Proof of Riemann hypothesis

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process.

1. We make one identity regarding x from one equation that gives Riemann zeta function $\zeta(s)$ analytic continuation and 2 formulas ($1/2+a\pm bi$, $1/2-a\pm bi$) that show non-trivial zero point of $\zeta(s)$.
2. We find that the above identity holds only at $a=0$.
3. Therefore non-trivial zero points of $\zeta(s)$ must be $1/2\pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $0 < Re(s)$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+\cdots\cdots=(1-2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$. i is $\sqrt{-1}$.

$$S_0 = 1/2+a\pm bi \qquad (0\leq a<1/2 \quad 14<b) \tag{2}$$

The following (3) also shows non-trivial zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1-S_0 = 1/2-a\mp bi \tag{3}$$

We define the range of a and b as $0\leq a<1/2$ and $14<b$ respectively. Then we can show all non-trivial zero points of $\zeta(s)$ by the above (2) and (3). Because non-trivial zero points of $\zeta(s)$ exist in the critical strip of $\zeta(s)$ ($0<Re(s)<1$) and non-trivial zero points of $\zeta(s)$ found until now exist in the range of $14<b$.
We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1=\frac{\cos(b\log 2)}{2^{1/2+a}}-\frac{\cos(b\log 3)}{3^{1/2+a}}+\frac{\cos(b\log 4)}{4^{1/2+a}}-\frac{\cos(b\log 5)}{5^{1/2+a}}+\cdots\cdots \tag{4}$$

$$0=\frac{\sin(b\log 2)}{2^{1/2+a}}-\frac{\sin(b\log 3)}{3^{1/2+a}}+\frac{\sin(b\log 4)}{4^{1/2+a}}-\frac{\sin(b\log 5)}{5^{1/2+a}}+\cdots\cdots \tag{5}$$

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We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. The identity regarding x

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from the above (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from the above (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) regarding real number x from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. The value of (11) is always zero at any value of x.

$$\begin{aligned} 0 &\equiv \cos x\{\text{the right side of (9)}\} + \sin x\{\text{the right side of (10)}\} \\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - \cdots\cdots\} \\ &\quad + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - \cdots\cdots\} \\ &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\ &\quad - f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots \end{aligned} \tag{11}$$

At $a = 0$ we have the following (8-1) and the above (11) holds at $a = 0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \equiv 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots \quad a = 0) \tag{8-1}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned} 0 = &f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\ &- f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots \end{aligned} \tag{12-1}$$

We have the following (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned} 0 = &f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\ &- f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots \end{aligned} \tag{12-2}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)\\ &- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \qquad (12\text{-}3)\end{aligned}$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, \cdots\cdots)$

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N)\\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \qquad (12\text{-N})\end{aligned}$$

3. The solution for the identity of (11)

We define $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots \qquad N = 1, 2, 3, 4, \cdots\cdots)$

$$\begin{aligned}g(k,N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cos(b\log k - b\log 3) + \cdots + \cos(b\log k - b\log N)\\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cos(b\log 3 - b\log k) + \cdots + \cos(b\log N - b\log k)\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \qquad (13)\end{aligned}$$

We can have the following (14) from N equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N) with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned}0 =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots + \cos(b\log 2 - b\log N)\}\\ &-f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots + \cos(b\log 4 - b\log N)\}\\ &-f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+\cdots\cdots\\ =& f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots \qquad (14)\end{aligned}$$

If (11) holds, the sum of the right sides of infinite number equations of (12-1), (12-2), (12-3), (12-4), (12-5), $\cdots\cdots$ becomes zero. The rightmost side of (14) is the sum of the right sides of N equations of (12-1), (12-2), (12-3),$\cdots\cdots$, (12-N) as shown in item 1.4 of [Appendix 1]. Thererfore if (11) holds, $\lim_{N\to\infty}$ {the rightmost side of (14)} $= 0$ must hold. Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + \cdots\cdots \qquad (15)$$

We have the following (22) in [Appendix 2 : Investigation of $g(k, N)$].

$$g(k,N) \quad \sim \quad \frac{N\cos(b\log N)}{\sqrt{1+b^2}} \qquad (N \to \infty \quad k = 2, 3, 4, 5, \cdots\cdots) \qquad (22)$$

From the above (15) and (22) we have the following (16).

$$
\begin{aligned}
&\text{The rightmost side of (14)}\\
&= f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots\\
&\sim \qquad f(2)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} - f(3)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} + f(4)\frac{N\cos(b\log N)}{\sqrt{1+b^2}}\\
&\qquad - f(5)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} + \cdots\cdots\\
&= \frac{N\cos(b\log N)}{\sqrt{1+b^2}}\{f(2) - f(3) + f(4) - f(5) + \cdots\cdots\}\\
&= F(a)\frac{N\cos(b\log N)}{\sqrt{1+b^2}} \qquad\qquad (N\to\infty) \qquad (16)
\end{aligned}
$$

$\lim_{N\to\infty}\frac{N\cos(b\log N)}{\sqrt{1+b^2}}$ diverges to $\pm\infty$. $0 < F(a)$ holds in $0 < a < 1/2$ as shown in [Appendix 3 : Investigation of $F(a)$]. Then $\lim_{N\to\infty}$ {the rightmost side of (14)} diverges to $\pm\infty$ in $0 < a < 1/2$ from the above (16) i.e. (11) does not hold in $0 < a < 1/2$. (11) holds at $a = 0$ as shown in item 2. Therefore the solution for the identity of (11) is only $a = 0$.

4. Conclusion

a has the range of $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero as shown in the above item 3. Therefore non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) is $1/2 \pm bi$ and other non-trivial zero point does not exist.

Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

If the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) hold.

$$\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A$$
$$\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B$$
$$\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B$$
$$\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B$$

1.1. Construction of (9)

We can have (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

1.2. Construction of (10)

We can have (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

1.3. Construction of (11)

We can have (11) as (Series 3) by regarding the following (11-1) and (11-2) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \cos x\{\text{the right side of (9)}\} \equiv 0 \tag{11-1}$$
$$\text{(Series 2)} = \sin x\{\text{the right side of (10)}\} \equiv 0 \tag{11-2}$$

1.4. Construction of (14)

1.4.1 We can have the following (12-1*2) as (Series 3) by regarding the following (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 1)} =& f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &+ f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &+ f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0\end{aligned} \tag{12-1}$$

$$\begin{aligned}\text{(Series 2)} =& f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &+ f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &+ f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0\end{aligned} \tag{12-2}$$

$$\begin{aligned}\text{(Series 3)} =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\\ &+ \cdots\cdots = 0 + 0\end{aligned} \tag{12-1*2}$$

1.4.2 We can have the following (12-1*3) as (Series 3) by regarding the above (12-1*2) and the following (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 2) = {} & f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) \\ & + f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3) \\ & + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \end{aligned} \tag{12-3}$$

$$\begin{aligned}&(\text{Series } 3) \\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\} \\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\} \\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\} \\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\} \\ &\quad + \cdots\cdots = 0 + 0 \end{aligned} \tag{12-1*3}$$

1.4.3 We can have the following (12-1*4) as (Series 3) by regarding the above (12-1*3) and the following (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 2) = {} & f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4) \\ & + f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4) \\ & + f(6)\cos(b\log 6 - b\log 4) - \cdots\cdots = 0 \end{aligned} \tag{12-4}$$

$$\begin{aligned}&(\text{Series } 3) \\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 4)\} \\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 4)\} \\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 4)\} \\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 4)\} \\ &\quad + \cdots\cdots = 0 + 0 \end{aligned} \tag{12-1*4}$$

1.4.4 In the same way as above we can have the following (12-1*N)=(14) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively. ($N = 5, 6, 7, 8, \cdots\cdots$) $g(k, N)$ is defined in page 3. ($k = 2, 3, 4, 5, \cdots\cdots$)

$$\begin{aligned}&(\text{Series } 3) = \\ &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots + \cos(b\log 2 - b\log N)\} \\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots + \cos(b\log 3 - b\log N)\} \\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots + \cos(b\log 4 - b\log N)\} \\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots + \cos(b\log 5 - b\log N)\} \\ &+ \cdots\cdots \\ &\quad = f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \cdots\cdots \\ &\quad = 0 + 0 \end{aligned} \tag{12-1*N}$$

Appendix 2. : Investigation of $g(k, N)$

2.1 We define G and H as follows. $(N = 1, 2, 3, 4, \cdots\cdots)$

$$G = \lim_{N\to\infty} \frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$= \int_0^1 \cos(b\log x)dx \tag{20-1}$$
$$H = \lim_{N\to\infty} \frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$
$$= \int_0^1 \sin(b\log x)dx \tag{20-2}$$

We calculate G and H by Integration by parts.

$$G = [x\cos(b\log x)]_0^1 + bH = 1 + bH$$
$$H = [x\sin(b\log x)]_0^1 - bG = -bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{1+b^2} \qquad H = \frac{-b}{1+b^2} \tag{21}$$

2.2 From (13) and the above (21) we have the following (22).

$$g(k,N) = \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k)$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{2}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{3}{N}\tfrac{N}{k}) + \cdots + \cos(b\log\tfrac{N}{N}\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N} + b\log\tfrac{N}{k}) + \cos(b\log\tfrac{2}{N} + b\log\tfrac{N}{k})$$
$$+ \cos(b\log\tfrac{3}{N} + b\log\tfrac{N}{k}) + \cdots\cdots + \cos(b\log\tfrac{N}{N} + b\log\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\cos(b\log\tfrac{N}{k})\{\cos(b\log\tfrac{1}{N}) + \cos(b\log\tfrac{2}{N}) + \cos(b\log\tfrac{3}{N}) + \cdots + \cos(b\log\tfrac{N}{N})\}$$
$$-N\tfrac{1}{N}\sin(b\log\tfrac{N}{k})\{\sin(b\log\tfrac{1}{N}) + \sin(b\log\tfrac{2}{N}) + \sin(b\log\tfrac{3}{N}) + \cdots + \sin(b\log\tfrac{N}{N})\}$$

$$\sim \qquad N\cos(b\log\frac{N}{k})G - N\sin(b\log\frac{N}{k})H$$
$$= N\cos(b\log\frac{N}{k})\frac{1}{1+b^2} + N\sin(b\log\frac{N}{k})\frac{b}{1+b^2}$$
$$= \frac{N}{\sqrt{1+b^2}}\{\cos(b\log\frac{N}{k})\frac{1}{\sqrt{1+b^2}} + \sin(b\log\frac{N}{k})\frac{b}{\sqrt{1+b^2}}\}$$
$$= \frac{N}{\sqrt{1+b^2}}\cos(b\log\frac{N}{k} - \tan^{-1} b)$$
$$= \frac{N}{\sqrt{1+b^2}}\cos\{b\log N(1 - \frac{\log k}{\log N} - \frac{\tan^{-1} b}{b\log N})\}$$
$$\sim \qquad \frac{N\cos(b\log N)}{\sqrt{1+b^2}} \qquad (N\to\infty \quad k = 2, 3, 4, 5, \cdots\cdots) \tag{22}$$

Appendix 3. : Investigation of $F(a)$

3.1. Investigation of $f(n)$

We have the following (8) and (15) in the text.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots \quad 0 \leq a < 1/2) \tag{8}$$

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{15}$$

$a = 0$ is the solution for $F(a) = 0$ due to $f(n) \equiv 0$ at $a = 0$. The alternating series $F(a)$ converges due to $\lim_{n\to\infty} f(n) = 0$.

We define the following (31) from the above (8) and we have the following (32) from (31).

$$f(r) = \frac{1}{r^{1/2-a}} - \frac{1}{r^{1/2+a}} \geq 0 \qquad (r : \text{real number} \quad 2 \leq r) \tag{31}$$

$$\frac{df(r)}{dr} = f'(r) = \frac{1/2+a}{r^{3/2+a}} - \frac{1/2-a}{r^{3/2-a}} = \frac{1/2+a}{r^{3/2+a}}\{1 - (\frac{1/2-a}{1/2+a})r^{2a}\} \tag{32}$$

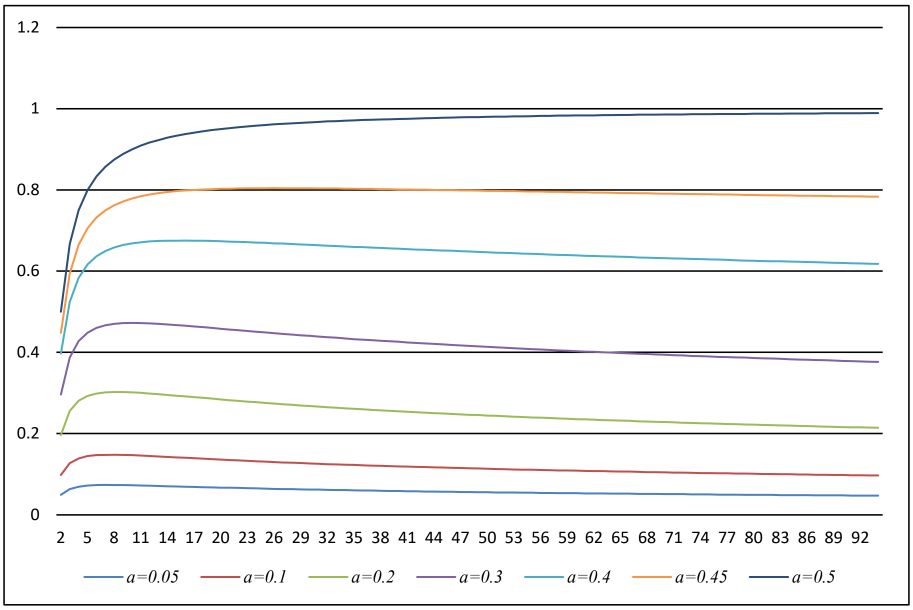
The value of $f(r)$ increases with increase of r and reaches the maximum value $f(r_{max})$ at $r = r_{max} = \left(\frac{1/2+a}{1/2-a}\right)^{1/(2a)}$. Afterward $f(r)$ decreases to zero with $r \to \infty$. $f(n)$ also has the maximum value $f(n_{max})$ at $n = n_{max}$ and n_{max} is either of $\lfloor r_{max} \rfloor$ and $\lfloor r_{max} \rfloor + 1$. Then we can have the following (34).

$$\begin{aligned} r_{max} &= (\frac{1/2+a}{1/2-a})^{1/(2a)} = (1 + 4a + 8a^2 + \cdots\cdots)^{1/(2a)} \\ &\sim \quad (1+4a)^{1/(2a)} = \{(1+4a)^{1/(4a)}\}^2 \\ &\sim \quad e^2 = 7.39 \qquad (a \to +0) \end{aligned} \tag{34}$$

From the above (34) we have the following (35).

$$7 \leq n_{max} \qquad (0 < a < 1/2) \tag{35}$$

The following (Graph 1) shows $f(n)$ in various value of a.



Graph 1 : $f(n)$ in various a

We have the following (36) from (32).

$$\begin{aligned}\frac{df'(r)}{dr} =& f''(r) = \frac{(1/2-a)(3/2-a)}{r^{5/2-a}} - \frac{(1/2+a)(3/2+a)}{r^{5/2+a}} \\ =& \frac{(1/2-a)(3/2-a)}{r^{5/2-a}}\{1 - \frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}r^{-2a}\} \qquad (36)\end{aligned}$$

We have the following (37) from the above (36) and $f''(r_0) = 0$.

$$r_0 = \{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{1/(2a)} = (1 + \frac{16}{3}a + \frac{128}{9}a^2 + \cdots\cdots)^{1/(2a)} \qquad (37)$$

Then we can have the following (37-1).

$$\begin{aligned} r_0 =& (1 + \frac{16}{3}a + \frac{128}{9}a^2 + \cdots\cdots)^{1/(2a)} \\ \sim \quad & (1 + \frac{16}{3}a)^{1/(2a)} = \{(1 + \frac{16}{3}a)^{3/(16a)}\}^{8/3} \\ \sim \quad & e^{8/3} = 14.39 \qquad (a \to +0) \qquad (37\text{-}1)\end{aligned}$$

We can confirm the property of $f(r)$ and $f'(r)$ from (32) and (36) as shown in the following (Table 1) and (Figure 1).

Item	Range of r	$f(r)$	$f'(r)$	The maximum value of $\lvert f'(r)\rvert$
3.1.1	$2 \leqq r \leqq r_{max}$	Positive value. Monotonically increasing and districtly concave function. The maximum value at $r=r_{max}$.	Positive value. Monotonically decreasing function. $f'(r)=0$ at $r=r_{max}$.	$f'(2)$
3.1.2	$r_{max} < r \leqq r_0$	Positive value. Monotonically decreasing and districtly concave function.	Negative value. Monotonically decreasing function. The minimum value at $r=r_0$.	$-f'(r_0)$
3.1.3	$r_0 \leqq r$	Positive value. Monotonically decreasing and districtly convex function. Converges to zero with $r \to \infty$.	Negative value. Monotonically increasing function. Converges to zero with $r \to \infty$.	$-f'(r_0)$

Table 1 : The property of $f(r)$ and $f'(r)$

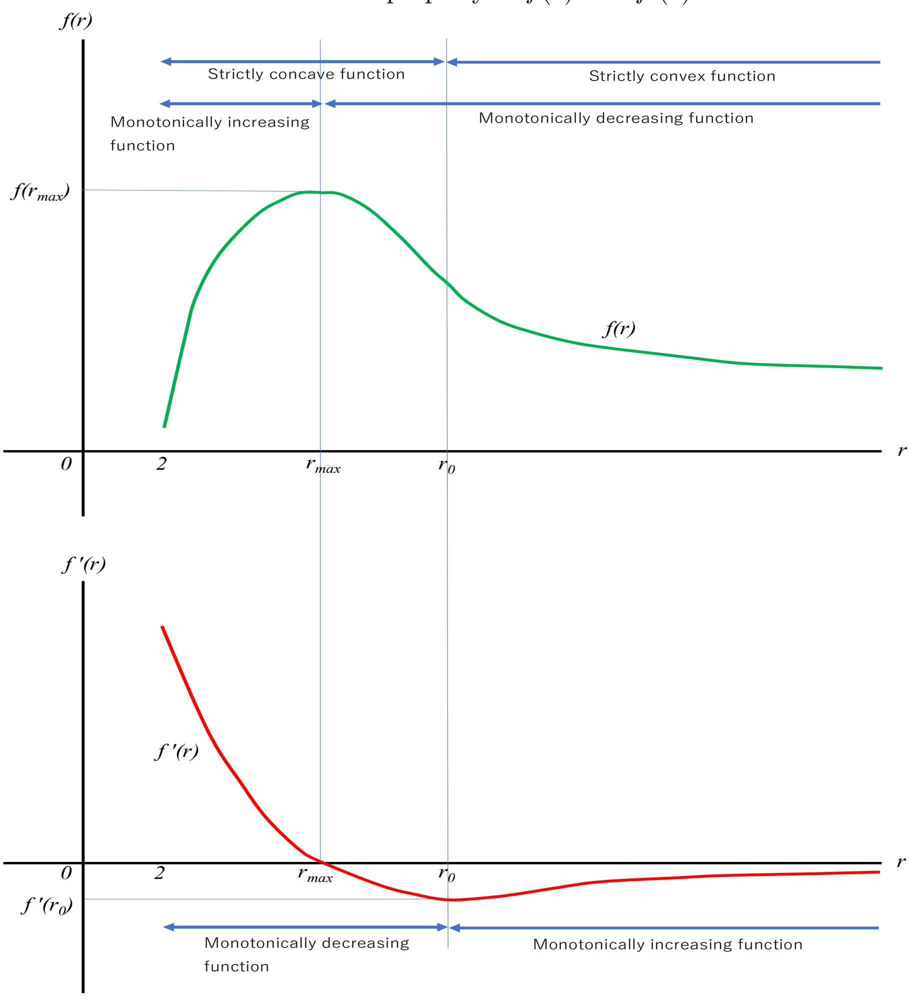


Figure 1 : The property of $f(r)$ and $f'(r)$

3.2. Verification method for $0 < F(a)$

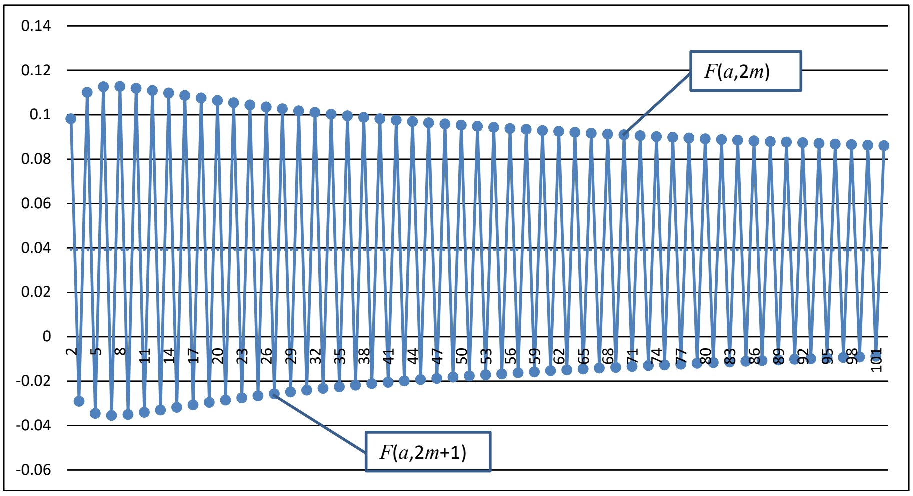
We define $F(a,n)$ as the following (38) and we have the following (39) from (38).

$$F(a,n) = f(2) - f(3) + f(4) - f(5) + \cdots + (-1)^n f(n) \tag{38}$$

$$\lim_{n\to\infty} F(a,n) = F(a) \tag{39}$$

$F(a)$ is an alternating series. So $F(a,n)$ repeats increase and decrease by $f(n)$ with increase of n as shown in the following (Graph 2). In (Graph 2) upper points mean $F(a,2m) \quad (m = 1,2,3,\cdots\cdots)$ and lower points mean $F(a,2m+1)$. $F(a,2m)$ decreases with increase of m in $n_{max} \le 2m$ and converges to $F(a)$ with $m \to \infty$ due to $\lim_{n\to\infty} f(n) = 0$. $F(a,2m+1)$ increases with increase of m in $n_{max} \le 2m+1$ and also converges to $F(a)$ with $m \to \infty$. From the above (39) we have the following (40).

$$\lim_{m\to\infty} F(a,2m) = \lim_{m\to\infty} F(a,2m+1) = F(a) \tag{40}$$



Graph 2 : $F(0.1,n)$ from $n = 2$ to $n = 100$

We define $F1(a)$ and $F1(a,2m+1)$ as the following (41) and (42-1). We have the following (42-2) from (42-1).

$$F1(a) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \{f(6) - f(7)\} + \cdots\cdots \tag{41}$$

$$F1(a,2m+1) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(2m) - f(2m+1)\} \tag{42-1}$$

$$= f(2) - f(3) + f(4) - f(5) + \cdots + f(2m) - f(2m+1) = F(a,2m+1) \tag{42-2}$$

We have the following (43) from the above (40), (41), (42-1) and (42-2).

$$F1(a) = \lim_{m\to\infty} F1(a,2m+1) = \lim_{m\to\infty} F(a,2m+1) = F(a) \tag{43}$$

Then we can use $F1(a)$ instead of $F(a)$ to verify $0 < F(a)$.

We enclose 2 terms of $F(a)$ each from the first term with { } as follows. If n_{max} is p or

$p+1$ (p: odd number), the inside sum of { } from $f(2)$ to $f(p)$ has a negative value and the inside sum of { } after $f(p+1)$ has a positive value as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - f(7) + \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(p-1) - f(p)\} + \{f(p+1) - f(p+2)\} + \cdots\cdots$$

(inside sum of { }) < 0 ⟵|⟶(inside sum of { }) > 0
(total sum of { }) = $-B$ ⟵|⟶(total sum of { }) = A

We define A and B as follows. n_{max} is p or $p+1$. (p: odd number)

$$\{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(p-1) - f(p)\} = -B < 0$$
$$\{f(p+1) - f(p+2)\} + \{f(p+3) - f(p+4)\} + \cdots\cdots = A > 0$$

We have the following (44) from the above definition.

$$F(a) = A - B \tag{44}$$

So we can verify $0 < F(a)$ by verifying $B < A$.

3.3. Investigation of $h(n) = f(n) - f(n+1)$

3.3.1 We define as follows from (8) and (31).

$$h(n) = f(n) - f(n+1) \qquad (n = 2, 3, 4, 5, \cdots\cdots \quad 0 \le a < 1/2) \tag{45}$$
$$h(r) = f(r) - f(r+1) \qquad (r : \text{real number} \quad 2 \le r) \tag{46}$$

We have the following (47) from the above (46) and (32).

$$\frac{dh(r)}{dr} = h'(r) = f'(r) - f'(r+1) \tag{47}$$

We can find the following item 3.3.3.1 — 3.3.3.4 from the above (47), (Table 1) and (Figure 1).

3.3.1.1 $f'(r)$ decreases monotonically in $2 \le r \le r_0$. Then we have the following (48) and we have the following (49) from (48).

$$f'(r) > f'(r+1) \qquad (2 \le r \le r_0 - 1) \tag{48}$$
$$h'(r) = f'(r) - f'(r+1) > 0 \qquad (2 \le r \le r_0 - 1) \tag{49}$$

Thererfore $h(r)$ increases monotonically in $2 \le r \le r_0 - 1$.

3.3.1.2 $f'(r)$ increases monotonically in $r_0 \le r$. Then we have the following (50) and we have the following (51) from (50).

$$f'(r) < f'(r+1) \qquad (r_0 \le r) \tag{50}$$
$$h'(r) = f'(r) - f'(r+1) < 0 \qquad (r_0 \le r) \tag{51}$$

Thererfore $h(r)$ decreases monotonically in $r_0 \le r$.

3.3.1.3 $f'(r+1)$ is the figure in which $f'(r)$ shifts to the left by 1 as shown in the following (Figure 2).

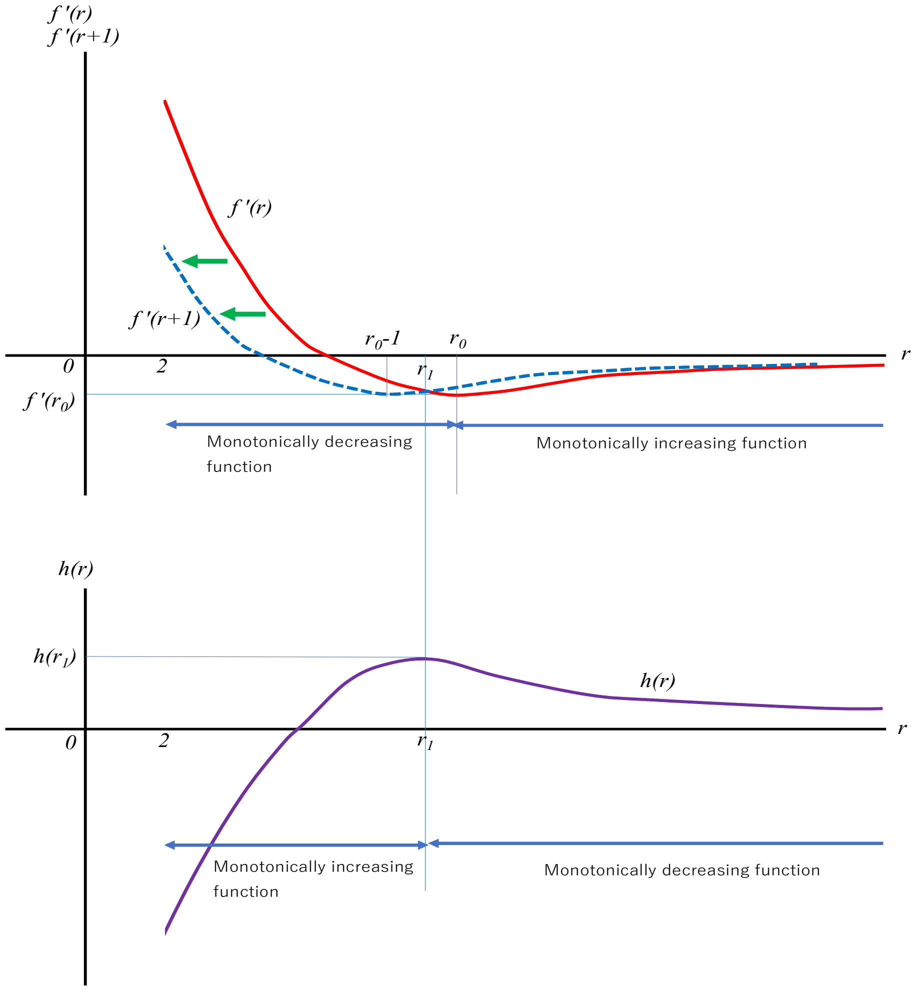
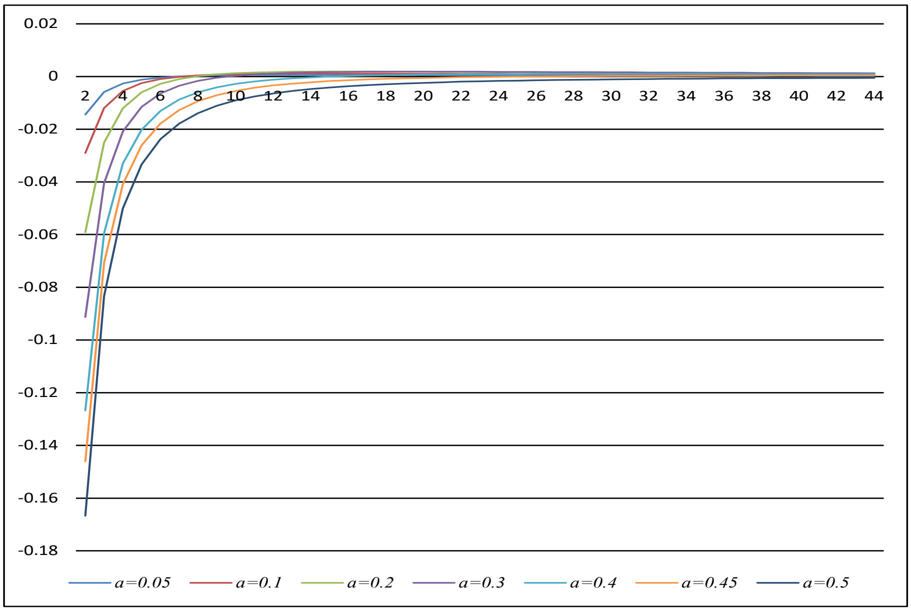


Figure 2 : The property of $f'(r)$ and $h(r)$

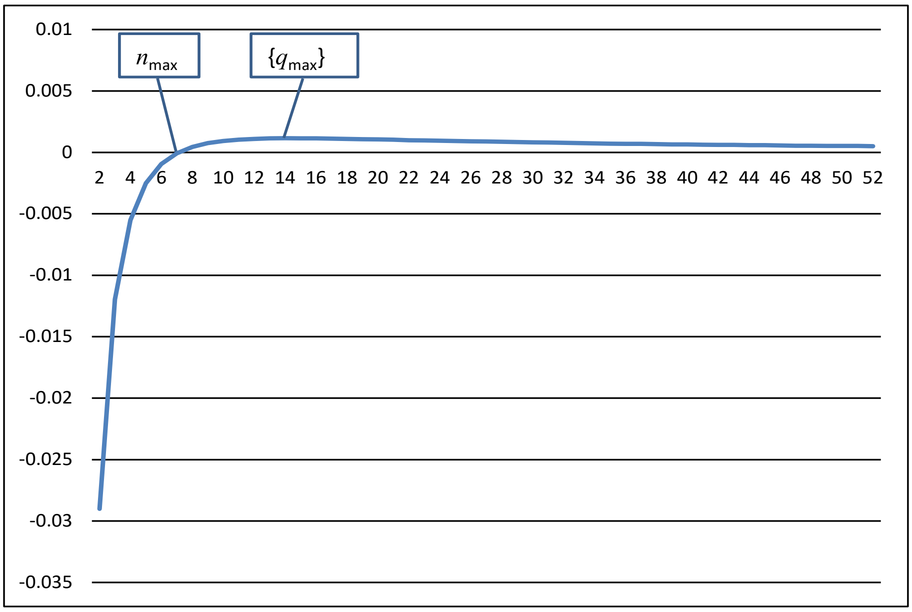
Then $f'(r)$ and $f'(r+1)$ have one intersection at r_1 $(r_0 - 1 < r_1 < r_0)$ i.e. $h'(r_1) = 0$ holds. Thererfore $h(r)$ has the maximum value $h(r_1)$ at $r = r_1$ from the above item 3.3.1.1 and 3.3.1.2. $h(n) = f(n) - f(n+1)$ also has the maximum value $f(n_1) - f(n_1 + 1) = \{q_{max}\}$ at $n = n_1$. n_1 is either of $\lfloor r_1 \rfloor$ and $\lfloor r_1 \rfloor + 1$.

3.3.1.4 The sign of $h(n)$ changes from minus to plus with increase of n at $n = n_{max}$. Afterward the value of $h(n)$ reaches the maximum value $\{q_{max}\}$ at $n = n_1$ and the value decreases to zero with $n \to \infty$.

The following (Graph 3) shows the value of $h(n)$ in various value of a. The following (Graph 4) shows the value of $h(n)$ at $a = 0.1$.



Graph 3 : $h(n)$ in various a



Graph 4 : $h(n)$ at $a = 0.1$

3.3.2 We have the following (52) and (53) from the above item 3.3.1.

$$f(3) - f(2) > f(4) - f(3) > f(5) - f(4) > \cdots\cdots > f(n_{max} - 1) - f(n_{max} - 2)$$
$$> f(n_{max}) - f(n_{max} - 1) > 0 \tag{52}$$

We abbreviate $\{f(n_{max}+q)-f(n_{max}+q+1)\}$ to $\{q\}$ for easy description. $(q=0,1,2,3,\cdots\cdots)$ All $\{q\}$ has a positive value from the above abbreviation.

$$\{0\}<\{1\}<\{2\}<\{3\}<\cdots<\{q_{max}-3\}<\{q_{max}-2\}<\{q_{max}-1\}$$
$$<\{q_{max}\}>\{q_{max}+1\}>\{q_{max}+2\}>\{q_{max}+3\}>\cdots\cdots \quad (53)$$

3.3.3 We can have the following (56) from (52).

$$0<f(n+1)-f(n)<f(3)-f(2) \qquad (3\le n\le n_{max}-1) \quad (56)$$

We can have the following (57) from (Table 1) and (Figure 1).

$$0<f(n)-f(n+1)=\int_n^{n+1}\{-f'(r)\}dr<\int_n^{n+1}\{-f'(r_0)\}dr=-f'(r_0)$$
$$(n_{max}\le r \quad n_{max}\le n) \quad (57)$$

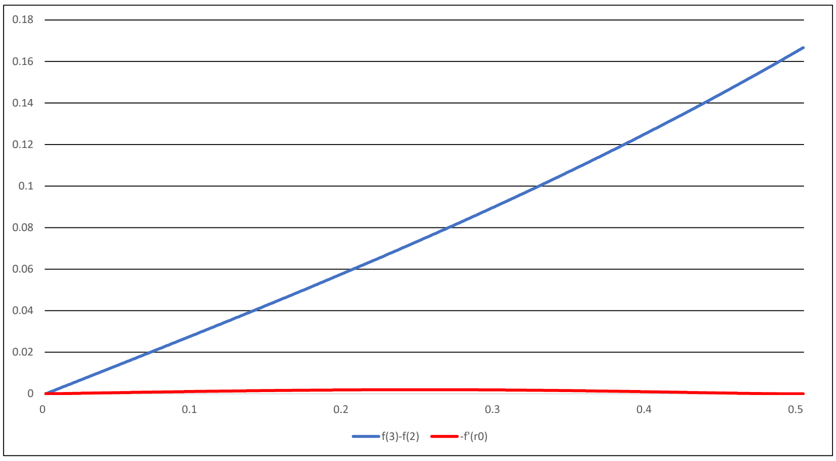
We can have the following (58) from the following item 3.3.4 — 3.3.6.

$$0<-f'(r_0)<f(3)-f(2) \qquad (0<a\le 1/2) \quad (58)$$

Then we can have the following (59) from the above (56), (57) and (58).

$$|f(n)-f(n+1)|<f(3)-f(2) \qquad (n=3,4,5,\cdots\cdots) \quad (59)$$

3.3.4 The following (Graph 5) is plotted by calculating $f(3)-f(2)$ and $-f'(r_0)$ for a every 0.01.



Graph 5 : $f(3)-f(2)$ and $-f'(r_0)$ regarding a

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
$f(3)$-$f(2)$	0	0.014438	0.029008	0.043844	0.05908	0.074851	0.091297	0.108555	0.126771	0.146091	0.166667
$-f'(r_0)$	0	0.000601	0.001149	0.001591	0.00188	0.001976	0.001852	0.001504	0.000968	0.000361	0

Table 2 : The values of $f(3)-f(2)$ and $-f'(r_0)$

If $f(3) - f(2)$ has a convex or a concave in $a_0 < a < a_0 + 0.01$, such a convex or a concave is not displayed in the above (Graph 5). (a_0=0, 0.01, 0.02, $\cdots\cdots$, 0.48, 0.49) If the function regarding a has the property shown in the following 3 items, the function does not have a convex or a concave in $a_0 \le a \le a_0 + 0.01$. Then the graph can display the function correctly although the graph is plotted for a every 0.01 i.e. we can imagine the shape of the function easily from the graph.

3.3.4.1 The function does not have a local maximum value or a local minimum value in $a_0 \le a \le a_0 + 0.01$.

3.3.4.2 When the function has a local maximum value in $a_0 \le a < a_0 + 0.01$ the function is districtly concave regarding a in $a_0 - 0.02 \le a \le a_0 + 0.03$.

3.3.4.3 When the function has a local minimum value in $a_0 \le a < a_0 + 0.01$ the function is districtly convex regarding a in $a_0 - 0.02 \le a \le a_0 + 0.03$.

For example, in the following (Figure 3) the blue line is the function that meets the above item 3.3.4.2 and the red line is the graph that is plotted for a every 0.01. We can imagine the shape of the function easily from the graph.

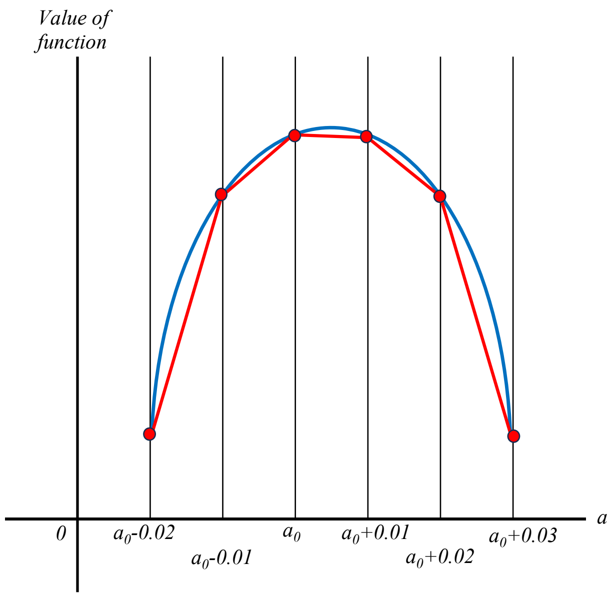


Figure 3 : The function and the graph

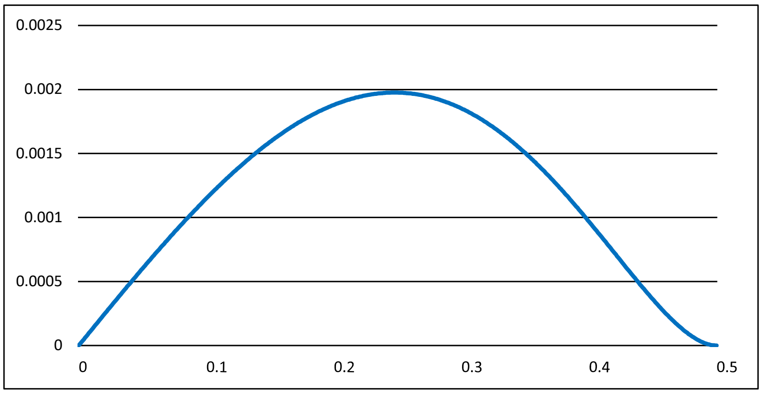
$f(n)$ is a monotonically increasing and districtly convex function regarding a in $0 < a \le 1/2$ from the following (60) and (61). Therefore $f(n)$ meets the above item 3.3.4.1.

$$\frac{df(n)}{da} = \log n\left(\frac{1}{n^{1/2-a}} + \frac{1}{n^{1/2+a}}\right) > 0 \tag{60}$$

$$\frac{d^2 f(n)}{da^2} = (\log n)^2\left(\frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}}\right) \ge 0 \tag{61}$$

Then $f(3)$ and $f(2)$ are monotonically increasing and districtly convex functions regarding a i.e. $f(3)$ and $f(2)$ do not have a convex or a concave in $a_0 \leq a \leq a_0+0.01$. $f(3)-f(2)$ also does not have a convex or a concave in $a_0 \leq a \leq a_0+0.01$ from the above property of $f(3)$ and $f(2)$. Therefore (Graph 5) shows $f(3)-f(2)$ correctly.

3.3.5 The following (Graph 6) is plotted by calculating $-f'(r_0)$ for a every 0.01. If $-f'(r_0)$ has a convex or a concave in $a_0 < a < a_0+0.01$, such a convex or a concave is not displayed in (Graph 6). (a_0=0, 0.01, 0.02, $\cdots\cdots$, 0.48, 0.49)



Graph 6 : $-f'(r_0)$ regarding a

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
$-f'(r_0)$	0	0.000601	0.001149	0.001591	0.00188	0.001976	0.001852	0.001504	0.000968	0.000361	0

Table 3 : The values of $-f'(r_0)$

We have the following (62) from (32) and (37).

$$
\begin{aligned}
&-f'(r_0) = (1/2-a)r_0^{a-3/2} - (1/2+a)r_0^{-a-3/2} \\
&= (1/2-a)\{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{1/2-3/(4a)} \\
&\quad -(1/2+a)\{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{-1/2-3/(4a)} \\
&= \{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{-3/(4a)}[(1/2-a)\{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{1/2} \\
&\quad -(1/2+a)\{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{-1/2}] \\
&= \{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{-3/(4a)}[\{\frac{(1/4-a^2)(3/2+a)}{3/2-a}\}^{1/2} \\
&\quad -\{\frac{(1/4-a^2)(3/2-a)}{3/2+a}\}^{1/2}] \\
&= \{\frac{(1/2-a)(3/2-a)}{(1/2+a)(3/2+a)}\}^{3/(4a)}(1/4-a^2)^{1/2}\{(\frac{3/2+a}{3/2-a})^{1/2}-(\frac{3/2-a}{3/2+a})^{1/2}\}
\end{aligned}
$$

$$= 2a\{\frac{(1/2-a)(3/2-a)}{(1/2+a)(3/2+a)}\}^{3/(4a)}\{\frac{(1/2+a)(1/2-a)}{(3/2+a)(3/2-a)}\}^{1/2}$$
$$= 2a\{u(a)\}^{3/(4a)}\{v(a)\}^{1/2} \tag{62}$$

$u(a)$ in the above (62) is a monotonically decreasing and districtly convex function regarding a in $0 \leq a \leq 1/2$ from the following (63-1) and (63-2).

$$\frac{du(a)}{da} = \frac{4a^2-3}{(1/2+a)^2(3/2+a)^2} < 0 \tag{63-1}$$
$$\frac{d^2u(a)}{da^2} = \frac{2(6+9a-4a^3)}{(1/2+a)^3(3/2+a)^3} > 0 \tag{63-2}$$

$v(a)$ in the above (62) is a monotonically decreasing and districtly concave function regarding a in $0 < a \leq 1/2$ from the following (63-3) and (63-4).

$$\frac{dv(a)}{da} = \frac{-4a}{(3/2+a)^2(3/2-a)^2} \leq 0 \tag{63-3}$$
$$\frac{d^2v(a)}{da^2} = \frac{-3(3+4a^2)}{(3/2+a)^3(3/2-a)^3} < 0 \tag{63-4}$$

$a, 3/(4a), u(a)$ and $v(a)$ compose $-f'(r_0)$ as shown in (62). These 4 functions do not have a convex or a concave in $a_0 \leq a \leq a_0 + 0.01$ respectively because they meet item 3.3.4.1. Then $-f'(r_0)$ also does not have a convex or a concave in $a_0 \leq a \leq a_0 + 0.01$ from the above property of $a, 3/(4a), u(a)$ and $v(a)$. Therefore (Graph 6) shows $-f'(r_0)$ correctly.
Now we can confirm that (Graph 5) and (Graph 6) show $f(3) - f(2)$ and $-f'(r_0)$ correctly and we can find that (58) holds from (Graph 5) and (Graph 6).

3.3.6 We can confirm that (58) holds also during $a \to +0$ from the following (64) and (65).
$f(3) - f(2)$ can be approximated in $a \to +0$ by performing Maclaurin expansion for $2^a, 2^{-a}, 3^a$ and 3^{-a} like the following (64).

$$\begin{aligned}
&f(3) - f(2)\\
&= (3^{a-1/2} - 3^{-a-1/2}) - (2^{a-1/2} - 2^{-a-1/2})\\
&= 3^{-1/2}(3^a - 3^{-a}) - 2^{-1/2}(2^a - 2^{-a})\\
&= 3^{-1/2}[\{1 + a\log 3 + (a\log 3)^2/2 + \cdots\cdots\} - \{1 - a\log 3 + (a\log 3)^2/2 - \cdots\cdots\}]\\
&\quad - 2^{-1/2}[\{1 + a\log 2 + (a\log 2)^2/2 + \cdots\cdots\} - \{1 - a\log 2 + (a\log 2)^2/2 - \cdots\cdots\}]\\
&= 2 * 3^{-1/2}\{a\log 3 + (a\log 3)^3/3! + (a\log 3)^5/5! + \cdots\cdots\}\\
&\quad - 2 * 2^{-1/2}\{a\log 2 + (a\log 2)^3/3! + (a\log 2)^5/5! + \cdots\cdots\}\\
&\sim \quad 2(3^{-1/2}\log 3 - 2^{-1/2}\log 2)a = 0.29a > 0.012a \qquad (a \to +0)
\end{aligned} \tag{64}$$

$-f'(r_0)$ can be approximated in $a \to +0$ from (32) and (37) by performing Maclaurin expansion for $(1 + \frac{16}{3}a)^{1/2}$ and $(1 + \frac{16}{3}a)^{-1/2}$ like the following (65).

$$-f'(r_0) = (1/2 - a)r_0^{a-3/2} - (1/2 + a)r_0^{-a-3/2}$$

$$
\begin{aligned}
=&(1/2-a)\{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{1/2-3/(4a)}\\
&-(1/2+a)\{\frac{(1/2+a)(3/2+a)}{(1/2-a)(3/2-a)}\}^{-1/2-3/(4a)}\\
=&(1/2-a)(1+\frac{16}{3}a+\frac{128}{9}a^2+\cdots\cdots)^{1/2-3/(4a)}\\
&-(1/2+a)(1+\frac{16}{3}a+\frac{128}{9}a^2+\cdots\cdots)^{-1/2-3/(4a)}\\
\sim\quad&(1/2-a)(1+\frac{16}{3}a)^{1/2-3/(4a)}-(1/2+a)(1+\frac{16}{3}a)^{-1/2-3/(4a)}\\
=&(1+\frac{16}{3}a)^{-3/(4a)}\{(1/2-a)(1+\frac{16}{3}a)^{1/2}-(1/2+a)(1+\frac{16}{3}a)^{-1/2}\}\\
=&(1+\frac{16}{3}a)^{-3/(4a)}\{(1/2-a)(1+\frac{8}{3}a-\frac{32}{9}a^2+\cdots\cdots)\\
&\qquad-(1/2+a)(1-\frac{8}{3}a+\frac{32}{3}a^2+\cdots\cdots)\}\\
\sim\quad&(1+\frac{16}{3}a)^{-3/(4a)}\{(1/2-a)(1+\frac{8}{3}a)-(1/2+a)(1-\frac{8}{3}a)\}\\
=&\{(1+\frac{16}{3}a)^{3/(16a)}\}^{-4}(\frac{8}{3}-2)a\\
\sim\quad&\frac{8/3-2}{e^4}a=0.012a<0.29a\qquad\qquad(a\to+0)\qquad\qquad(65)
\end{aligned}
$$

3.4. Verification of $B < A$ (n_{max} is odd number.)

n_{max} is odd number as follows.

$$
\begin{aligned}
F(a)&=f(2)-f(3)+f(4)-f(5)+f(6)-\cdots\cdots\\
&=\{f(2)-f(3)\}+\{f(4)-f(5)\}+\cdots+\{f(n_{max}-3)-f(n_{max}-2)\}+\{f(n_{max}-1)-f(n_{max})\}\\
&\quad+\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+\cdots\cdots
\end{aligned}
$$

We can have A and B as follows. A and B are defined in item 3.2.

$$B=\{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+\cdots+\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$$

$$A=\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+\cdots\cdots$$

3.4.1. Condition for B

We define as follows.

{ (yellow) } : the term which is included within B.

{ (grey) } : the term which is not included within B.

We have the following (66).

$$
\begin{aligned}
f(n_{max})-f(2)=&\{f(n_{max})-f(n_{max}-1)\}+\{f(n_{max}-1)-f(n_{max}-2)\}+\{f(n_{max}-2)-f(n_{max}-3)\}\\
&+\cdots+\{f(7)-f(6)\}+\{f(6)-f(5)\}+\{f(5)-f(4)\}+\{f(4)-f(3)\}+\{f(3)-f(2)\}\qquad(66)
\end{aligned}
$$

And we have the following (67) from (52) in item 3.3.2.

$$
\begin{aligned}
&\{f(3)-f(2)\}>\{f(4)-f(3)\}>\{f(5)-f(4)\}>\{f(6)-f(5)\}>\{f(7)-f(6)\}>\cdots\cdots\\
&\quad>\{f(n_{max}-2)-f(n_{max}-3)\}>\{f(n_{max}-1)-f(n_{max}-2)\}>\{f(n_{max})-f(n_{max}-1)\}>0
\end{aligned}
$$

(67)

From the above (66) and (67) we have the following (68).

$$f(n_{max}) - f(2) + \{f(3) - f(2)\}$$

$$= \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 2) - f(n_{max} - 3)\} + \{f(n_{max}) - f(n_{max} - 1)\}$$

$\|$ $\wedge$ $\wedge$ $\wedge$ ←Value comparison→ $\wedge$

$$+\{f(3) - f(2)\} + \{f(4) - f(3)\} + \{f(6) - f(5)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$

$$> 2B \tag{68}$$

The above (68) shows the following inequality.

$$\{\text{Total sum of upper row of (68)}\} = B < \{\text{Total sum of lower row of (68)}\}$$

Then we have the following (69).

$$2B < f(n_{max}) - f(2) + \{f(3) - f(2)\} \tag{69}$$

3.4.2. Condition for A ($\{q_{max}\}$ is included within A.)

We abbreviate $\{f(n_{max} + q) - f(n_{max} + q + 1)\}$ to $\{q\}$ for easy description. $(q = 0, 1, 2, 3, \cdots\cdots)$ All $\{q\}$ has a positive value from the above abbreviation.
We define as follows.

{ (yellow) } : the term which is included within A.

{ (gray) } : the term which is not included within A.

$\{q_{max}\}$ has the maximum value in all $\{q\}$. And $\{q_{max}\}$ is included within A. Then value comparison of $\{q\}$ is as follows from (53) in item 3.3.2.

$$\{1\} < \{2\} < \{3\} < \cdots < \{q_{max} - 3\} < \{q_{max} - 2\} < \{q_{max} - 1\} < \{q_{max}\} > \{q_{max} + 1\} > \{q_{max} + 2\} > \{q_{max} + 3\} > \cdots\cdots$$

We have the following (70).

$$f(n_{max} + 1) = \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\}$$

$$+\{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{70}$$

From the above (70) we have the following (71).

$$f(n_{max} + 1) - \{q_{max} - 1\}$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{71}$$

← · · · · · · · · · · · Range 1 · · · · · · · · · · · · →|← · · · · · · · · · · · · Range 2 · · · · · · · · · · · ·

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{1\} < \{2\} < \{3\} < \{4\} < \cdots < \{q_{max} - 4\} < \{q_{max} - 3\} < \{q_{max} - 2\}$$

And we can find the following.

$$\text{Total sum of } \{\ \} = \{1\} + \{3\} + \{5\} + \{7\} + \cdots + \{q_{max}-4\} + \{q_{max}-2\}$$
$$\vee \quad \vee \quad \vee \quad \vee \quad \vee \quad \leftarrow \text{Value comparison}$$
$$\text{Total sum of } \{\ \} = \{2\} + \{4\} + \{6\} + \cdots + \{q_{max}-5\} + \{q_{max}-3\}$$

Therefore [Total sum of $\{\ \}$ > Total sum of $\{\ \}$] holds.
In (Range 2) value comparison is as follows.

$$\{q_{max}\} > \{q_{max}+1\} > \{q_{max}+2\} > \{q_{max}+3\} > \{q_{max}+4\} > \{q_{max}+5\} > \{q_{max}+6\} > \cdots\cdots$$

We have the following (71-1) and (71-2). The right sides of (71-1) and (71-2) are alternating series regarding $f(n)$ and they converge due to $\lim_{n\to\infty} f(n) = 0$.

$$\text{Total sum of } \{\ \} = \{q_{max}\} + \{q_{max}+2\} + \{q_{max}+4\} + \{q_{max}+6\} + \cdots\cdots \quad (71\text{-}1)$$
$$\vee \quad \vee \quad \vee \quad \vee \quad \leftarrow \text{Value comparison}$$
$$\text{Total sum of } \{\ \} = \{q_{max}+1\} + \{q_{max}+3\} + \{q_{max}+5\} + \{q_{max}+7\} + \cdots\cdots \quad (71\text{-}2)$$

Therefore [Total sum of $\{\ \}$ > Total sum of $\{\ \}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\ \} = A >$ Total sum of $\{\ \}$].
We have the following (72).

$$f(n_{max}+1) - \{q_{max}-1\} < 2A \tag{72}$$

3.4.3. Condition for A ($\{q_{max}\}$ is not included within A.)

We have the following (73). $\{q_{max}\}$ is not included within A.

$$f(n_{max}+1) = \{f(n_{max}+1) - f(n_{max}+2)\} + \{f(n_{max}+2) - f(n_{max}+3)\} + \{f(n_{max}+3) - f(n_{max}+4)\}$$
$$+ \{f(n_{max}+4) - f(n_{max}+5)\} + \cdots\cdots$$
$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max}-3\} + \{q_{max}-2\} + \{q_{max}-1\} + \{q_{max}\} + \{q_{max}+1\} + \{q_{max}+2\} + \{q_{max}+3\} + \cdots\cdots \tag{73}$$

From the above (73) we have the following (74).

$$f(n_{max}+1) - \{q_{max}\}$$
$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max}-3\} + \{q_{max}-2\} + \{q_{max}-1\} + \{q_{max}+1\} + \{q_{max}+2\} + \{q_{max}+3\} + \cdots\cdots \tag{74}$$
$$\leftarrow \cdots\cdots \text{Range 1} \cdots\cdots \rightarrow | \leftarrow \cdots\cdots \text{Range 2} \cdots\cdots$$

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{1\} < \{2\} < \{3\} < \{4\} < \cdots < \{q_{max}-4\} < \{q_{max}-3\} < \{q_{max}-2\} < \{q_{max}-1\}$$

And we can find the following.

$$\text{Total sum of } \{\ \} = \{1\} + \{3\} + \{5\} + \{7\} + \cdots + \{q_{max}-3\} + \{q_{max}-1\}$$
$$\vee \quad \vee \quad \vee \quad \vee \quad \vee \quad \leftarrow \text{Value comparison}$$
$$\text{Total sum of } \{\ \} = \{2\} + \{4\} + \{6\} + \cdots + \{q_{max}-4\} + \{q_{max}-2\}$$

Therefore [Total sum of $\{\ \}$ > Total sum of $\{\ \}$] holds.
In (Range 2) value comparison is as follows.

$$\{q_{max}+1\} > \{q_{max}+2\} > \{q_{max}+3\} > \{q_{max}+4\} > \{q_{max}+5\} > \{q_{max}+6\} > \{q_{max}+7\} > \cdots\cdots$$

And we can find the following.

$$\begin{array}{l} \text{Total sum of } \{\colorbox{yellow}{}\} = \{\colorbox{yellow}{$q_{max}+1$}\} + \{\colorbox{yellow}{$q_{max}+3$}\} + \{\colorbox{yellow}{$q_{max}+5$}\} + \{\colorbox{yellow}{$q_{max}+7$}\} + \cdots\cdots \\ \qquad\qquad\qquad\qquad\quad \vee \qquad\qquad\quad \vee \qquad\qquad\quad \vee \qquad\qquad\quad \vee \qquad \leftarrow\text{Value comparison} \\ \text{Total sum of } \{\colorbox{lightgray}{}\} = \{\colorbox{lightgray}{$q_{max}+2$}\} + \{\colorbox{lightgray}{$q_{max}+4$}\} + \{\colorbox{lightgray}{$q_{max}+6$}\} + \{\colorbox{lightgray}{$q_{max}+8$}\} + \cdots\cdots \end{array}$$

Therefore [Total sum of $\{\colorbox{yellow}{}\}$ > Total sum of $\{\colorbox{lightgray}{}\}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\colorbox{yellow}{}\} = A >$ Total sum of $\{\colorbox{lightgray}{}\}$].
We have the following (75).

$$f(n_{max}+1) - \{q_{max}\} < 2A \tag{75}$$

3.4.4. Condition for $B < A$

From (72) and (75) we have the following inequality.

$$f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] < 2A$$

Then the following inequalities hold from (59).

$$[\{q_{max}\} \text{ or } \{q_{max}-1\}] < f(3) - f(2)$$
$$f(n_{max}) - f(n_{max}+1) < f(3) - f(2)$$

We have the following (76) from the above 3 inequalities.

$$\begin{aligned} 2A >& f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] > f(n_{max}+1) - \{f(3)-f(2)\} \\ >& f(n_{max}) - \{f(3)-f(2)\} - \{f(3)-f(2)\} = f(n_{max}) - 2\{f(3)-f(2)\} \end{aligned} \tag{76}$$

We have the following (77) for $B < A$ from (69) and (76).

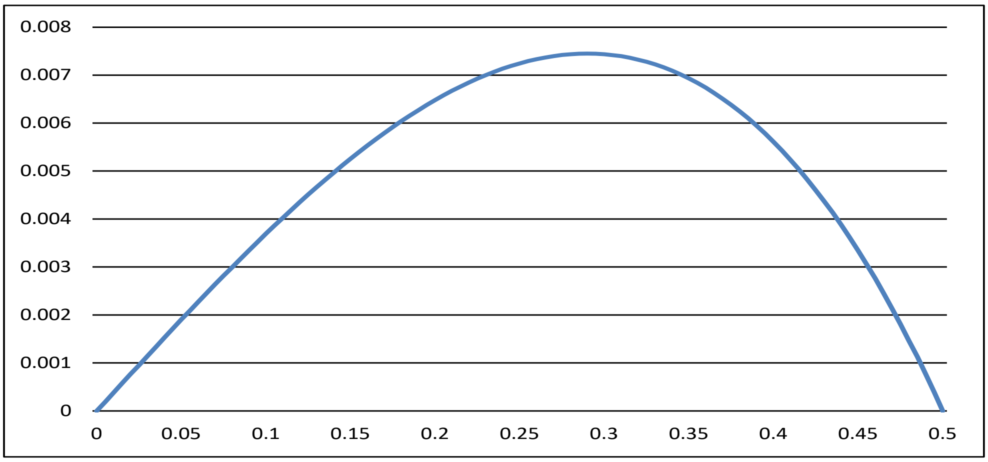
$$2A > f(n_{max}) - 2\{f(3)-f(2)\} > f(n_{max}) - f(2) + \{f(3)-f(2)\} > 2B \tag{77}$$

From the above (77) we can have the final condition for $B < A$ as follows.

$$f(3) < (4/3)f(2) \tag{78}$$

The following (Graph 7) is plotted by calculating the following (79) for a every 0.01.

$$J(a) = (4/3)f(2) - f(3) = (4/3)\left(\frac{1}{2^{1/2-a}} - \frac{1}{2^{1/2+a}}\right) - \left(\frac{1}{3^{1/2-a}} - \frac{1}{3^{1/2+a}}\right) \tag{79}$$



Graph 7 : $J(a) = (4/3)f(2) - f(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)-f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

Table 4 : The values of $J(a)$

$f(2)$ and $f(3)$ do not have a convex or a concave in $a_0 \le a \le a_0 + 0.01$ as shown in item 3.3.4. (a_0=0, 0.01, 0.02, $\cdots\cdots$, 0.48, 0.49) $J(a)$ also does not have a convex or a concave in $a_0 \le a \le a_0 + 0.01$ from the above property of $f(2)$ and $f(3)$. Therefore (Graph 7) shows $J(a)$ correctly. We can confirm that $0 < J(a)$ holds also during $a \to +0$ and $a \to 1/2 - 0$ from the following item 3.4.4.1 and 3.4.4.2. From (Graph 7), item 3.4.4.1 and 3.4.4.2 we can find that $0 < J(a)$ holds in $0 < a < 1/2$. Therefore $B < A$ holds in $0 < a < 1/2$ i.e. $0 < F(a)$ holds in $0 < a < 1/2$ from (44).

3.4.4.1 $J(a)$ can be approximated in $a \to +0$ by performing Maclaurin expansion for $2^a, 2^{-a}, 3^a$ and 3^{-a} like the following (80).

$$
\begin{aligned}
&J(a) = (4/3)f(2) - f(3) \\
&= (4/3)(2^{a-1/2} - 2^{-a-1/2}) - (3^{a-1/2} - 3^{-a-1/2}) \\
&= (4/3)2^{-1/2}(2^a - 2^{-a}) - 3^{-1/2}(3^a - 3^{-a}) \\
&= (4/3)2^{-1/2}[\{1 + a\log 2 + (a\log 2)^2/2 + \cdots\cdots\} - \{1 - a\log 2 + (a\log 2)^2/2 - \cdots\cdots\}] \\
&\quad - 3^{-1/2}[\{1 + a\log 3 + (a\log 3)^2/2 + \cdots\cdots\} - \{1 - a\log 3 + (a\log 3)^2/2 - \cdots\cdots\}] \\
&= 2 * (4/3)2^{-1/2}\{a\log 2 + (a\log 2)^3/3! + (a\log 2)^5/5! + \cdots\cdots\} \\
&\quad - 2 * 3^{-1/2}\{a\log 3 + (a\log 3)^3/3! + (a\log 3)^5/5! + \cdots\cdots\} \\
&\sim \quad (4/3)2^{-1/2}(2a\log 2) - 3^{-1/2}(2a\log 3) = 0.038a > 0 \qquad (a \to +0) \qquad (80)
\end{aligned}
$$

3.4.4.2 Let $(1/2 - a)$ be t. $J(a)$ can be approximated in $a \to 1/2 - 0$ by performing Maclaurin expansion for $2^t, 2^{-t}, 3^t$ and 3^{-t} like the following (81).

$$J(a) = (4/3)f(2) - f(3)$$

$$= (4/3)(2^{a-1/2} - 2^{-a-1/2}) - (3^{a-1/2} - 3^{-a-1/2})$$
$$= (4/3)(2^{-t} - 2^{t-1}) - (3^{-t} - 3^{t-1}) = (4/3)(2^{-t} - 2^{t}/2) - (3^{-t} - 3^{t}/3)$$
$$= (4/3)[\{1 - t\log 2 + (t\log 2)^2/2 - \cdots\cdots\}$$
$$- (1/2)\{1 + t\log 2 + (t\log 2)^2/2 + \cdots\cdots\}]$$
$$- [\{1 - t\log 3 + (t\log 3)^2/2 - \cdots\cdots\}$$
$$- (1/3)\{1 + t\log 3 + (t\log 3)^2/2 + \cdots\cdots\}]$$
$$\sim \quad (4/3)\{(1 - t\log 2) - (1 + t\log 2)/2\} - \{(1 - t\log 3) - (1 + t\log 3)/3\}$$
$$= (4/3)\{1/2 - (3/2)t\log 2\} - \{2/3 - (4/3)t\log 3\} = 0.0785t$$
$$= 0.0785(1/2 - a) > 0 \qquad (t \to +0 \quad a \to 1/2 - 0) \tag{81}$$

3.5. Verification of $B < A$ (n_{max} is even number.)

n_{max} is even number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 4) - f(n_{max} - 3)\} + \{f(n_{max} - 2) - f(n_{max} - 1)\}$$
$$+\{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\}$$
$$+ \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$
$$A = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$
$$f(n_{max}) = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\}$$
$$+\{f(n_{max} + 3) - f(n_{max} + 4)\} + \cdots\cdots$$
$$= \{0\} + \{1\} + \{2\} + \{3\} + \{4\}$$
$$+ \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots$$

After the same process as in item 3.4.1 we can have the following (82).

$$f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{82}$$

After the same process as in item 3.4.2 and item 3.4.3 we can have the following inequalty.

$$f(n_{max}) - [\{q_{max}\} \text{ or } \{q_{max} - 1\}] < 2A$$

The following inequality holds from (59).

$$[\{q_{max}\} \text{ or } \{q_{max} - 1\}] < f(3) - f(2)$$

We have the following (83) from the above 2 inequalities.

$$2A > f(n_{max}) - [\{q_{max}\} \text{ or } \{q_{max} - 1\}] > f(n_{max}) - \{f(3) - f(2)\}$$

$$>f(n_{max}-1)-\{f(3)-f(2)\} \tag{83}$$

We have the following (84) for $B < A$ from (82) and (83).

$$2A > f(n_{max}-1)-\{f(3)-f(2)\} > f(n_{max}-1)-f(2)+\{f(3)-f(2)\} > 2B \tag{84}$$

From the above (84) we can have the final condition for $B < A$ as follows.

$$f(3) < (3/2)f(2) \tag{85}$$

In the following (86), $(4/3)f(2) < (3/2)f(2)$ is true due to $0 < f(2)$ in $0 < a < 1/2$ and we already confirmed in item 3.4.4 that the following (78) was true in $0 < a < 1/2$.

$$0 < f(3) < (4/3)f(2) < (3/2)f(2) \tag{86}$$

$$f(3) < (4/3)f(2) \tag{78}$$

Therefore the above (85) is true in $0 < a < 1/2$. Now we can confirm $0 < F(a)$ in $0 < a < 1/2$.

3.6. Conclusion

$0 < F(a)$ holds in $0 < a < 1/2$ as shown in the above item 3.4 and item 3.5.

Appendix 4. Graph of $F(a)$

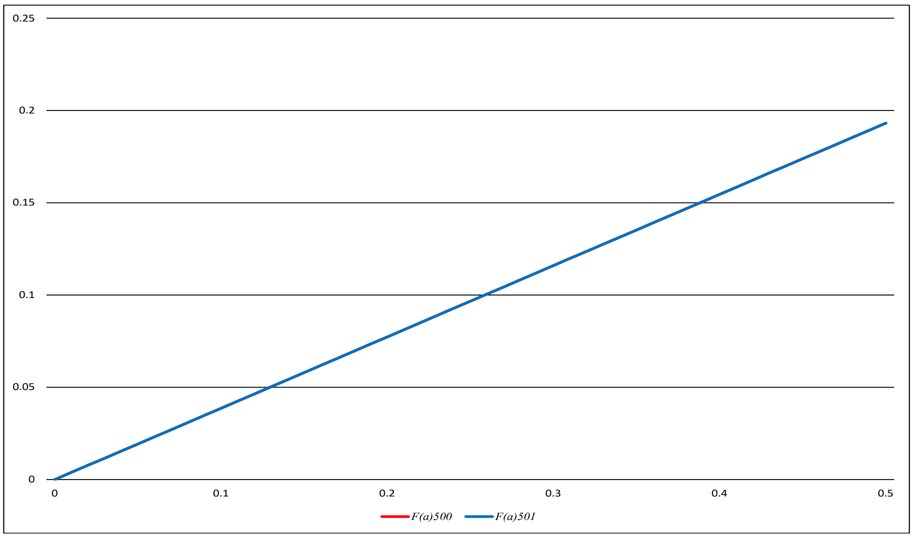
4.1 We can approximate $F(a)$ like the following (91) from (38). We have the following (92) and (93) from (91).

$$F(a)_n = \frac{F(a,n) + F(a,n+1)}{2} \tag{91}$$

$$\lim_{n\to\infty} F(a)_n = F(a) \tag{92}$$

$$F(a)_n = F(a)_{n-1} + (-1)^n \frac{f(n) - f(n+1)}{2} \tag{93}$$

The following (Graph 8) is plotted by calculating $F(a)_{500}$ and $F(a)_{501}$ for a every 0.01. $F(a)_{500}$ and $F(a)_{501}$ almost overlap because the values of $F(a)_{500}$ and $F(a)_{501}$ are equal up to 3 digits after the decimal point as shown in the following (Table 5).



Graph 8：$F(a)_{500}$ and $F(a)_{501}$

a	0	0.01	0.1	0.2	0.3	0.4	0.5
F(a)500	0	0.0038667	0.038666	0.077326	0.115971	0.154587	0.193146
F(a)501	0	0.0038648	0.038647	0.077289	0.115919	0.154537	0.193148
F(a)	0	0.00386	0.0386	0.077	0.1159	0.1545	–

Table 5：The values of $F(a)_{500}$ and $F(a)_{501}$

The range of a is $0 \le a < 1/2$. $a = 1/2$ is not included in the range. But we added $F(1/2)_n$ to calculation due to the following reason.
$f(n)$ at $a = 1/2$ is $(1 - 1/n)$ and $F(1/2)$ fluctuates due to $\lim_{n\to\infty} f(n) = 1$. The above (93) shows that $F(a)_n$ is partial sum of alternating series which has the term of $\frac{f(n)-f(n+1)}{2}$. Then $\lim_{n\to\infty} F(1/2)_n$ can converge to the fixed value on the condition of $\lim_{n\to\infty}\{f(n)-f(n+1)\} = 0$. The condition holds due to $f(n)-f(n+1) = -1/(n^2+n)$.

4.2 r_0 in (37) has the value of 217 at $a = 0.49$. Then $h(n) = f(n) - f(n+1)$ has a positive value and decreases monotonically to zero with $n \to \infty$ in $217 < n$ and $0 < a \leq 0.49$. $F(a)_n$ converges to $F(a)$ with $n \to \infty$ as (92) shows. Then we can have the following (94) from (93).

$$F(a)_{219} < F(a)_{221} < F(a)_{223} < \cdots < F(a)_{501} < \cdots\cdots\cdots$$
$$< F(a) < \cdots\cdots\cdots < F(a)_{500} < \cdots < F(a)_{222} < F(a)_{220} < F(a)_{218}$$
$$(0 < a \leq 0.49) \qquad (94)$$

Therefore (Graph 8) shows $F(a)$ as well as $F(a)_{500}$ and $F(a)_{501}$ in $0 \leq a \leq 0.49$. Because $F(a)_{500}$ and $F(a)_{501}$ almost overlap and $F(a)$ exists between $F(a)_{500}$ and $F(a)_{501}$.

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

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