The Proofs of Legendre's Conjecture and Three Related Conjectures

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Abstracts

In this paper, we are going to prove Legendre's Conjecture: There is a prime number between n^2 and $(n+1)^2$ for every positive integer n. We will also prove three related conjectures. The method that we use is to analyze a binomial coefficient. It has been developed from the method of analyzing a central binomial coefficient that was used by Paul Erdős to prove Bertrand's postulate - Chebyshev's theorem.

Table of Contents

1. Introduction

Legendre's Conjecture was proposed by Andrien-Marie Legendre [1]. The conjecture is one of Legendre's problems (1912) on prime numbers. It states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n.

In this paper, we will prove Legendre's Conjecture by analyzing the binomial coefficient $\binom{\lambda n}{n}$ where λ is an integer and $\lambda \geq 3$. It is developed from the method that was used by Paul Erdős [2] to prove Bertrand's postulate - Chebyshev's theorem [3].

In Section 1, we will define the prime number factorization operator and clarify some terms and concepts. In Section 2, we will derive some lemmas. In Section 3, we will develop a theorem to be used in the proofs of the conjectures in the later sections. In Section 4, we will prove Legendre's conjecture, and in Section 5, we will prove Oppermann's conjecture [4], Brocard's conjecture [5], and Andrica's conjecture [6].

Definition: $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\}$ denotes the prime number factorization operator of $\binom{\lambda n}{n}$, an integer expression. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \geq p > b$. In this operator, p is a prime number, a and b are real numbers, $\lambda n \geq a \geq p > b \geq 1$.
It has some properties:
It is always true that $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} \geq 1$ **— (1.1)**

If there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$, then $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} = 1$,
or vice versa, if $\Gamma_{a\geq p>b}\{\binom{\lambda n}{n}\} = 1$, then there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$. **— (1.2)**
For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16\geq p>10}\left\{\binom{20}{4}\right\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ in the range of $16 \geq p > 10$.

If there is at least one prime number in $\binom{\lambda n}{n}$ in the range of $a \geq p > b$, then $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} > 1$,
or vice versa, if $\Gamma_{a\geq p>b}\left\{\binom{\lambda n}{n}\right\} > 1$, then there is at least one prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$. **— (1.3)**

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{18\geq p>16}\left\{\binom{20}{4}\right\} = 17 > 1$. A prime number 17 is in $\binom{20}{4}$ within the range of $18 \geq p > 16$.

Let $v_p(n)$ be the $p\text{-}adic\ valuation$ of n, the exponent of the highest power of p that divides n. Similar to Paul Erdős' paper [2], we define *R(p)* by the inequalities $p^{R(p)} \leq \lambda n < p^{R(p)+1}$, and determine the $p\text{-}adic\ valuation$ of $\binom{\lambda n}{n}$.

$$v_p\left(\binom{\lambda n}{n}\right) = v_p((\lambda n)!) - v_p(((\lambda-1)n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{\lambda n}{p^i}\right\rfloor - \left\lfloor\frac{(\lambda-1)n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \leq R(p)$$

because for any real numbers a and b, the expression of $\lfloor a+b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \leq p^{R(p)} \leq \lambda n$ **— (1.4)**

And if $\lambda n \geq p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \leq v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq 1$ **— (1.5)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(n) \leq \left\lfloor\frac{n}{3}\right\rfloor+2 \leq \frac{n}{3}+2$. **— (1.6)**

From the prime number decomposition,

when $n > \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} = \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

when $n \leq \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

Thus, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} = \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to **(1.5)**, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \leq \prod_{n \geq p} p$. It has been proved [7] that for $n \geq 3$, $\prod_{n \geq p} p < 2^{2n-3}$. Thus, for $n \geq 3$, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \leq \prod_{n \geq p} p < 2^{2n-3}$.

Referred to **(1.4)** and **(1.6)**, $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

Thus, for $n \geq 3$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$ **— (1.7)**

2. Lemmas

Lemma 1: If a real number $x \geq 3$, then $\frac{2(2x-1)}{x-1} > \left(\frac{x}{x-1}\right)^x$ **— (2.1)**

Proof:

Let $f_1(x) = \frac{2(2x-1)}{x-1}$, then $f_1{}'(x) = \frac{2(x-1)(2x-1)'-2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus, $f_1(x)$ is a strictly decreasing function for $x > 1$.

Since $f_1(3) = 5$ and $\lim_{x\to\infty} f_1(x) = 4$, for $x \geq 3$, we have $5 \geq f_1(x) = \frac{2(2x-1)}{x-1} \geq 4$.

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2{}'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(e^{x\cdot ln\frac{x}{x-1}}\right)' = e^{x\cdot ln\frac{x}{x-1}} \cdot \left(x \cdot ln\frac{x}{x-1}\right)'$

$f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(ln\frac{x}{x-1} + x \cdot \left(ln\frac{x}{x-1}\right)'\right) = \left(\frac{x}{x-1}\right)^x \cdot \left(ln\frac{x}{x-1} + x \cdot \frac{x-1}{x} \cdot \frac{x-1-x}{(x-1)^2}\right)$

$f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(ln\frac{x}{x-1} - \frac{1}{x-1}\right)$ **— (2.1.1)**

In **(2.1.1)**, $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots$

Using the formula: $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$, we have

$ln\frac{x}{x-1} = ln\frac{1}{1+\frac{-1}{x}} = -\,ln\left(1+\frac{-1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots$

Thus for $x \geq 3$, $ln\frac{x}{x-1} - \frac{1}{x-1} < 0$

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x \geq 3$, $f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(ln\frac{x}{x-1} - \frac{1}{x-1}\right) < 0$.

Thus $f_2(x)$ is a strictly deceasing function for $x \geq 3$.

Since $f_2(3) = 3.375$ and $\lim_{x\to\infty} f_2(x) = e \approx 2.718$, for $x \geq 3$, $3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e$ **— (2.1.2)**

Since for $x \geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$ is proven. **— (2.1.3)**

Lemma 2: For $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ **— (2.2)**

Proof:

When $\lambda \geq 3$ and $n = 2$, $\binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1)$ **— (2.2.1)**

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^\lambda$ **— (2.2.2)**

In **(2.1)** when $x = \lambda \geq 3$, we have $\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^\lambda$ **— (2.2.3)**

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \geq 3$, referring to **(2.2.1)** and **(2.2.2)**, when $\frac{\lambda(\lambda-1)}{2}$ multiplies to both sides of **(2.2.3)**, we have

$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n} > \left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^\lambda = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \geq 3$ and $n = 2$. **— (2.2.4)**

By induction on n, when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for n, then for n+1, we have

$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \binom{\lambda n}{n}$

$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} \cdot \frac{\lambda n+1}{n} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Notice $\frac{\lambda n+1}{n} > \lambda$, and $\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} > \left(\frac{\lambda}{\lambda-1}\right)^{(\lambda-1)}$

because $\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1} = \frac{\lambda}{\lambda-1}$; $\frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2} > \frac{\lambda}{\lambda-1}$; $\cdots$ $\frac{\lambda n+2}{\lambda n-n+1} > \frac{\lambda}{\lambda-1}$.

Thus $\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda-1}}{(\lambda-1)^{(\lambda-1)}} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}}$ **— (2.2.5)**

From **(2.2.4)** and **(2.2.5)**, we have for $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Thus, **Lemma 2** is proven.

3. A Prime Number between $(\lambda-1)n$ and λn when $n \geq (\lambda-2) \geq 25$

Proposition:

For $n \geq \lambda - 2 \geq 25$, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.1)**

Proof:

Referring to **(1.7)**, when $n \geq (\lambda-2) \geq 3$, if there is a prime number p in $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$,

then $p \geq n+1 = \sqrt{(n+2)n+1} > \sqrt{\lambda n}$. From **(1.5)**, $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right) \leq R(p) \leq 1$.

Then every prime number in $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ has the power of 0 or 1. **— (3.2)**

From **(1.7)**, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

Applying this inequality to **(2.2)**,

when $n \geq (\lambda-2) \geq 3$, $\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$. Since $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > 1$ and $2^{2n-3} > 1$,

$$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} \cdot 2^{2n-3} \cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$$

Referring to **(2.1.2)**, when $\lambda \geq 3$, $\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} \geq e$. Thus, when $n \geq (\lambda-2) \geq 3$,

$$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \geq \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right) \cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_3(n, \lambda)$$ **— (3.3)**

Let $x \geq 3$ and $y \geq 5$ both be real numbers.

When $x = y-2$, $f_3(x, y) = \frac{2(x+2)^2 \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)}}{((x+2)\cdot x)^{\frac{\sqrt{x\cdot(x+2)}}{3}+3}} > f_4(x) = \frac{2(x+2)^2 \cdot \left(\left(\frac{x+1}{4}\right) \cdot e\right)^{(x-1)}}{((x+2)\cdot x)^{\frac{x+1}{3}+3}} > 0$ **— (3.4)**

$f_4{}'(x) = f_4(x) \cdot \left(\frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} ln\big((x+2)\cdot x\big) - \frac{10}{3x} - \frac{8}{3(x+2)} \right) = f_4(x) \cdot f_5(x)$

where $f_5(x) = \frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} ln\big((x+2)\cdot x\big) - \frac{10}{3x} - \frac{8}{3(x+2)}$

$f_5{}'(x) = \frac{4x+6}{(x+1)^2\cdot(x+2)^2} + \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{10}{3x^2} + \frac{8}{3(x+2)^2} > 0$ when $x \geq 3$.

Thus, $f_5(x)$ is a strictly increasing function for $x \geq 3$.

When $x = 9$, $f_5(x) = \frac{2}{9+2} + ln\left(\frac{9+1}{4}\right) + \frac{4}{3} - \frac{2}{9+1} - \frac{1}{3} ln(9) - \frac{1}{3} ln(9+2) - \frac{10}{27} - \frac{8}{33} > 0$. Thus, for $x \geq 9$, $f_5(x) > 0$. Then, $f_4{}'(x) = f_4(x) \cdot f_5(x) > 0$.

Thus, $f_4(x)$ is a strictly increasing function for $x \geq 9$.

Let $x_1 = 9$ and $y_1 = 11$. From **(3.4)**, when $x = y - 2$, $f_3(x, y) > f_4(x) > 0$. Thus, when $x = y - 2 \geq 9$, then $xy \geq x_1 y_1 = 99$, $f_3(x, y)$ is an increasing function respect to the product of xy. **— (3.5)**

$\frac{\partial f_3(x,y)}{\partial x} = f_3(x, y) \cdot \left(ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x} \right) = f_3(x, y) \cdot f_6(x, y)$ **— (3.6)**

where $f_6(x, y) = ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}$

When $x = y - 2$, then $f_6(x, y) = f_7(x) = ln\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \big(ln(x+2) + ln(x) + 2\big) - \frac{3}{x}$

When $x \geq 3$, $f_7{}'(x) = \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \left(\frac{1}{x+2} + \frac{1}{x}\right) + \frac{ln(x+2)+ln(x)+2}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$

$f_7{}'(x) = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}} \right) + \frac{ln(x+2)+ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} > 0.$

Thus, when $x \geq 3$, $f_7(x)$ is a strictly increasing function.

When $x = y - 2 \geq 3$, since $f_6(x, y) = f_7(x)$, $f_6(x, y)$ is an increasing function respect to xy.

When $x = y - 2 = 9$, $f_6(x, y) = ln\left(\frac{11-1}{4}\right) + 1 - \frac{\sqrt{11}}{6\sqrt{9}} \cdot ln(99) - \frac{\sqrt{11}}{3\sqrt{9}} - \frac{3}{9} > 0$.

$\frac{\partial f_6(x,y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}} \cdot ln(y) + \frac{\sqrt{y}}{12x\sqrt{x}} \cdot ln(x) + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{3}{x^2} > 0$ when $x \geq (y - 2) \geq 3$.

Thus, when $x \geq (y - 2) \geq 9$, $f_6(x, y) > 0$, and it is an increasing function with respect to x and to the product of xy, then, $\frac{\partial f_3(x,y)}{\partial x} = f_3(x, y) \cdot f_6(x, y) > 0$.

Thus, when $x \geq y - 2 \geq 9$, $f_3(x, y)$ is an increasing function with respect to x. **— (3.7)**

Referring to **(3.5)** and **(3.7)**, when $x \geq y - 2 \geq 9$, then $xy \geq x_1 y_1 = 99$, $f_3(x, y)$ is an increasing function respect to the product of xy. **— (3.8)**

Let $x = n$ and $y = \lambda$. Then when $n \geq (\lambda - 2) \geq 9$, $f_3(n, \lambda)$ is an increasing function respect to the product of λn and respect to n. **— (3.9)**

When $n = (\lambda - 2) = 25$, $f_3(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = \frac{2\cdot 27^2 \cdot \left(\left(\frac{27-1}{4}\right)\cdot e\right)^{(25-1)}}{(27\cdot 25)^{\frac{\sqrt{27\cdot 25}}{3}+3}} \approx \frac{1.249\text{E}+33}{9.784\text{E}+32} > 1$.

Since $f_3(n, \lambda)$ is an increasing function to the product of λn, when $n = (\lambda - 2) \geq 25$, $f_3(n, \lambda) > 1$.

Since $f_3(n,\lambda)$ is an increasing function respect to n, when $n \geq (\lambda - 2) \geq 25$, $f_3(n,\lambda) > 1$.

Thus, referring to **(3.3)**, when $n \geq (\lambda - 2) \geq 25$, $\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_3(n,\lambda) > 1$.

Let integer $m \geq n$. When $m \geq n \geq (\lambda - 2) \geq 25$, $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > f_3(m,\lambda) > 1$. **— (3.10)**

$\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} =$

$= \Gamma_{\lambda m \geq p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$, for every distinct prime number p in these ranges, the numerator $(\lambda m)!$ has the product of $p \cdot 2p \cdot 3p \ldots ip = (i)! \cdot p^i$. The denominator $((\lambda-1)m)!$ also has the same product of $(i)! \cdot p^i$. Thus, they cancel to each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1$.

Thus, $\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \Gamma_{\lambda m \geq p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$

$\Gamma_{\lambda m \geq p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$. **— (3.11)**

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is the product of $(\lambda - 1)$ sectors from $i = 1$ to $i = (\lambda - 1)$.
Each of these sectors is the prime number factorization of the product of the consecutive integers between $\frac{(\lambda-1)m}{i}$ and $\frac{\lambda m}{i}$.

From **(3.10)** and **(3.11)**, when $m \geq n \geq \lambda - 2 \geq 25$, $\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1$.

Referring to **(1.1)**, $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \geq 1$. Thus, when $m \geq n \geq \lambda - 2 \geq 25$, at least one of the sectors in $\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1$.

Let $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$ be such a sector and let $m = ni$ where $(\lambda - 1) \geq i \geq 1$ from **(3.11)**.

Thus, when $m = ni \geq n \geq \lambda - 2 \geq 25$, $\Gamma_{\frac{\lambda ni}{i} \geq p > \frac{(\lambda-1)ni}{i}}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1$.
— (3.12)

Referring to **(1.3)**, when $m = ni \geq n \geq (\lambda - 2) \geq 25$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.13)**

There is another way to prove the results of **(3.13)**.

Referring to the definition, all prime numbers in $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the ranges of $\lambda ni \geq p > \lambda n$ and $(\lambda-1)n > p$ do not contribute to $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$, nor does i for $(\lambda-1) \geq i \geq 1$. Only the prime numbers in the prime factorization of $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the range of $\lambda n \geq p > (\lambda-1)n$ present in $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$.

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{(\boldsymbol{\lambda ni})\cdot(\lambda ni-1)\cdots(\boldsymbol{\lambda ni-i})\cdots(\boldsymbol{\lambda ni-2i})\cdots(\boldsymbol{\lambda ni-(n-1)i})\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}.$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{\boldsymbol{i}\cdot(\boldsymbol{\lambda n})\cdot(\lambda ni-1)\cdots\boldsymbol{i}\cdot(\boldsymbol{\lambda n-1})\cdots\boldsymbol{i}\cdot(\boldsymbol{\lambda n-2})\cdots\boldsymbol{i}\cdot(\boldsymbol{\lambda n-n+1})\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains all the factors of (λn), $(\lambda n-1)$, $(\lambda n-2)$,… $(\lambda n-n+1)$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

These factors make up of all the consecutive integers in the range of $\lambda n \geq p > (\lambda-1)n$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$. Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

Referring to **(3.12)**, when $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1$, then there exists at least one prime number p in the range of $\lambda n \geq p > (\lambda-1)n$. Since $\frac{(\lambda n)!}{((\lambda-1)n)!}$ is the product of all the consecutive integers in this range. These integers include all the possible prime numbers in this range.

Thus, when $n \geq \lambda-2 \geq 25$, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$. Referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.14)**

Referring to **(3.13)** or **(3.14)**, **Proposition (3.1)** is proven. It becomes a theorem: **Theorem (3.1)**.

4. The Proof of Legendre's Conjecture

Legendre's Conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. **— (4.1)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k-2 \geq 25$, there exists at least a prime number p such that $j(k-1) < p \leq jk$. **— (4.2)**

When $k = j+1 \geq 27$, then $j = k-1 \geq 26$

Applying $k = j+1$ into **(4.2)**, then $j^2 < p \leq j(j+1) < (j+1)^2$

Let $n = j \geq 26$, then we have $n^2 < p < (n+1)^2$. **— (4.3)**

For $1 \leq n \leq 26$, we have a table, **Table 1**, that shows Legendre's conjecture valid. **— (4.4)**

Table 1: For $1 \leq n \leq 26$, there is a prime number between n^2 and $(n+1)^2$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p	3	5	11	19	29	41	53	67	83	103	127	149	173
$(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196
n	14	15	16	17	18	19	20	21	22	23	24	25	26
n^2	196	225	256	289	324	361	400	441	484	529	576	625	676
p	199	229	263	307	331	373	409	449	491	541	587	641	683
$(n+1)^2$	225	256	289	324	361	400	441	484	529	576	625	676	729

Combining **(4.3)** and **(4.4)**, we have proven Legendre's conjecture.

Extension of Legendre's conjecture

There are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where p_n is the n^{th} prime number, p_m is the m^{th} prime number, and $m \geq n + 1$. **— (4.5)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k-2 \geq 25$, there exists at least a prime number p such that $j(k-1) < p \leq jk$.

When $k - 1 = j \geq 26$, then $j(k-1) = j^2 < p_n \leq jk = j(j+1)$. Thus, there is at least a prime number p_n such that $j^2 < p_n \leq j(j+1)$ when $j = k - 1 \geq 26$.

When $j = k - 2 \geq 26$, then $k = j + 2$. Thus, $j(k-1) = j(j+1) < p_m \leq jk = j(j+2) < (j+1)^2$. Thus, there is at least another prime number p_m such that $j(j+1) < p_m < (j+1)^2$ when $j = k - 2 \geq 26$. Thus, when $j \geq 26$, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n + 1$ for $p_m > p_n$. **— (4.6)**

For $1 \leq j \leq 26$, we have a table, **Table 2**, that shows **(4.5)** valid. **— (4.7)**

Table 2: For $1 \leq j \leq 26$, there are 2 prime numbers such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$.

j	1	2	3	4	5	6	7	8	9	10	11	12	13
j^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p_n	2	5	11	19	29	41	53	67	83	103	127	149	173
$j(j+1)$	2	6	12	20	30	42	56	72	90	110	132	156	182
p_m	3	7	13	23	31	43	59	73	97	113	137	163	191
$(j+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196
j	14	15	16	17	18	19	20	21	22	23	24	25	26
j^2	196	225	256	289	324	361	400	441	484	529	576	625	676
p_n	199	229	263	393	331	373	409	449	491	541	587	641	683
$j(j+1)$	210	240	272	306	342	380	420	462	506	552	600	650	702
p_m	211	251	277	311	349	389	431	467	521	557	613	659	709
$(j+1)^2$	225	256	289	324	361	400	441	484	529	576	625	676	729

Combining **(4.6)** and **(4.7)**, we have proven **(4.5)**. It becomes a theorem: **Theorem (4.5)**.

5. The Proofs of Three Related Conjectures

Oppermann's conjecture was proposed by Ludvig Oppermann [4] in March 1877. It states that for every integer $x > 1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime between x^2 and $x(x+1)$. **— (5.1)**

Proof:

Theorem (4.5) states there are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$.

$j(j+1)$ is a composite number except $j = 1$. Since $j^2 < p_n \leq j(j+1)$ is valid for every positive integer j, when we replace j with $j+1$, we have $(j+1)^2 < p_v < (j+1)(j+2)$.

Thus, we have $j(j+1) < p_m < (j+1)^2 < p_v < (j+1)(j+2)$. **— (5.2)**

When $x > 1$, then $(x-1) \geq 1$. Substitute j with $(x-1)$ in **(5.2)**, we have

$x(x-1) < p_m < x^2 < p_v < x(x+1)$ **— (5.3)**

Thus, we have proven Oppermann's conjecture.

Brocard's conjecture is after Henri Brocard [5]. It states that there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, where p_n is the n^{th} prime number, for every $n > 1$. **— (5.4)**

Proof:

Theorem (4.5) states there are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$. When $j > 1$, $j(j+1)$ is a composite number. Then **Theorem (4.5)** can be written as $j^2 < p_n < j(j+1)$ and $j(j+1) < p_m < (j+1)^2$.

In the series of prime numbers: $p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, $p_5=11$... all prime numbers except p_1 are odd numbers. Their gaps are two or more. Thus when $n > 1$, $(p_{n+1} - p_n) \geq 2$.

Thus, we have $p_n < (p_n + 1) < p_{n+1}$ when $n > 1$. **— (5.5)**

Applying **Theorem (4.5)** to **(5.5)**, when $n > 1$, we have at least two prime numbers p_{m1} , p_{m2} in between $(p_n)^2$ and $(p_n+1)^2$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2$, and at least two more prime numbers p_{m3} , p_{m4} in between $(p_n+1)^2$ and $(p_{n+1})^2$ such that

$(p_n+1)^2 < p_{m3} < p_{n+1}(p_n+1) < p_{m4} < (p_{n+1})^2$.

Thus, there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$ for $n > 1$ such that

$(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2 < p_{m3} < p_{n+1}(p_n+1) < p_{m4} < (p_{n+1})^2$ **— (5.6)**

Thus, Brocard's conjecture is proven.

Andrica's conjecture is named after Dorin Andrica [6]. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n where p_n is the n^{th} prime number. If $g_n = p_{n+1} - p_n$ denotes the n^{th} prime gap, then Andrica's conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. **— (5.7)**

Proof:

From **Theorem (4.5)**, for every positive integer j, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$.

Since $m \geq n+1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$. **— (5.8)**

And $p_{n+1} \leq p_m < (j+1)^2$. **— (5.9)**

Since j, p_n , p_{n+1} and $(j+1)$ are positive integers,

$j < \sqrt{p_n}$ **— (5.10)**

And $\sqrt{p_{n+1}} < j + 1$ **— (5.11)**

Applying **(5.10)** to **(5.11)**, we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. **— (5.12)**

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n since in **Theorem (4.5)**, j holds for all positive integers.

Using the prime gap to prove the conjecture, from **(5.8)** and **(5.9)**, we have

$g_n = p_{n+1} - p_n < (j+1)^2 - j^2 = 2j + 1$. From **(5.10)**, $j < \sqrt{p_n}$.

Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. **— (5.13)**

Thus, Andrica's conjecture is proven.

6. References

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[5] *Wikipedia,* https://en.wikipedia.org/wiki/Brocard%27s_conjecture

[6] *Wikipedia,* https://en.wikipedia.org/wiki/Andrica%27s_conjecture

[7] Wikipedia, https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate, Lemma 4.