(k+1)!}B_k$$

$$S'_k(0) = (-1)^k B_k$$

Reordering:

$$B_k = (-1)^k S'_k(0)$$

the ecuation (1) has been proofed

Note that from the function $C_p(k)$ when evaluated at $k = -1$ we obtain:

$$C_p(-1) = \zeta(2-p) \tag{11}$$

The equation (11) *will be proofed later on.*

5 The funtion $S_k^*(n)$.

Knowing that the function $C_p(k)$ allows us to obtain the coefficients of $S_k(n)$, then it is possible to propose a function $S_k^*(n)$, which is the extension of the function $S_k(n)$ but this time instead of adding $k+1$ terms, the sum of terms will be infinite.

Formula to calculate the sum of k-th power:

$$S_k(n) = \sum_{p=1}^{k+1} C_p(k)n^{k+2-p}$$

Proposed function:

$$S_k^*(n) = \sum_{p=1}^{\infty} C_p(k)n^{k+2-p} \tag{12}$$

Note 3 *We will use the symbol * to distinguish the sum of powers $S_k(n)$, from the proposed function $S_k^*(n)$.*

Now let's define $\Delta_{-k}(n)$ as the difference between the two functions:

$$\Delta_{-k}(n) = \sum_{n=1}^{n} n^k - S_k^*(n) \tag{13}$$

And Δ_{-k} as the limit when $n \to \infty$ of $\Delta_{-k}(n)$:

$$\Delta_{-k} = \lim_{n\to\infty}\left[\sum_{n=1}^{n} n^k - S_k^*(n)\right] \tag{14}$$

Replacing equation (12) in equation(14) we obtain:

$$\Delta_{-k} = \lim_{n\to\infty}\left[\sum_{n=1}^{n} n^k - \sum_{p=1}^{\infty} C_p(k)n^{k+2-p}\right] \tag{15}$$

5.1 Verification of the convergence of the function Δ_{-k}.

It is known that the sum of k-th power is convergent for values of $k < -1$, therefore it is possible to verify if convergence exists for Δ_{-k} in this range:

Let $k < -1$:

The exponent of n is:

$$k < -1$$
$$k + 2 - p < -1 + 2 - p$$
$$k + 2 - p < 1 - p$$

Since p is always positive it is demonstrated that the power of n will always be negative:

$$k + 2 - p < 0$$

Therefore: when $n \to \infty$ and $k < -1$ it is satisfied that the value of $n^{k+2-p} = 0$. Replacing n^{k+2-p} in equation (15) gives:

$$\Delta_{-k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^k - C_1 * 0 - C_2 * 0 - C_3 * 0 - \cdots - C_p * 0 \right]$$
$$\Delta_{-k} = \left[\sum_{n=1}^{\infty} n^k - 0 - 0 - 0 - \cdots \right]$$
$$\Delta_{-k} = \sum_{n=1}^{\infty} n^k \ When\ k < -1 \tag{16}$$

Making a change of variable of k by $-k$ one can write equation (16) as follows:

$$\Delta_k = \sum_{n=1}^{\infty} \frac{1}{n^k} \quad when \quad k > 1 \tag{17}$$

Therefore it is concluded that: the value of Δ_k converges, and is equal to the sum of k-th power inverses when $k > 1$.

Let's see what happens to Δ_{-k} when $k = -1$, in equation (15):

$$\Delta_1 = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^{-1} - C_1 n^0 - C_2 n^{-1} - C_2 n^{-2} - \cdots \right]$$
$$\Delta_1 = \left[\sum_{n=1}^{\infty} n^{-1} - C_1 - 0 - 0 - \cdots \right]$$

Since the sum of powers when $k = -1$ is infinite then:

$$\Delta_1 = \left[\infty - \frac{(-1)^{-1+1}}{(-1+1)(-1+1)!} B_{-1+1} - 0 - 0 - \cdots \right]$$
$$\Delta_1 = \left[\infty - \frac{(-1)^0}{(0)(0)!} B_0 \right]$$
$$\Delta_1 = undetermined \tag{18}$$

We conclude that Δ_k is undetermined, so it has no convergence when $k = 1$.

Now let's analyze what happens with Δ_{-k} in equation (15) when $k \geq 0$:

Let us first separate the summation of $C_p(k)n^{k+2-p}$ into two, where the first rank of p will be from 1 to $(k+1)$ and the second rank, from $(k+2)$ to (∞):

$$\Delta_{-k} = \lim_{n\to\infty}\left[\sum_{n=1}^{n} n^k - \sum_{p=1}^{k+1} C_p(k)n^{k+2-p} - \sum_{p=k+2}^{\infty} C_p(k)n^{k+2-p}\right]$$

Since:

$$\sum_{n=1}^{n} n^k = \sum_{p=1}^{k+1} C_p(k)n^{k+2-p}$$

We obtain:

$$\Delta_{-k} = -\lim_{n\to\infty}\left[\sum_{p=k+2}^{\infty} C_p(k)n^{k+2-p}\right]$$

Developing the summation:

$$\Delta_{-k} = -\lim_{n\to\infty}\left[C_{k+2}(k)n^0 + C_{k+3}(k)n^{-1} + C_{k+4}(k)n^{-2} + \cdots\right]$$

Replacing the limit:

$$\Delta_{-k} = -[C_{k+2}(k) + C_{k+3}(k)0 + C_{k+4}(k)0 + \cdots]$$

Simplifying:

$$\Delta_{-k} = -C_{k+2}(k) \quad para \quad k < 0 \tag{19}$$

Replacing the value of $C_{k+2}(k)$ gives:

$$\Delta_{-k} = -\frac{(-1)^{k+1}}{(k+1)(k+1)!}\prod_{m=1}^{k+1}(k+2-m)B_{k+1} \tag{20}$$

Solving the product:

$$\Delta_{-k} = -\frac{(-1)^{k+1}}{(k+1)(k+1)!}B_{k+1}\left[(k+1) * k * (k-1) * (k-2) * \cdots * 1\right]$$

$$\Delta_{-k} = -\frac{(-1)^{k+1}}{(k+1)(k+1)!}B_{k+1}(k+1)!$$

Simplifying terms:

$$\Delta_{-k} = \frac{(-1)^k B_{k+1}}{(k+1)} \tag{21}$$

This equation is equivalent to the well-known formula for the function $\zeta(-k)$ when $k \in \mathbb{N}$, and also satisfies when $k = 0$.

Theorem 1 *Let $S_k^*(n)$ be defined as:*

$$S_k^*(n) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(k+1)(p-1)!} \prod_{m=1}^{p-1} (k+2-m) B_{p-1} n^{k+2-p} \tag{22}$$

And the function Δ_{-k}:

$$\Delta_{-k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^k - S_k^*(n) \right]$$

It can be written:

$$\Delta_{-k} = \lim_{n\to\infty} \left[\sum_{n=1}^{n} n^k - \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(k+1)(p-1)!} \prod_{m=1}^{p-1} (k+2-m) B_{p-1} n^{k+2-p} \right] \tag{23}$$

Since the limit when $n \to \infty$ of the function converges for all $k \in \mathbb{Z}$, and is expressed as a power series, it also converges for any $k \in \mathbb{C}$, moreover its value coincides with that of the function $\zeta(-k)$ therefore one can define the function $\zeta(k)$ as follows:

$$\zeta(k) = \lim_{n\to\infty} \left[\sum_{n=1}^{n} \frac{1}{n^k} - S_{-k}^*(n) \right] \tag{24}$$

Where $S_{-k}^*(n)$ is the related function of $\sum_{n=1}^{n} \frac{1}{n^k}$.

Now let us compare the summation of k-th powers for any real value of k in the function $S_k^*(n)$, giving increasingly larger n values:

Example 5 *Let $k = \frac{1}{4}$:*

To facilitate the calculations without losing much precision, as $k < 1$, and n will become larger and larger, we will use the first 3 terms of the function $S_k^*(n)$, since the value of the rest of the terms tends quickly to 0:

$$S_k^*(n) \approx \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \frac{k}{12} n^{k-1} \tag{25}$$

For $n = 4$

$$\sum_{n=1}^{4} n^{\frac{1}{4}} = 1^{\frac{1}{4}} + 2^{\frac{1}{4}} + 3^{\frac{1}{4}} + 4^{\frac{1}{4}} \cong 4.91949469$$

$$S_{\frac{1}{4}}^*(n) \approx \frac{1}{\frac{1}{4}+1} n^{\frac{1}{4}+1} + \frac{1}{2} n^{\frac{1}{4}} + \frac{\frac{1}{4}}{12} n^{\frac{1}{4}-1}$$

$$S_{\frac{1}{4}}^*(n) \approx \frac{4}{5} n^{\frac{5}{4}} + \frac{1}{2} n^{\frac{1}{4}} + \frac{1}{48} n^{-\frac{3}{4}}$$

$$S_{\frac{1}{4}}^*(4) \approx \frac{4}{5} 4^{\frac{5}{4}} + \frac{1}{2} 4^{\frac{1}{4}} + \frac{1}{48} 4^{-\frac{3}{4}} \approx 5.23995588$$

The difference $\Delta_{-k}(n)$ between $\sum_{n=1}^{n} n^k$ and $S_k^*(n)$ is:

$$\Delta_{-\frac{1}{4}}(4) = \sum_{n=1}^{4} n^{\frac{1}{4}} - S_{\frac{1}{4}}^*(4) \approx -0.32046119$$

For $n = 20$
The sum is:

$$\sum_{n=1}^{20} n^{\frac{1}{4}} \cong 34.57500317$$

The function $S^*(n)$ is :

$$S_{\frac{1}{4}}^*(20) = 34.89545455$$

And $\Delta(n)$*:*

$$\Delta_{-\frac{1}{4}}(20) = \sum_{n=1}^{20} n^{\frac{1}{4}} - S^*_{\frac{1}{4}}(20) \approx -0.32045138$$

For $n = 1000$
The sum is:

$$\sum_{n=1}^{1000} n^{\frac{1}{4}} \cong 4501.221974$$

The function $S^*(n)$ *is:*

$$S^*_{\frac{1}{4}}(1000) = 4501.542425$$

y $\Delta(n)$*:*

$$\Delta_{-\frac{1}{4}}(1000) = \sum_{n=1}^{1000} n^{\frac{1}{4}} - S^*_{\frac{1}{4}}(1000) \approx -0.320451264$$

If we compute the value of $\zeta(-\frac{1}{4})$ *by some numerical method we obtain the approximate value* -0.320451264*, and by comparing with the value of* $\Delta_{-\frac{1}{4}} = \zeta(-\frac{1}{4})$

Example 6 *Now let's calculate a second example, but this time when* $k \in \mathbb{C}$

Let $k = -0.4 - 7i$*:*

This time we will tabulate the results and compare with $\zeta(0.4 + 7i)$*:*

n	$\sum_{n=1}^{n} n^{-0.4-7i}$	$S^*_{-0.4-7i}(n)$	$\Delta_{0.4+7i}$
10	$0.57811497 + 0.019131365i$	$-0.441562086 - 0.398323033i$	$1.019677056 + 0.417454399i$
100	$2.847975959 + 1.750845166i$	$1.82847357 + 1.333500916i$	$1.019502389 + 0.417344249i$
1000	$36.34167952 - 5.123159998i$	$35.32217705 - 5.540504261i$	$1.01950247 + 0.417344263i$

If we calculate the value of $\zeta(0.4+7i)$ *by some numerical method we obtain the approximate value of* $1.01950247 + 0.417344263i$*, and comparing with the value of* $\Delta_{0.4+7i}$ *we again verify that* $\Delta_{0.4+7i} = \zeta(0.4 + 7i)$*.*

6 The related function $S^*_{-k}(n)$, the sum series of k-th power inverses, the function $\zeta(k)$, and its conjugates.

6.1 The related function $S^*_{-k}(n)$ to the sum series of k-th power inverses.

Since the function $S^*_{-k}(n)$ is the function appearing in equation (24) it is necessary to rewrite the equation (22):

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(1-k)(p-1)!} \prod_{m=1}^{p-1} (2-k-m) B_{p-1} n^{2-k-p} \tag{26}$$

Developing and rearranging expression:

$$S^*_{-k}(n) = \frac{1}{(1-k)0!} B_0 n^{1-k} - \frac{1}{1!} B_1 n^{-k} - \frac{k}{2!} B_2 n^{-1-k} - \frac{k(k+1)}{3!} B_3 n^{-2-k} - \frac{k(k+1)(k+2)}{4!} B_4 n^{-3-k} - \cdots$$

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} \frac{B_{p-1}}{(1-k)(p-1)!} \prod_{m=1}^{p-1} (k+m-2) n^{2-k-p} = \sum_{p=1}^{\infty} \frac{B_{p-1}(k+p-3)!}{(k-1)!(p-1)!} n^{2-k-p} \tag{27}$$

Since $k \in \mathbb{C}$, and let $a, b \in \mathbb{R}$, it is convenient to use the product operator, to avoid factorials and the Gamma function, so that: $k = a + bi$, and introduce it in equation (26):

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} \frac{B_{p-1}}{(1-a-bi)(p-1)!} \prod_{m=1}^{p-1} (a+m-2+bi) n^{2-a-p-bi} \tag{28}$$

Applying properties of complex numbers:

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} \frac{(1-a+bi)B_{p-1}}{[(1-a)^2+b^2]\,(p-1)!} \prod_{m=1}^{p-1} (a+m-2+bi) n^{2-a-p} e^{-ib\ln n} \tag{29}$$

Where $C_p(-a-bi)$ is:

$$C_p(-a-bi) = \frac{(1-a+bi)B_{p-1}}{[(1-a)^2+b^2]\,(p-1)!} \prod_{m=1}^{p-1} (a+m-2+bi) \tag{30}$$

Let u_p y v_p defined as:

$$u_p(-a-bi) = Re\,[C_p(-a-bi)] \tag{31}$$

$$v_p(-a-bi) = Im\,[C_p(-a-bi)] \tag{32}$$

Then:

$$C_p(-a-bi) = u_p + v_p i \tag{33}$$

The function can be written in its polar form:

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} n^{2-a-p}\,[u_p + v_p i] e^{-ib\ln n} \tag{34}$$

It can also be expressed in its Cartesian form:

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} n^{2-a-p}\,[u_p + v_p i]\,[\cos(b\ln n) - i\sin(b\ln n)]$$

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} n^{2-a-p}\,\{[u_p[\cos(b\ln n) + v_p\sin(b\ln n)] + i\,[v_p\cos(b\ln n) - u_p\sin(b\ln n)]\} \tag{35}$$

6.2 $S^*_{-k}(n)$ conjugated.

Applying the conjugate property of the product of two complex numbers:

$$\overline{z * w * y} = \overline{z} * \overline{w} * \overline{y}$$

Since we know that $C_p(-a-bi)$ is a product of complex numbers that have in common the same imaginary part bi, the following deduction can be made:

$$C_p(-a-bi) = \frac{B_{p-1}}{[(1-a)^2+b^2]\,(p-1)!}(1-a+bi)*(a-1+bi)*(a+bi)(a+1+bi)*(a+2+bi)*\cdots$$

Applying the conjugate product property:

$$\overline{C_p(-a-bi)} = \frac{B_{p-1}}{[(1-a)^2+b^2]\,(p-1)!}\overline{(1-a+bi)}*\overline{(a-1+bi)}*\overline{(a+bi)}*\overline{(a+1+bi)}*\overline{(a+2+bi)}*\cdots$$

$$\overline{C_p(-a-bi)} = \frac{B_{p-1}}{[(1-a)^2+b^2]\,(p-1)!}(1-a-bi)*(a-1-bi)*(a-bi)(a+1-bi)*(a+2-bi)*\cdots$$

Since:

$$C_p(-a+bi) = \frac{B_{p-1}}{[(1-a)^2+b^2]\,(p-1)!}(1-a-bi)*(a-1-bi)*(a-bi)(a+1-bi)*(a+2-bi)*\cdots$$

We infer that:

$$\overline{C_p(-a-bi)} = C_p\overline{(-a-bi)} = u_p - v_p i \tag{36}$$

Replacing equation (36) in equation (34) and finding its conjugate:

$$\overline{S^*_{-a-bi}(n)} = \sum_{p=1}^{\infty} n^{2-a-p}\overline{[u_p + v_p i]}\;\overline{e^{-ib\ln n}}$$

Finally:

$$\overline{S^*_{-a-bi}(n)} = S^*_{\overline{-a-bi}}(n) = S^*_{-a+bi}(n) \tag{37}$$

6.3 The series of k-th power inverse Conjugate.

Applying the conjugate property of the sum of complex numbers to the inverse power series:

$$\overline{z} + \overline{y} + \overline{w} = \overline{z + y + w}$$

It is inferred that:

$$\overline{\sum_{n=1}^{\infty} \frac{1}{n^{a+bi}}} = \sum_{n=1}^{\infty} \frac{1}{n^{a-bi}} \tag{38}$$

6.4 Conjugate of the function $\zeta(k)$.

From the conjugates of $S_k^*(n)$ and $\sum_{n=1}^{\infty} \frac{1}{n^k}$ the conjugate of $\zeta(k)$ can be deduced:

$$\overline{\zeta(k)} = \overline{\sum_{n=1}^{\infty} \frac{1}{n^k} - \lim_{nto\infty} S_{-k}^*(n)} = \overline{\sum_{n=1}^{\infty} \frac{1}{n^k}} - \overline{\lim_{nto\infty} S_{-k}^*(n)} = \sum_{n=1}^{\infty} \frac{1}{n^{\overline{k}}} - \lim_{nto\infty} S_{-\overline{k}}^*(n)$$

$$\overline{\zeta(k)} = \zeta(\overline{k}) \tag{39}$$

7 Graphical interpretation of the function $\zeta(k)$.

7.1 Logarithmic spirals.

A logarithmic spiral is described in its polar form as follows:

$$r = a\ b^{\theta}$$

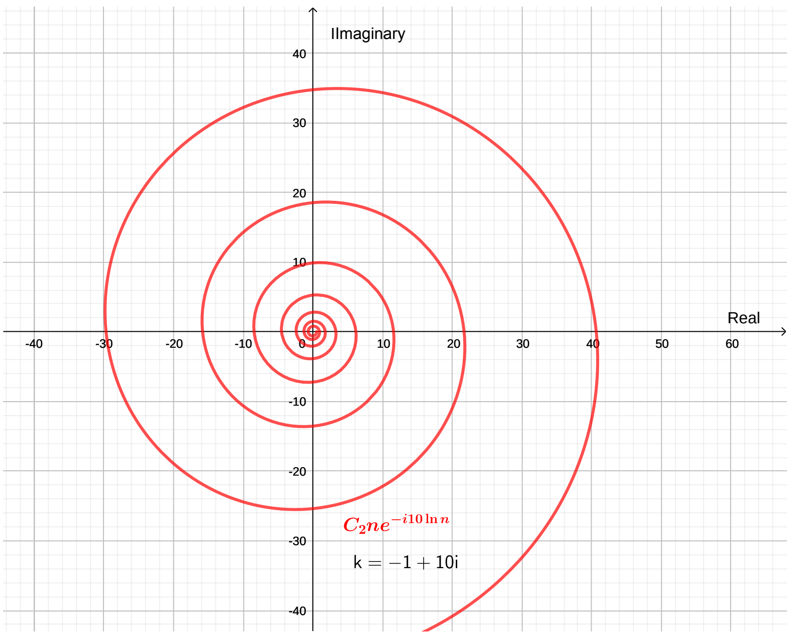

Figure 3: *Example of logarithmic spiral* $C_p n^{2-a-p} e^{-ib \ln n}$ *when* $k = -1 + 10i$ *and* $p = 2$

And the sum of logarithmic spirals is also a logarithmic spiral, although with certain degeneracies, for example, if one of the spirals converges to its center and another diverges from its center, the resulting spiral will try to converge to its center, but then it will move away again.

Logarithmic spirals appear very often in nature, as for example in the snail shell, the shape of galaxies, the spider web, or the turbulence of a fluid in the form of a tornado. It is not surprising that it also appeared in the study of the sum of power series and the function $\zeta(k)$.

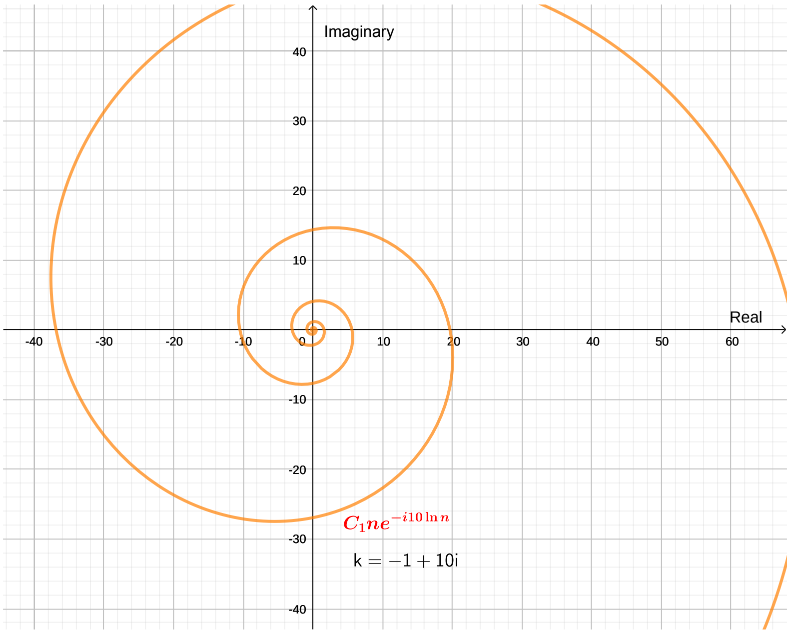

Figure 4: *Example of logarithmic spiral* $C_p n^{2-a-p} e^{-ib\ln n}$ *when* $k = -1 + 10i$ *and* $p = 1$

Considering that $k = a + bi$, equation (34) shows us that the function $S^*_{-k}(n)$ is a sum of functions of the family of logarithmic spirals within the complex plane, and that it results in another spiral of the family of logarithmic spirals, and the center of this spiral is at the origin. $SpiralCenter = (0 + 0i) = 0$:

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} n^{2-a-p}\,[u_p + v_p i]e^{-ib\ln n}$$

If you would like to move the center of the function $S^*_{-k}(n)$ you just have to add a complex number to the function z:

$$S^*_{-k}(n) = \sum_{p=1}^{\infty} n^{2-a-p}\,[u_p + v_p i]e^{-ib\ln n} + z$$

Then the graph of the spiral would shift and its new center would be: $Re(z) + Im(z)$. That is just what happens in the limit of the of Eq.(24) when the inverse k-th power sum series is cleared:

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \lim_{n\to\infty} S^*_{-k}(n) + \zeta(k)$$

This shows us that the series: sum of k-th power inverses is graphed as a sequence of points that are part of a function of the family of logarithmic spirals whose center has coordinates $\zeta(k)$ in the complex plane.

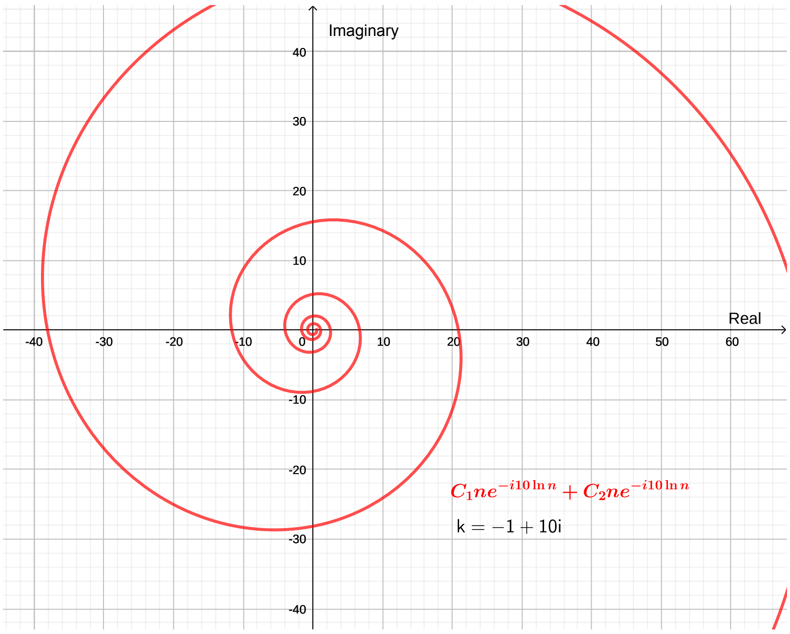

Figure 5: *Example of logarithmic spiral* $C_1 n^{2-a-p} e^{-ib\ln n} + C_2 n^{2-a-p} e^{-ib\ln n}$ *when* $k = -1 + 10i$

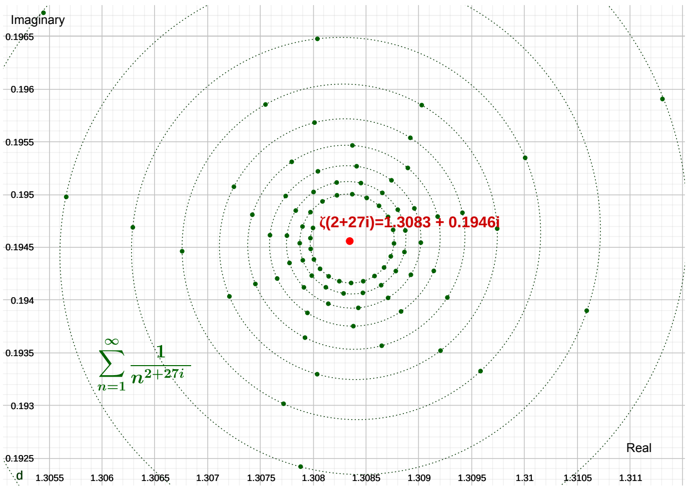

Figure 6: *Example of a logarithmic spiral of the series* $\sum_{n=1}^{\infty} \frac{1}{n^k}$ for $k = 2 + 27i$ where $\zeta(k)$ is the center of the spiral.

Now we understand that for values of $Re(k) > 1$ the spiral of the series $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges to its center, that is why it coincides with the Riemann Zeta function $\zeta(k)$, but when $Re(k) < 1$ the spiral diverges from its center, so it is necessary to calculate it with equation (24).
When: $Re(k) = 1$ the spiral becomes a quasi-circumference:

$$S^*_{-1-bi}(n) = \frac{bi}{b^2 0!} [\cos(b\ln n) - i\sin(b\ln n)]\, n^0$$

$$S^*_{-1-bi}(n) = \frac{i}{b\ 0!} [\cos(b\ln n) - i\sin(b\ln n)]$$

$$S^*_{-1-bi}(n) = \frac{1}{b} [\sin(b\ln n) + i\cos(b\ln n)] \tag{40}$$

Graphing the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+bi}}$ yields a quasi-circumference spiral with center at $\zeta(1 + bi)$.

If we plot the functions of $S^*_{-k}(n)$ and $\sum_{n=1}^{\infty} \frac{1}{n^k}$ in the space $\mathbb{R} \to \mathbb{C}$ we observe two similar helices, but the first with its axis at n and the second shifted where $\zeta(k)$ is its axis. The domino of the functions is: $D =]0, \infty[$

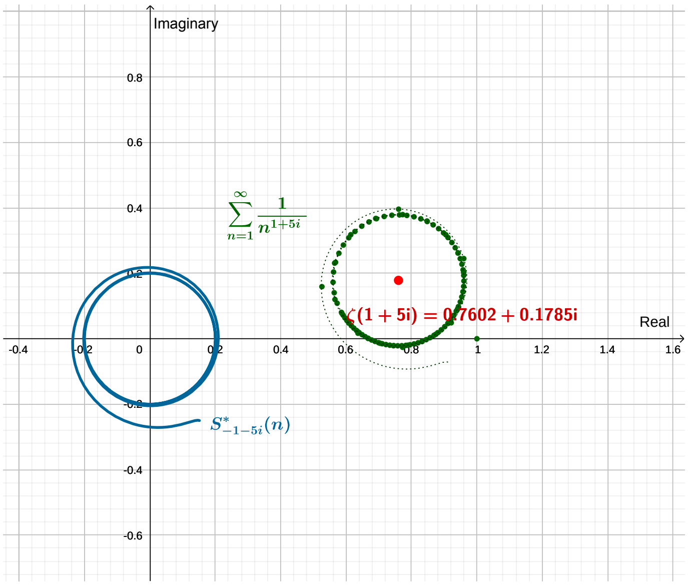

Figure 7: *Example of a quasi-circumferential logarithmic spiral of the series* $\sum_{n=1}^{\infty}\frac{1}{n^{1+5i}}$ *where* $\zeta(1+5i)$ *is the center of the spiral.*

When $n \to \infty$ and $Re(k) > 1$, the helices converge to their axis (Figure 8), if $Re(k) = 1$ the helices tend to be circular (Figure 9), and if $Re(k) < 1$ the helices diverge from their axis (Figure 10).

When k is a *"Nontrivial zero"*, both helixs coincide and their axis coincides with the axis n (Figure 11).

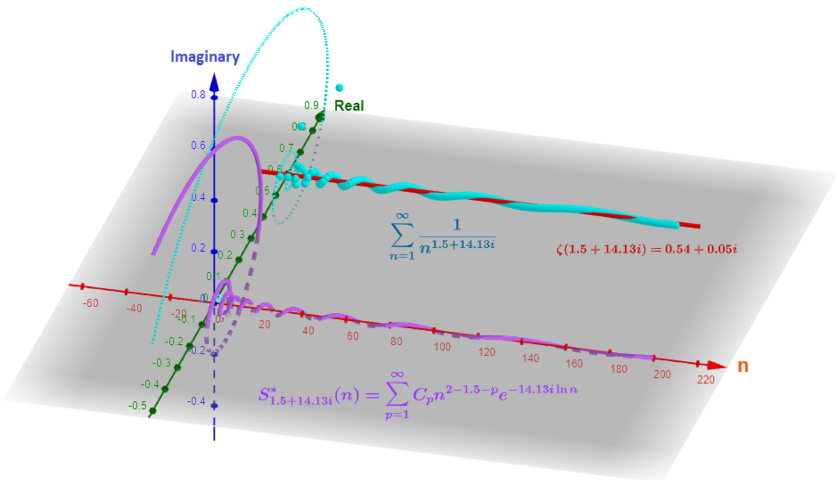

Figure 8: *Example of logarithmic helixs in the space* $\mathbb{C} \to \mathbb{R}$ *of* $\sum_{n=1}^{\infty}\frac{1}{n^k}$ *(Series of points in light blue) and the related function* $S^*_{-k}(n)$ *(purple) for* $Re(k) > 1$.

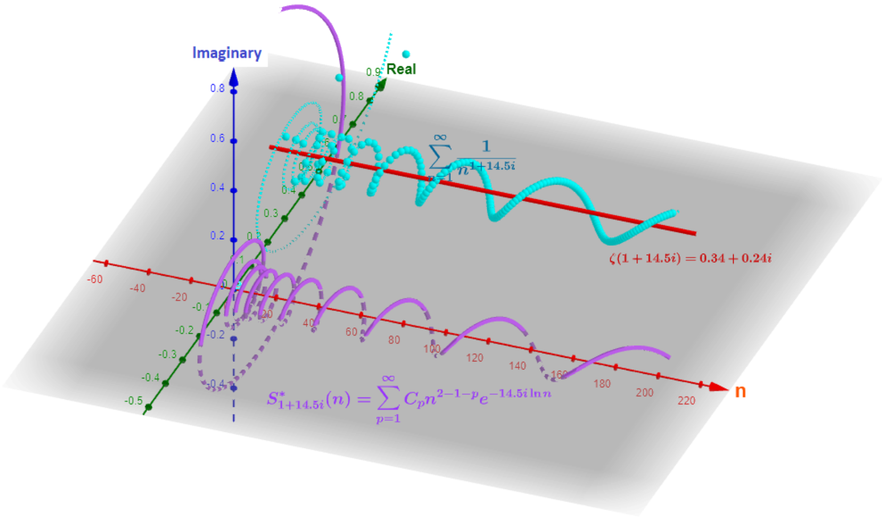

Figure 9: *Example of logarithmic helix in the space* $\mathbb{C} \to \mathbb{R}$ *of* $\sum_{n=1}^{\infty} \frac{1}{n^k}$ *(Series of points in light blue) and the related function* $S^*_{-k}(n)$ *(purple) for* $Re(k)_(= 1$.

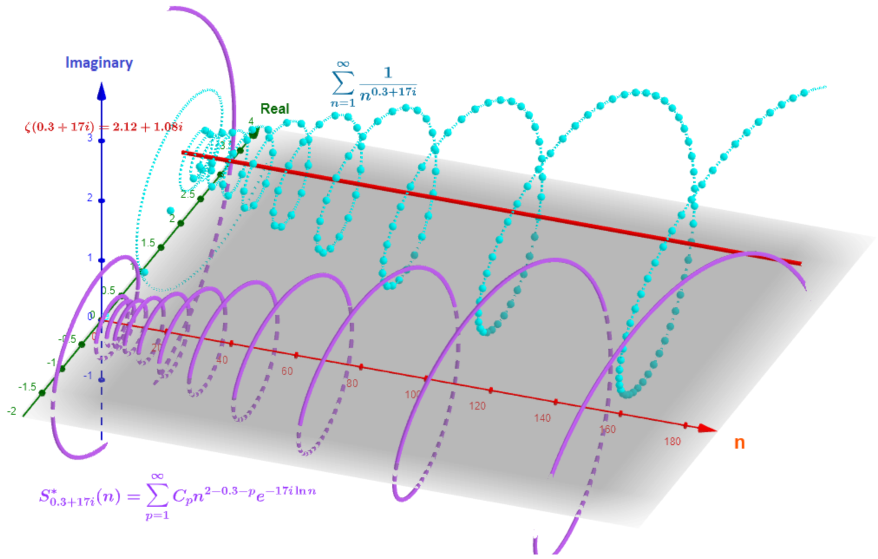

Figure 10: *Example of logarithmic helixs in the space* $\mathbb{C} \to \mathbb{R}$ *of* $\sum_{n=1}^{\infty} \frac{1}{n^k}$ *(Series of points in light blue) and the related function* $S^*_{-k}(n)$ *(purple) for* $Re(k) < 1$.

8 Period and variable amplitude curves.

Now that it is known that the function $\zeta(k)$ is the center of a spiral in the complex plane, we can also plot separately the sum of powers in both the real part and the imaginary part:

$$Re\left[\zeta(k)\right] = \lim_{n\to\infty} \left\{ \sum_{n=1}^{n} \frac{1}{n^a} \cos(b \ln n) - \sum_{n=1}^{\infty} \left[u_p \cos(b \ln n) + v_p \sin(b \ln n)\right] n^{2-p-a} \right\} \tag{41}$$

$$Im\left[\zeta(k)\right] = \lim_{n\to\infty} \left\{ \sum_{n=1}^{n} \frac{1}{n^a} \sin(-b \ln n) - \sum_{n=1}^{\infty} \left[v_p \cos(b \ln n) - u_p \sin(b \ln n)\right] n^{2-p-a} \right\} \tag{42}$$

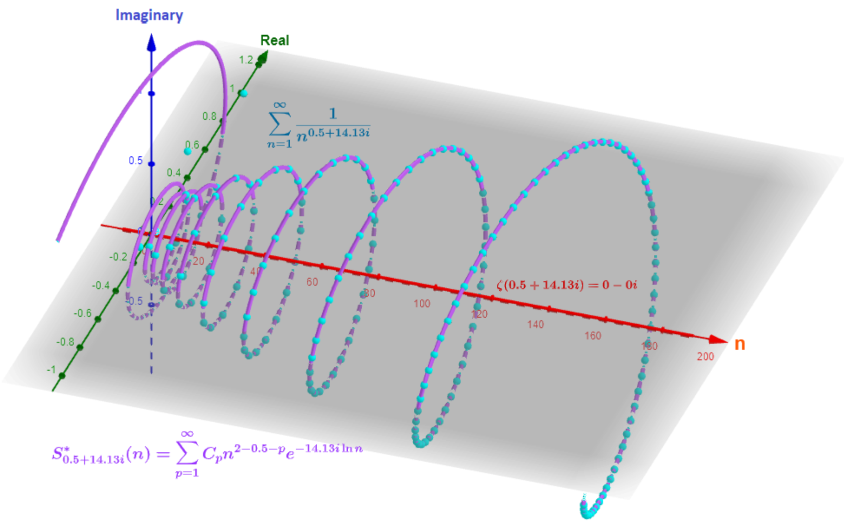

Figure 11: *Example of logarithmic helix in the space* $\mathbb{C} \to \mathbb{R}$ *of* $\sum_{n=1}^{\infty} \frac{1}{n^k}$ *(Series of points in light blue) and the related function* $S^*_{-k}(n)$ *(purple) for* $k = 0.5 + 14.13$ *first non-trivial zero. The spirals coincide.*

Reordering:

$$\lim_{n\to\infty} \left\{ \sum_{n=1}^{\infty} \left[u_p \cos(b \ln n) + v_p \sin(b \ln n) \right] n^{2-p-a} \right\} = \sum_{n=1}^{\infty} \frac{1}{n^a} \cos(b \ln n) - Re\left[\zeta(k)\right] \tag{43}$$

$$\lim_{n\to\infty} \left\{ \sum_{n=1}^{\infty} \left[v_p \cos(b \ln n) - u_p \sin(b \ln n) \right] n^{2-p-a} \right\} = \sum_{n=1}^{\infty} \frac{1}{n^a} \sin(-b \ln n) - Im\left[\zeta(k)\right] \tag{44}$$

From equations (43) and (44), it is observed that the functions $Re\left[S^*_{-k}(n)\right]$ and $Im\left[S^*_{-k}(n)\right]$ oscillate about the horizontal n-coordinate axis, on the other hand the series $\sum_{n=1}^{\infty} \frac{1}{n^a} \cos(b \ln n)$ and $\sum_{n=1}^{\infty} \frac{1}{n^a} \sin(-b \ln n)$ oscillate about their mean value $-Re\left[\zeta(k)\right]$ and $-Im\left[\zeta(k)\right]$ respectively:

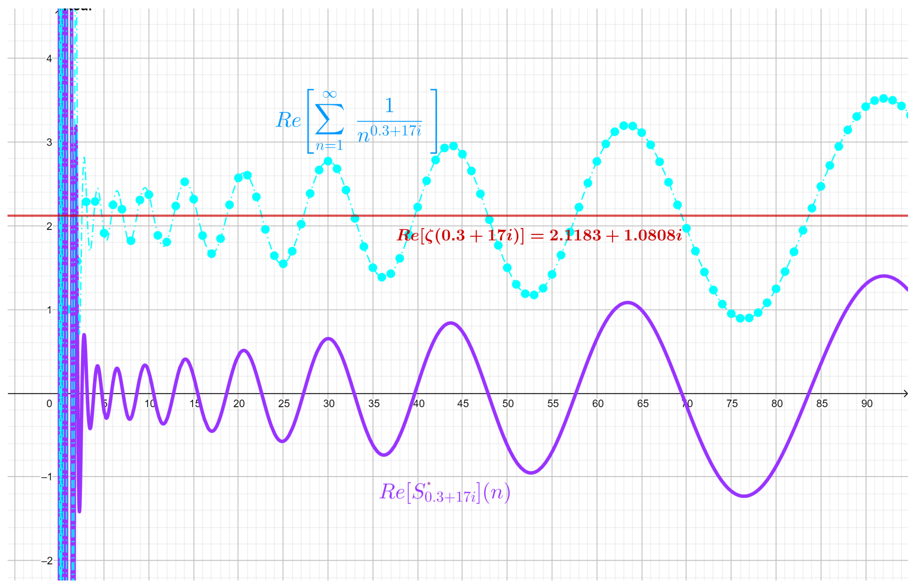

Figure 12: *Graphical representation of functions:* $Re\left[S^*_{-k}(n)\right]$ *(purple) and serie* $\sum_{n=1}^{\infty} \frac{1}{n^a} \cos(b \ln n)$ *(Series of points in light blue) that oscillates with respect to its mean value* $-Re\left[\zeta(k)\right]$

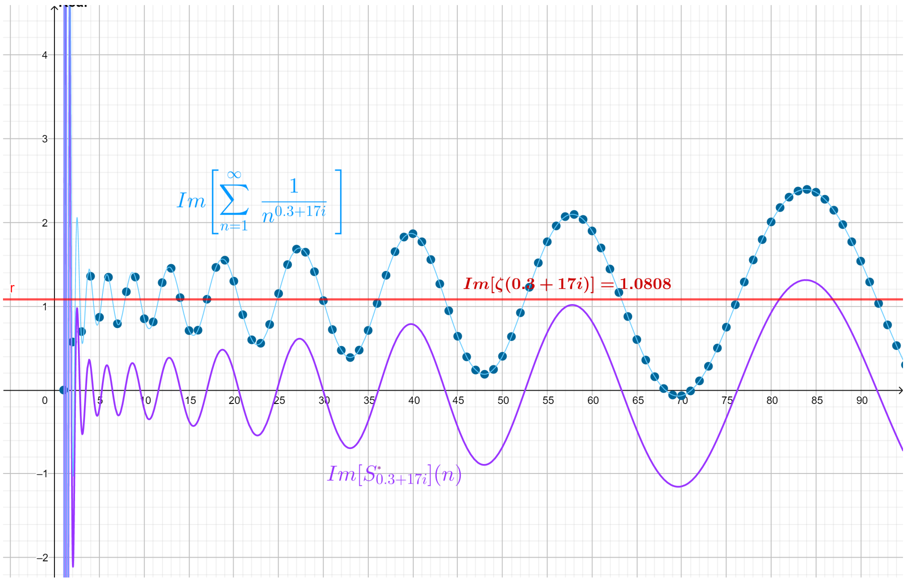

Figure 13: *Graphical representation of functions:* $Im\left[S^{*}_{-k}(n)\right]$ *(purple)y la serie* $\sum_{n=1}^{\infty}\frac{1}{n^{a}}\sin(-b\ln n)$ *(Series of points in light blue) that oscillates with respect to its mean value* $-Im\left[\zeta(k)\right]$

By the mean value theorem it can be written:

$$\zeta(k) = -\frac{1}{d-c}\int_{c}^{d}\sum_{n=1}^{k+1}\frac{1}{n^{k}}dn \tag{45}$$

$$\zeta(k) = -\frac{1}{d-c}\int_{c}^{d}S_{-k}(n)dn \tag{46}$$

Where $d-c$ is the period of the series.

Equation (46) is the general form of equation (2) which will be demonstrated below:

Demonstration of equation (2) If we integrate the function $S_k(n)$ on the interval $[-1,0]$ we obtain:

$$-\int_{-1}^{0}S_k(n)dn = -\int_{-1}^{0}\sum_{p=1}^{k+1}\frac{(-1)^{p-1}B_{p-1}}{(k+1)(p-1)!}\prod_{m=1}^{p-1}(k+2-m)n^{k+2-p}dn$$

$$-\int_{-1}^{0}S_k(n)dn = -\sum_{p=1}^{k+1}\int_{-1}^{0}\frac{(-1)^{p-1}B_{p-1}}{(k+1)(p-1)!}\prod_{m=1}^{p-1}(k+2-m)n^{k+2-p}dn$$

$$-\int_{-1}^{0}S_k(n)dn = -\sum_{p=1}^{k+1}\left[\frac{(-1)^{p-1}B_{p-1}}{(k+1)(p-1)!(k+3-p)}\prod_{m=1}^{p-1}(k+2-m)n^{k+3-p}\right]_{-1}^{0}$$

$$-\int_{-1}^{0}S_k(n)dn = \left[\sum_{p=1}^{k+1}\frac{(-1)^{p-1}B_{p-1}}{(k+1)(p-1)!(k+3-p)}\prod_{m=1}^{p-1}(k+2-m)(-1)^{k+3-p}\right]$$

$$-\int_{-1}^{0}S_k(n)dn = \left[\sum_{p=1}^{k+1}\frac{(-1)^{k+2}B_{p-1}}{(k+1)(p-1)!(k+3-p)}\prod_{m=1}^{p-1}(k+2-m)\right]$$

$$-\int_{-1}^{0}S_k(n)dn = -\left[\sum_{p=1}^{k+1}\frac{(-1)^{k+2}B_{p-1}}{(k+1)(p-1)!(k+3-p)} * \frac{(k+1)!}{(k+2-p)!}\right]$$

$$-\int_{-1}^{0}S_k(n)dn = -\frac{(-1)^{k+2}}{(k+1)}\left[\sum_{p=1}^{k+1}\frac{B_{p-1}(k+1)!}{(p-1)!(k+3-p)!}\right] \tag{47}$$

On the other hand, if from the known equation to find the Bernoulli numbers

$$B_k = -\frac{1}{k+1}\sum_{m=0}^{k-1}\binom{k+1}{m}B_m$$

we can find B_{k+1}:

$$B_{k+1} = -\frac{1}{k+2}\sum_{p=1}^{k+1}\binom{k+2}{p-1}B_{p-1}$$

$$B_{k+1} = -\frac{1}{k+2}\sum_{p=1}^{k+1}\frac{(k+2)!}{(p-1)!(k+3-p)!}B_{p-1}$$

$$B_{k+1} = -\sum_{p=1}^{k+1}\frac{(k+1)!}{(p-1)!(k+3-p)!}B_{p-1} \tag{48}$$

Substituting equation (48) into (47):

$$-\int_{-1}^{0} S_k(n)dn = (-1)^{k+1}\frac{B_{k+1}}{(k+1)}$$

And the second term is: $-\zeta(-k)$:

$$-\zeta(-k) = -\int_{-1}^{0} S_k(n)dn$$

Finally:

$$\zeta(-k) = \int_{-1}^{0} S_k(n)dn \tag{49}$$

Equation (2) has been proved

9 Proof of the Riemann Hypothesis.

Let k be a complex number such that $k = a + bi$ where $a, b \in \mathbb{R} \wedge \zeta(k) = 0$.
From the Riemann functional equation:

$$\zeta(a+bi) = 2^{a+bi}\pi^{a+bi-1}\sin\left(\frac{\pi(a+bi)}{2}\right)\Gamma(1-a-bi)\zeta(1-a-bi)$$

It is known that for the critical band range $0 < a < 1$, the terms:

$$2^{a+bi}\pi^{a+bi-1}\sin\left(\frac{\pi(a+bi)}{2}\right)\Gamma(1-a-bi) \neq 0$$

Therefore, it must comply:

$$\zeta(a+bi) = \zeta(1-a-bi) = 0 \tag{50}$$

On another hand, from the property of conjugates of $\zeta(k)$, it must be fulfilled:

$$\overline{\zeta(a+bi)} = \zeta(\overline{a+bi}) = \zeta(a-bi) = 0 \tag{51}$$

Therefore it can be written:

$$\zeta(a-bi) = \zeta(1-a+bi) = 0 \tag{52}$$

Replacing (24) in (50) and (52) the following 4 equations can be written:

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{a+bi}} - S^*_{-a-bi}(n)\right] = 0 \tag{53}$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{a-bi}} - S^*_{-a+bi}(n)\right] = 0 \tag{54}$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{1-a-bi}}-S^{*}_{a-1+bi}(n)\right]=0 \quad (55)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{1-a+bi}}-S^{*}_{a-1-bi}(n)\right]=0 \quad (56)$$

If we make a change of variable from $a=\frac{1}{2}+\delta$ we can rewrite the four equations:

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta+bi}}-S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]=0 \quad (57)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta-bi}}-S^{*}_{-\frac{1}{2}-\delta+bi}(n)\right]=0 \quad (58)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta-bi}}-S^{*}_{-\frac{1}{2}+\delta+bi}(n)\right]=0 \quad (59)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta+bi}}-S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]=0 \quad (60)$$

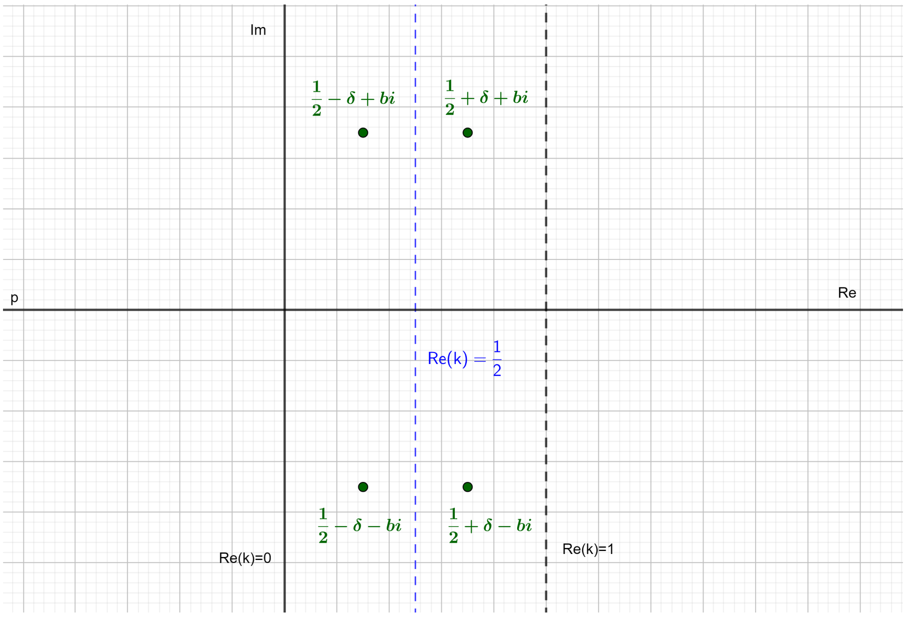

Figure 14: *Graphical representation in the complex plane of values of k for which the function $\zeta(k)=0$.*

Applying power properties:

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}e^{-ib\ln n}-S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]=0 \quad (61)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}e^{ib\ln n}-S^{*}_{-\frac{1}{2}-\delta+bi}(n)\right]=0 \quad (62)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}e^{ib\ln n}-S^{*}_{-\frac{1}{2}+\delta+bi}(n)\right]=0 \quad (63)$$

$$\lim_{n\to\infty}\left[\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}e^{-ib\ln n}-S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]=0 \quad (64)$$

Transforming the equations to Cartesian mode:

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\left[\cos(b\ln n)-i\sin(b\ln n)\right]-S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right\}=0 \tag{65}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\left[\cos(b\ln n)+i\sin(b\ln n)\right]-S^{*}_{-\frac{1}{2}-\delta+bi}(n)\right\}=0 \tag{66}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\left[\cos(b\ln n)+i\sin(b\ln n)\right]-S^{*}_{-\frac{1}{2}+\delta+bi}(n)\right\}=0 \tag{67}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\left[\cos(b\ln n)-i\sin(b\ln n)\right]-S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right\}=0 \tag{68}$$

By separating the real part and the imaginary part, the following 8 equations are formed:

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\cos(b\ln n)-Re\left[S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]\right\}=0 \tag{69}$$

$$\lim_{n\to\infty}\left\{-\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\sin(b\ln n)-Im\left[S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]\right\}=0 \tag{70}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\cos(b\ln n)-Re\left[S^{*}_{-\frac{1}{2}-\delta+bi}(n)\right]\right\}=0 \tag{71}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\sin(b\ln n)-Im\left[S^{*}_{-\frac{1}{2}-\delta+bi}(n)\right]\right\}=0 \tag{72}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\cos(b\ln n)-Re\left[S^{*}_{-\frac{1}{2}+\delta+bi}(n)\right]\right\}=0 \tag{73}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\sin(b\ln n)-Im\left[S^{*}_{-\frac{1}{2}+\delta+bi}(n)\right]\right\}=0 \tag{74}$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\cos(b\ln n)-Re\left[S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]\right\}=0 \tag{75}$$

$$\lim_{n\to\infty}\left\{-\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\sin(b\ln n)-Im\left[S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]\right\}=0 \tag{76}$$

By properties of conjugates in complex numbers, the following equivalences are verified:

$$(69)\equiv(71)$$

$$(70)\equiv(72)$$

$$(73)\equiv(75)$$

$$(74)\equiv(76)$$

If we add the equations (69) with (75) and factoring:

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\cos(b\ln n)+\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\cos(b\ln n)-Re\left[S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]-Re\left[S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]\right\}=0$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}}}\cos(b\ln n)\left[\frac{1}{n^{\delta}}+\frac{1}{n^{-\delta}}\right]-Re\left[S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]-Re\left[S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]\right\}=0 \tag{77}$$

And similarly we add (70) with (76):

$$\lim_{n\to\infty}\left\{-\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+\delta}}\sin(b\ln n)-\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-\delta}}\sin(b\ln n)-Im\left[S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]-Im\left[S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]\right\}=0$$

$$\lim_{n\to\infty}\left\{-\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}}}\sin(b\ln n)\left[\frac{1}{n^{\delta}}+\frac{1}{n^{-\delta}}\right]-Im\left[S^{*}_{-\frac{1}{2}-\delta-bi}(n)\right]-Im\left[S^{*}_{-\frac{1}{2}+\delta-bi}(n)\right]\right\}=0 \quad (78)$$

Now let us consider $\delta=0$ in equation (77):

$$\lim_{n\to\infty}\left\{2\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}}}\cos(b\ln n)-2\ Re\left[S^{*}_{-\frac{1}{2}-bi}(n)\right]\right\}=0 \quad (79)$$

Let us similarly consider $\delta=0$ in equation (78):

$$\lim_{n\to\infty}\left\{-2\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}}}\sin(b\ln n)-2\ Im\left[S^{*}_{-\frac{1}{2}-bi}(n)\right]\right\}=0 \quad (80)$$

Adding (79) with (80) and then simplifying we obtain the following expression:

$$\lim_{n\to\infty}\left\{2\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}}}\left[\cos(b\ln n)-i\sin(b\ln n)\right]-2\ S^{*}_{-\frac{1}{2}-bi}(n)\right\}=0 \quad (81)$$

$$2\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+bi}}-S^{*}_{-\frac{1}{2}-bi}(n)\right\}=0 \quad (82)$$

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}+bi}}-S^{*}_{-\frac{1}{2}-bi}(n)\right\}=0 \quad (83)$$

And by properties of conjugates in complex numbers, the equation is also obtained:

$$\lim_{n\to\infty}\left\{\sum_{n=1}^{n}\frac{1}{n^{\frac{1}{2}-bi}}-S^{*}_{-\frac{1}{2}+bi}(n)\right\}=0 \quad (84)$$

Therefore, for all b that satisfy the equality:

$$\zeta(a\pm b\ i)=0 \quad (85)$$

There exists a number $a\in\mathbb{R}$ such that $a=\frac{1}{2}$, and it satisfies the equality:

$$\zeta(\frac{1}{2}\pm b\ i)=0 \quad (86)$$

Theorem 2 has been demonstrated:

Theorem 2 *All non trivial zeros of the Riemann zeta function have real part equal to* $\frac{1}{2}$.

Therefore, the Riemann Hypothesis is true.

10 Conclusion and final comments.

There remained several findings to be shown, around the Euler-Riemann zeta function $\zeta(k)$, but these were far from the objective of this work, which is to prove the Riemann Hypothesis. However, the proof of equation (11) remained pending, which is achieved by making $k = -1$ in equation (9), from which we obtain:

$$C_p(-1) = -\frac{B_{p-1}}{p-1}$$

With a change of variable of $-k = 2 - p$, equation (11) is proved.

$$C_p(-1) = \zeta(2-p)$$

This demonstrates that the values of $\zeta(k)$, B_k and $C_p(k)$ are interdependent, so for example as B_k can be calculated from a summation that depends on the Bernoulli numbers, similarly C_p can be written as a function of $\zeta(k)$, B_k and the same function $\zeta(k)$ can be expressed as an infinite sum of values depending on $\zeta(-m)$ where $\mathbb{N}$.
Is interesting to know that equation (9) can also be expressed as a summation of higher order derivatives, although we do not use it in this paper:

$$S_k^*(n) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1} B_{p-1}}{(k+1)(p-1)!} * \frac{d^{p-1}}{dn^{p-1}} \left(n^{k+2-p}\right) \tag{87}$$

Finally, this research work gives us insights to make the following two conjectures about the zeta function:

- If the amplitude and period of the function $S_{-k}^*(n)$ could be computed, $\zeta(k)$ could be computed exactly with the mean value integral.
- One can write B_k as a continuous function in the complex plane, which depends on $\zeta(k)$. This function has "trivial zeros" when K is an odd integer greater than 1, and has "nontrivial zeros" that coincide with the "nontrivial zeros" of $\zeta(k)$.

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