Demonstrating the equivalence of different expressions for vector rotations

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Abstract

Because newcomers to GA may have difficulty applying its identities to real problems, we use those identities to prove the equivalence of two expressions for rotations of a vector. Rather than simply present the proof, we first review the relevant GA identities, then formulate and explore reasonable conjectures that lead, promptly, to a solution.

1 Introduction

A particularly useful feature of GA is its ability to express rotations conveniently. For example (Fig. 1), the vector $v'$ that results from the rotation of vector $v$ through the angle $\theta$ about an axis perpendicular to the bivector $\hat{B}$, and in the sense of the rotation of $\hat{B}$ itself, is

$$v' = \left[ e^{-B\theta/2} \right] v \left[ e^{-B\theta/2} \right].$$

(1.1)

Macdonald ([1], p. 89) begins the derivation of that formula by expressing $v$ as the sum of its components parallel and perpendicular to $\hat{B}$ ($v_\parallel$ and $v_\perp$, respectively). Then, Macdonald notes that while the vertical component is unaffected by the rotation, the parallel component becomes $v_\parallel e^{B\theta}$. Thus, $v'$ is also
How might we demonstrate that Eqs. (1.1) and (1.2) are equivalent? We begin by expanding Eq. 1.1:

\[
v' = \left[ \cos \frac{\theta}{2} - \hat{B} \sin \frac{\theta}{2} \right] v \left[ \cos \frac{\theta}{2} + \hat{B} \sin \frac{\theta}{2} \right] = v \cos^2 \frac{\theta}{2} + v \hat{B} \cos \frac{\theta}{2} \sin \frac{\theta}{2} - \hat{B} v \cos \frac{\theta}{2} \sin \frac{\theta}{2} - \hat{B} v \hat{B} \sin^2 \frac{\theta}{2}. \tag{1.3}
\]

To make further progress, we need to review a bit.

2 From 3D Euclidean GA: some identities that we will use . . .

For any vector \( \mathbf{v} \) and any unit bivector \( \hat{\mathbf{B}} \),

1. The multiplicative inverse of \( \hat{\mathbf{B}} \): \( \hat{\mathbf{B}}^{-1} = -\hat{\mathbf{B}} \)
2. \( \hat{\mathbf{B}} \cdot \mathbf{v} = \mathbf{v} \cdot \hat{\mathbf{B}} \)
3. \( \hat{\mathbf{B}} \wedge \mathbf{v} = \mathbf{v} \wedge \hat{\mathbf{B}} \)
4. \( \mathbf{v} \hat{\mathbf{B}} = \mathbf{v} \cdot \hat{\mathbf{B}} + \mathbf{v} \wedge \hat{\mathbf{B}} \)
5. \( \hat{\mathbf{B}} \mathbf{v} = \hat{\mathbf{B}} \cdot \mathbf{v} + \mathbf{v} \wedge \hat{\mathbf{B}} = \mathbf{v} \cdot \hat{\mathbf{B}} + \mathbf{v} \wedge \hat{\mathbf{B}} \)
6. The components of \( \mathbf{v} \) parallel to and perpendicular to \( \hat{\mathbf{B}} \) are:

(a) \( \mathbf{v}_\parallel = (\mathbf{v} \cdot \hat{\mathbf{B}}) \hat{\mathbf{B}}^{-1} = (\mathbf{v} \cdot \hat{\mathbf{B}}) (-\hat{\mathbf{B}}) \)

(b) \( \mathbf{v}_\perp = (\mathbf{v} \wedge \hat{\mathbf{B}}) \hat{\mathbf{B}}^{-1} = (\mathbf{v} \wedge \hat{\mathbf{B}}) (-\hat{\mathbf{B}}) \)

7. From 3, 4, and 5 (above),

(a) \( \hat{\mathbf{B}} \mathbf{v} = -\mathbf{v} \hat{\mathbf{B}} + 2 \mathbf{v} \wedge \hat{\mathbf{B}} \)

(b) \( \hat{\mathbf{B}} \mathbf{v} = \mathbf{v} \hat{\mathbf{B}} - 2 \mathbf{v} \cdot \hat{\mathbf{B}} \)

8. The component of \( \mathbf{v} \) perpendicular to \( \hat{\mathbf{B}} \): \( \mathbf{v}_\perp = (\mathbf{v} \wedge \hat{\mathbf{B}}) (-\hat{\mathbf{B}}) \)

9. From trigonometry:

(a) \( 2 \sin \alpha / 2 \cos \alpha / 2 = \sin \alpha \)

(b) \( \cos^2 \alpha / 2 - \sin^2 \alpha / 2 = \cos \alpha \)

### 3 Demonstration of the Equivalence of Our Two Expressions for \( \mathbf{v}' \)

After reviewing the identities in Section 2, several possible routes might suggest themselves. For example, we can combine the two \( \cos \theta / 2 \sin \theta / 2 \) terms in Eq. (1.2) to obtain

\[
\mathbf{v}' = \mathbf{v} \cos^2 \theta / 2 + (\mathbf{v} \hat{\mathbf{B}} - \hat{\mathbf{B}} \mathbf{v}) \sin \theta / 2 \cos \theta / 2 - \hat{\mathbf{B}} \mathbf{v} \hat{\mathbf{B}} \sin^2 \theta / 2.
\]

Now, from point 7b in Section 2, we see that \( \mathbf{v} \hat{\mathbf{B}} - \hat{\mathbf{B}} \mathbf{v} = 2 \mathbf{v} \cdot \hat{\mathbf{B}} \). Therefore,

\[
\mathbf{v}' = \mathbf{v} \cos^2 \theta / 2 + 2 \mathbf{v} \cdot \hat{\mathbf{B}} \sin \theta / 2 \cos \theta / 2 - \hat{\mathbf{B}} \mathbf{v} \hat{\mathbf{B}} \sin^2 \theta / 2
\]

\[
= \mathbf{v} \cos^2 \theta / 2 + \mathbf{v} \cdot \hat{\mathbf{B}} \left[ 2 \sin \theta / 2 \cos \theta / 2 \right] - \hat{\mathbf{B}} \mathbf{v} \hat{\mathbf{B}} \sin^2 \theta / 2
\]

\[
= \mathbf{v} \cos^2 \theta / 2 + \mathbf{v} \cdot \hat{\mathbf{B}} \sin \theta - \hat{\mathbf{B}} \mathbf{v} \mathbf{B} \sin^2 \theta / 2 . \quad (3.1)
\]

We now have a \( \sin \theta \) term in this expression for \( \mathbf{v}' \), just as we do in Eq. (1.2). We can demonstrate the equality of those terms (i.e., that \( \mathbf{v}_\parallel = \mathbf{v} \wedge \hat{\mathbf{B}} \)) by noting that \( \mathbf{v}_\parallel = (\mathbf{v} \cdot \hat{\mathbf{B}}) (\hat{\mathbf{B}}^{-1}) \), so that \( \mathbf{v}_\parallel \hat{\mathbf{B}} = (\mathbf{v} \cdot \hat{\mathbf{B}}) (\hat{\mathbf{B}}^{-1}) \hat{\mathbf{B}} = \mathbf{v} \cdot \hat{\mathbf{B}} (\hat{\mathbf{B}}^{-1} \hat{\mathbf{B}}) = \mathbf{v} \cdot \mathbf{B} \).

What to do with the factor \( \hat{\mathbf{B}} \mathbf{v} \hat{\mathbf{B}} \) in Eq. (1.3) may not be clear. One idea is to “reverse” the product \( \hat{\mathbf{B}} \mathbf{v} \) to obtain \( \mathbf{v} \), so that the \( \hat{\mathbf{B}} \) in that part will
cancel with the second $\hat{B}$. We can do this in either of two ways, using items 7a and 7b:

$$
\hat{B}v\hat{B} = \left[ v\hat{B} + 2v \wedge \hat{B} \right] \hat{B}
= v\hat{B}B + 2 \left( v \wedge \hat{B} \right) \hat{B}
= v + 2 \left( v \wedge \hat{B} \right) \hat{B},
$$

and

$$
\hat{B}v\hat{B} = \left[ v\hat{B} - 2v \cdot \hat{B} \right] \hat{B}
= v\hat{B}B - 2 \left( v \cdot \hat{B} \right) \hat{B}
= v - 2 \left( v \cdot \hat{B} \right) \hat{B}.
$$

These approaches will work, but—at least when I attempted them—they turned out to be tedious, and not at all insightful. So, let’s look for a different idea. First, let’s note that we’re trying to demonstrate the equivalence between (1) a relation that’s expressed in terms of the two vectors $v_\parallel$ and $v_\perp$ (i.e., Eq. (1.2)), and (2) a relation that’s expressed in terms of products of $v$ and $\hat{B}$ (i.e., Eq. (3.1)). If we recall the derivations of items 6a and 6b, ([1], p. 119) we can see that the product $\hat{B}v\hat{B}$ is indeed a sum or difference of $v_\parallel$ and $v_\perp$. Let’s find out what that specific sum/difference is:

$$
\hat{B}v\hat{B} = \left[ \hat{B} \cdot v + v \wedge \hat{B} \right] \hat{B}
= \left[ v \cdot \hat{B} + v \wedge \hat{B} \right] \hat{B}
= \left( v \cdot \hat{B} \right) \hat{B} + \left( v \wedge \hat{B} \right) \hat{B}
= \left( v \cdot \hat{B} \right) \left( \hat{B} \right) - \left( v \wedge \hat{B} \right) \left( \hat{B} \right)
= v_\parallel - v_\perp.
$$

Substituting this result into Eq. (3.1),

$$
v' = v \cos^2 \frac{\theta}{2} + v \cdot \hat{B} \sin \theta - \left( v_\parallel - v_\perp \right) \sin^2 \frac{\theta}{2}.
$$

Now we can see that the terms $\cos^2 \frac{\theta}{2}$ and $\sin^2 \frac{\theta}{2}$ might be combined per the double-angle formulas (items 9a and 9b) if we write $v$ as $v_\parallel + v_\perp$ in the $\cos^2 \frac{\theta}{2}$ term:

$$
v' = (v_\parallel + v_\perp) \cos^2 \frac{\theta}{2} + v \cdot \hat{B} \sin \theta - \left( v_\parallel - v_\perp \right) \sin^2 \frac{\theta}{2}.
$$

The rest is simple:

$$
v' = v_\perp \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) + v_\parallel \left( \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) v \cdot \hat{B} \sin \theta
= v_\perp + v_\parallel \cos \theta + v \cdot \hat{B} \sin \theta.
$$

(3.2)
References