

QUANTUM MECHANICS (Chapter 25)

D.I. Blokhintsev

Abstract

The last English edition of widely acclaimed and highly popular “Quantum Mechanics” of D.I. Blokhintsev was arranged based on 3rd and 4th Russian editions and published by Reidel in 1964. Since then, D.I. Blokhintsev prepared the revised and updated 5th edition (1976), with main changes affecting the concluding Ch. 25 “The Conclusions”. As the revised edition was not available so far to an English-speaking audience and to bridge this gap, we undertook an English translation of that concluding chapter. 25. In the process, §137, The formalism of quantum mechanics, which underwent no changes in a new edition, was reproduced from the 4th 1964 Reidel edition, and §141, dedicated mostly to elementary particles, was omitted as an outdated one.

There is a little doubt that the original text, created almost half a century ago, would look differently if written in our days and age. However, we introduced no edits, except for just a few editorial notes, to preserve Author’s unique style and render a spirit of the legendary times.

Needless to say, D.I. Blokhintsev was among founding fathers (along with L.E. Ballentine and K.V. Nikolsky) and a passionate proponent and advocate of the Statistical Interpretation of Quantum Mechanics. His concept of Quantum Ensembles, tracing back to earlier works of von Mises and von Neumann, proved instrumental for a modern understanding of Quantum Mechanics and demystifying the paradox of Quantum Non-locality / “spooky action” at the distance. Considering a nowadays progress in Quantum Computing and related need in a clarification of foundations of Quantum Mechanics, we hope this translation will be welcomed by a physics community working at all stripes of Quantum Mechanics.

L.D. Blokhintsev
M.I. Syrkin
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§137. The formalism of quantum mechanics

In this exposition of the fundamental ideas of quantum mechanics no attempt has been made to preserve a strict sequence of deduction. The orderly logic of a deductive account would inevitably involve some degree of abstractness which would obscure the experimental foundation of any given general result. However, to conclude the book it is appropriate to summarise briefly the fundamental ideas and problems of quantum mechanics.

Quantum mechanics deals with statistical ensembles of microparticles, and solves three main problems: (1) to determine the possible values (spectrum of values) of physical quantities; (2) to calculate the probability of any particular value of these quantities in the ensemble of microparticles; (3) to examine the variation of an ensemble with time (the motion of microparticles).

In quantum mechanics the *wave function* ψ represents the fact that a microparticle belongs to a particular ensemble. It is a function of a *complete set of quantities*, which we denote ¹ by x . The number of quantities in a complete set is determined by the nature of the system and is equal to *the number of its degrees of freedom*. The choice of the set of quantities which appear as arguments of the wave function is said to determine a particular *representation*.

The wave function also has a suffix (often omitted), such as n in $\psi_n(x)$, indicating another set which determines the wave function itself.

A statistical ensemble described by a particular wave function is called a *pure ensemble*; one which does not have a particular wave function is called a *mixed ensemble*, and is described by a *density matrix*.

The fundamental property of pure quantum ensembles is given by the principle of superposition: *if two possible states are described by wave functions ψ_1 and ψ_2 , there exists a third state described by the wave function*

$$\psi = c_1\psi_1 + c_2\psi_2, \tag{I}$$

where c_1 and c_2 are *arbitrary amplitudes*.

¹ Here x does not necessarily denote one or more co-ordinates. We use it to signify any group of variables, discrete or continuous, which form a complete set.

All relations between physical quantities are expressed in quantum mechanics in terms of linear self-adjoint operators so that to every real physical quantity L there corresponds a linear self-adjoint operator \hat{L} . The representation of quantities by means of operators is related to measurable quantities by a formula giving the mean value \bar{L} of a quantity in the state ψ . This formula is

$$\bar{L} = (\psi, \hat{L}\psi) \quad (\text{II})$$

with the normalisation condition ²

$$1 = (\psi, \psi).$$

This definition of the mean value enables us to find the spectrum of the quantity L , i.e. its possible values. For this purpose we seek states in which the quantity L has only a single definite value, i.e. states in which $\overline{\Delta L^2} = 0$. This requirement leads to an equation for the eigenfunctions of the operator \hat{L} (cf. Section 20):

$$\hat{L}\psi_L(x) = L\psi_L(x). \quad (\text{III})$$

Hence we find the spectrum (continuous or discrete) of L and the corresponding eigenstates $\psi_L(x)$. It is assumed that *the eigenvalues of the operator \hat{L} are those values of the quantity L which are experimentally observed.*

Since the eigenfunctions form an orthogonal set, any wave function $\psi(x)$ can be expanded as a series of eigenfunctions $\psi_L(x)$:

$$\psi(x) = \sum_L c(L)\psi_L(x), \quad (1)$$

where

$$c(L) = (\psi_L, \psi), \quad (2)$$

and the sum is to be regarded as an integral $\int dL$... if the spectrum of L is continuous.

² The symbol $(u, \hat{L}v)$ denotes the ‘scalar product’ of u and $\hat{L}v$, which for continuous variables is the integral

$$(u, \hat{L}v) = \int u^* \cdot \hat{L}v \cdot dx,$$

and for discrete variables is the sum

$$(u, \hat{L}v) = \sum_n \sum_m u_n^* L_{nm} v_m.$$

This spectral resolution is in fact performed in an apparatus which resolves the ensemble $\psi(x)$ into sub-ensembles $\psi_L(x)$, and in particular in a measuring apparatus which measures the quantity L .

The probability of finding a value L in an ensemble described by a wave function $\psi(x)$ is $|c(L)|^2$ (for a continuous spectrum, $|c(L)|^2$ is the probability density); $c(L)$ is also the wave function of the ensemble in the L representation. That is, $c(L)$ and $\psi(x)$ represent the same ensemble.

A fourth fundamental point in quantum mechanics relates to the variation of ensembles with time. The variation with time of the wave function describing an ensemble is given by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \quad (\text{IV})$$

where the operator \hat{H} is the Hamiltonian of the system and depends only on the nature of the system and the kinds of external field acting on it. The operator \hat{H} is the total-energy operator if the external fields are independent of time. Usually

$$\hat{H} = \hat{T} + \hat{U}, \quad (3)$$

where \hat{T} is the kinetic-energy operator and \hat{U} an operator representing the potential energy or force function. The operator \hat{T} is a function of the momentum operator $\hat{\mathbf{P}}$. Experiment shows that, in the absence of magnetic forces,

$$\hat{T} = \sum_k \frac{\hat{\mathbf{P}}_k^2}{2m_k}, \quad (4)$$

where $\hat{\mathbf{P}}_k$ is the momentum of the k -th particle and m_k its mass. When a magnetic field is present, $\hat{\mathbf{P}}_k$ must be replaced by

$$\hat{\mathbf{\Pi}}_k = \hat{\mathbf{P}}_k - \frac{e}{c} \mathbf{A}_k, \quad (5)$$

where \mathbf{A}_k is the vector potential at the position of the k -th particle.

From the Schrödinger equation (IV) and the definition of the mean value (II) it follows that

$$\frac{d\bar{L}}{dt} = \left(\psi, \frac{\partial \hat{L}}{\partial t} \psi \right) + (\psi, [\hat{H}, \hat{L}] \psi). \quad (6)$$

The operator $d\hat{L}/dt$ which represents the time derivative of the quantity L is therefore

$$\frac{d\hat{L}}{dt} = \frac{\partial \hat{L}}{\partial t} + [\hat{H}, \hat{L}], \quad (7)$$

where $[\hat{H}, \hat{L}] = (i/\hbar)(\hat{H}\hat{L} - \hat{L}\hat{H})$ is the quantum Poisson bracket. The integrals of the motion are such that

$$\frac{d\hat{L}}{dt} = 0. \quad (8)$$

In the absence of external forces the most important integrals of the motion are the energy, the total momentum of the system

$$\hat{\mathbf{P}} = \sum_k \hat{\mathbf{P}}_k = -i\hbar \sum_k \nabla_k \quad (9)$$

and the angular momentum

$$\hat{\mathbf{M}} = \sum_k [\mathbf{r}_k \cdot \hat{\mathbf{P}}_k] + \sum_k \hat{\mathbf{S}}_k, \quad (10)$$

where $\hat{\mathbf{S}}_k$ is the spin angular momentum of the k -th particle.

The form of the operator $\hat{\mathbf{P}}$ can be determined from the very fact that it represents a quantity which is an integral of the motion, i.e. commutes with the operator \hat{H} in the absence of external forces. Other more complex operators, whose physical significance may be highly specialised, can be constructed from the operators $\hat{\mathbf{P}}_k$ and \mathbf{r}_k . Thus the form of the principal operators is automatically determined if the form of the Hamiltonian (or the Schrödinger equation) is postulated.

The last of the fundamental ideas of quantum mechanics relates to systems of identical particles, and is the *principle of indistinguishability*, according to which the interchange of any pair of identical particles (k, j) does not lead to a physically different state. Mathematically this is expressed as a condition on the wave functions:

$$\hat{P}_{kj}\Psi = \lambda\Psi, \quad (V)$$

where $\lambda = \pm 1$ is an eigenvalue of the interchange operator \hat{P}_{kj} . This condition leads to a division of states into two classes:

$$\Psi = \Psi_s \text{ (symmetric),} \quad (11)$$

$$\Psi = \Psi_a \text{ (antisymmetric).} \quad (12)$$

It also follows from the Schrödinger equation that the symmetry cannot alter in the course of time. Hence the nature of particles alone determines whether they belong to the s type or the a type. Particles whose states are described by antisymmetric wave functions Ψ_a

are fermions, and obey the Pauli principle, which is a consequence of the properties of an ensemble described by antisymmetric wave functions. Particles whose states are described by symmetric wave functions Ψ_s are called bosons.

Thus we see that quantum mechanics is based on five fundamental ideas: the principle of superposition of states (**I**), the definition of the mean value (**II**), the interpretation of eigenvalues as the only possible values (**III**), the Schrödinger equation (**IV**), and the principle of indistinguishability of identical particles (**V**). The physical foundations of these ideas have been discussed in detail in the relevant chapters.

§138. The Feynman formulation of quantum mechanics

In the preceding paragraph there was presented a formal framework of quantum mechanics which became a commonly accepted one. What underlies this framework is the Schrödinger equation, and when transiting from a classical description to the quantum Hamiltonian approach is used.

However, there exists yet another formulation of quantum mechanics proposed by Feynman in 1942³. Feynmans approach is not based on the Schrödinger equation, and the Lagrange method is used instead of Hamiltonian one⁴. And even though the former is not as popular, it nevertheless has a number of advantages.

The main object in the Feynman formulation is a propagator $K(q, t; q_0, t_0)$, which enables to express a wave function $\psi(q, t)$ in terms of its initial value $\psi(q_0, t_0)$ at the time $t = t_0$.

Here q can stand for any dynamic variables describing our system at time t , and q_0 – the same variables at time t_0 . With that notations the propagator K is defined via

$$\psi(q, t) = \int K(q, t; q_0, t_0)\psi(q_0, t_0)dq_0. \quad (13)$$

Obviously, the propagator K should comply with the Schrödinger equation because $\psi(q, t)$ complies with that equation. It must turn into $\delta(q - q_0)$ at $t = t_0$ for the equation (13) to make sense yet at $t = t_0$. Further, at $t < t_0$ one typically sets $K = 0$ (the causality

³ A complete exposition of this method is found in the book by R. Feynman and A. Hibbs Quantum Mechanics and Path Integrals, Emended edition, Dover, 2005.

⁴ For the first time the prospects of using the Lagrange approach in quantum mechanics were pointed out to by Dirac in 1933. See P.A.M. Dirac, The Principles of Quantum Mechanics, Oxford, 1958, 32.

principle). These conditions result in the propagator K coinciding with the retarding Green function \mathcal{G} of the complete (i.e. including an interaction) Schrödinger's equation.

We, however, will not refer now to the Schrödinger equation, and choose another way to obtain the propagator K , more adequate to this new notion.

Consider first basic properties of the operator K . Let dynamic variables q at the moment $t = t_0$ assume one certain value $q = q_0$. Then $\psi(q'_0, t_0) = \delta(q'_0 - q_0)$. If at time t $q = q'$, then according to (13) we have

$$\psi(q', t) = K(q', t; q_0, t_0).$$

It follows from here that a quantity

$$P(q', t; q_0, t_0) = |\psi(q', t)|^2 = |K(q', t; q_0, t_0)|^2$$

is a system transition probability from the state $q = q_0$ into the state $q = q'$ over time period $t - t_0$ ($t > t_0$). The propagator K has an important property: the product of propagators is a propagator again. Indeed, by taking $\psi(q', t)$ as an initial one and inserting it into (13) we obtain

$$K(q, t; q_0, t_0) = \int K(q, t; q'', t'')K(q'', t''; q_0, t_0)dq''. \quad (14)$$

From (14) it is seen that the system transition from the state q_0 which it occupied at time t_0 to the state q at time t ($t > t_0$), can be viewed as happening in two steps. Initially the system passes into an arbitrary intermediate state q'' at time t'' ($t_0 < t'' < t$), and only after that there effectuates the transition to a final state q at time t .

Apparently, one can continue on slicing an interval (t, t_0) . Let's break it up into N intervals: $(t_0, t_1), (t_1, t_2), \dots, (t_k, t_{k+1}), \dots, (t_{N-1}, t_N), t_N = t$. Denote the value of dynamic variables at times t_k as q_k ($k = 0, 1, \dots, N$), so that the propagator K for the l -interval, will be

$$K_l = K(q_{l+1}, t_{l+1}; q_l, t_l).$$

Applying sequentially the propagator K_l to any initial function $\psi(q_0, t_0)$ we obtain the following expression of the propagator for the time interval (t_0, t) :

$$K(q, t; q_0, t_0) = \int \dots \int K(q, t; q_{N-1}, t_{N-1})K(q_{N-1}, t_{N-1}; q_{N-2}, t_{N-2}) \dots \dots K(q_2, t_2; q_1, t_1)K(q_1, t_1; q_0, t_0)dq_{N-1}dq_{N-2} \dots dq_1, \quad (15)$$

where an integration occurs over all intermediate states (an integral of multiplicity $N - 1$). The process of successive transitions through all admissible states is known as the Markov

chain. However, in the classical theory this chain forms via not transition amplitudes (as we obtained in (15)), but via transition probabilities $P(q_{k+1}, t_{k+1}; q_k, t_k)$:

$$P(q, t; q_0, t_0) = \int \dots \int P(q, t; q_{N-1}, t_{N-1}) P(q_{N-1}, t_{N-1}; q_{N-2}, t_{N-2}) \dots \dots P(q_2, t_2; q_1, t_1) P(q_1, t_1; q_0, t_0) dq_{N-1} dq_{N-2} \dots dq_1. \quad (16)$$

On Fig. 1 there are shown several “trajectories” emerging in Markov’s chain. We took the word trajectories into quotes because any final time interval $\Delta t = t_{k+1} - t_k$ can be broken down into shorter subintervals $\Delta t' \ll \Delta t$. In turn, these subintervals can be broken down even further, so that the Markov chain trajectories do not have continuous tangent lines.

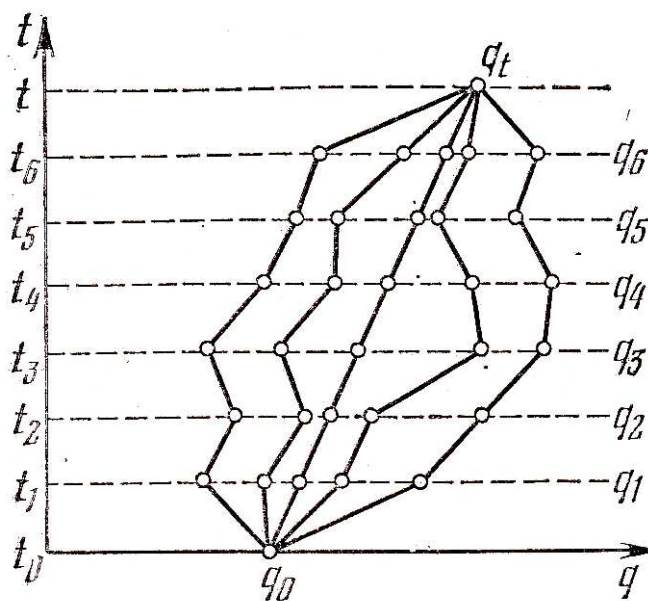


Figure 1: Particle trajectories which are subject for integration over in the Markov chain.

The time interval (t_0, t) is broken down into seven subintervals, q the particle coordinate.

Note in passing, that the difference between quantum (15) and classical (16) chains is a further manifestation that the central role in quantum mechanics belongs to probability amplitudes, and not to probabilities themselves. This fact does not allow, even in principle, to reduce quantum mechanics to any kind of statistical mechanics.

It goes without saying, that a classical Markov chain also makes sense even in quantum mechanics. However, that chain describes the motion of a quantum system which is interrupted at times $t = t_k$ ($k = 1, 2, \dots, N - 1$) by the measurement of its dynamic variables

q , that is, by the intervention of a measuring device. Along with that a coherence of the system motion is disrupted on intervals (t_{k-1}, t_k) and (t_k, t_{k+1}) .

To find the explicit expression for a propagator $K(q, t; q_0, t_0)$ let's resort, for a simplicity sake, to a particular case of a one-dimensional motion in the external potential $V(x)$. In that case $q = x$ and a classical Lagrange function is

$$L(x, \dot{x}) = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x).$$

Here m is a particle mass, $\dot{x} = \frac{dx}{dt}$ – its velocity. The action S over an interval (t_k, t_{k+1}) is equal to

$$S(x_{k+1}, t_{k+1}; x_k, t_k) = \int_{t_k}^{t_{k+1}} L(x, \dot{x}) dt.$$

Let's show now, that if the quantum propagator K for an infinitesimal time interval $\Delta t = t_{k+1} - t_k$ is taken as

$$K(x_{k+1}, t_{k+1}; x_k, t_k) = C \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2} \left(\frac{x_{k+1} - x_k}{\Delta t} \right)^2 - V(x_k) \right] \Delta t \right\}, \quad (17)$$

then the wave function $\psi(q, t)$ given by the formula (13) will satisfy the Schrödinger's equation

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(x, t) + V(x) \psi(x, t). \quad (18)$$

Note, that the quantity $\frac{x_{k+1} - x_k}{\Delta t}$ approximates a particle velocity for the interval $\Delta t = (t_k, t_{k+1})$, and C is a normalization factor stemming from the condition $K = \delta(x_{k+1} - x_k)$ if $\Delta t \rightarrow 0$. It is easy to find that

$$C = \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{1/2}. \quad (19)$$

Let's substitute now (17) into (13) and set there $q_0 = x - \xi$, $q - q_0 = x - x_0 = \xi$, $t = t_0 + \Delta t$. Further

$$\psi(x_0, t_0) = \psi(x - \xi, t_0) = \psi(x, t_0) - \frac{\partial \psi(x, t_0)}{\partial x} \xi + \frac{1}{2} \frac{\partial^2 \psi(x, t_0)}{\partial x^2} \xi^2 + \dots$$

and

$$\exp \left\{ -\frac{i}{\hbar} V(x) \Delta t \right\} = 1 + \frac{1}{i\hbar} V(x) \Delta t + \dots$$

The expression (13) then takes the form

$$\begin{aligned} \psi(x, t_0 + \Delta t) = & C \int_{-\infty}^{+\infty} d\xi \exp\left(\frac{i}{\hbar} \frac{m}{2\Delta t} \xi^2\right) \left[1 + \frac{1}{i\hbar} V(x)\Delta t + \dots\right] \\ & \times \left[\psi(x, t_0) - \frac{\partial\psi(x, t_0)}{\partial x} \xi + \frac{1}{2} \frac{\partial^2\psi(x, t_0)}{\partial x^2} \xi^2 + \dots\right]. \end{aligned} \quad (20)$$

Employing the fact that $\int_{-\infty}^{+\infty} e^{iaz^2} dz = \sqrt{\frac{i\pi}{a}}$ it is easy to obtain the right hand side of the formula (20). The integral comprising the factor $\psi(x, t_0)$ equals 1 in virtue of the normalization condition (19). The integration of the term that is linear over ξ gives zero. The integral containing ξ^2 equals $-\frac{1}{i\hbar} \frac{\hbar^2}{2m} \Delta t$. Terms of a higher power over ξ vanish faster than $(\Delta t)^{3/2}$. Collecting now together the results of integration and noting that $\frac{1}{\Delta t} [\psi(x, t_0 + \Delta t) - \psi(x, t_0)] \rightarrow \frac{\partial\psi(x, t)}{\partial t}$ (we replaced t_0 with t because they coincide in the limit of $\Delta t \rightarrow 0$), we obtain the Schrödinger's equation (18) for the wave function $\psi(x, t)$ defined by means of (13) and (17). It is proved thereby that the propagator method (the Lagrange method) is equivalent to the method of the Schrödinger equation – the counterpart of the Hamilton-Jacobi method of the classical mechanics.

Having said all that, one can write the propagator as well for the total time interval (t_0, t) . Multiplying propagators (17) for intermediate intervals (t_k, t_{k+1}) and integrating over intermediate variables x_k , we find

$$\begin{aligned} K(x, t; x_0, t_0) = & \lim_{N \rightarrow \infty, \Delta t \rightarrow 0} \int \dots \int \exp \left\{ \frac{i}{\hbar} \sum_{k=1}^{N-1} \left[\frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\Delta t} - V(x_k) \Delta t \right] \right\} \\ & \times C^{\frac{N}{2}} dx_1 dx_2 \dots dx_{N-1}. \end{aligned} \quad (21)$$

This limit of the multiple integral is called a functional integral. Noting, that for an infinitesimal slicing of the interval (t_0, t) the quantity $\frac{x_{k+1} - x_k}{\Delta t}$ is tractable as a velocity $\frac{dx}{dt} = \dot{x}$ and denoting the element of the integration volume $C^{N/2} dx_1 \dots dx_{N-1}$ as $d\{x\}$, we can write the result (21) in the compact form

$$K(x, t; x_0, t_0) = \int d\{x\} \exp \left[\frac{i}{\hbar} \int_{t_0}^t L(x, \dot{x}) dt \right]. \quad (22)$$

The integral appearing here in the exponent is the classical action

$$S = \int_{t_0}^t L(x, \dot{x}) dt. \quad (23)$$

The integration in the formula (22) occurs not only over trajectories delivering an extremum to the integral (23), but extends over all trajectories connecting points (t_0, x_0) and (t, x) .

The propagator representation as a functional integral over trajectories (22) enables a clear understanding why in a classical limit it is permissible to consider only classical trajectories. Indeed, if one describes the given system via a classical mechanics, then the action S is much bigger than a Planck constant \hbar . Consider now a trajectory which is not a solution to classical motion equations. Every small change of such a trajectory causes a big change of a ratio S/\hbar in (22) and fast oscillations of the amplitude. As a result, contributions from all that trajectories cancel each other. Therefore, in the classical limit these trajectories can be omitted.

However, in the vicinity of a trajectory coming out of classical motion equations, the situation is different. Since the action here reaches an extremum $\delta S = 0$, then small deviations from that trajectory do not lead to changes in S . Therefore, contributions from such trajectories into the propagator do not cancel each other as they are all close in phase, which is equal here to S_{cl}/\hbar . This way in the classical approximation the propagator (22) will differ from zero only there, where the action is extreme. But this is exactly a classical result, namely, any object moves along the path of an extreme action $\delta S = 0$.

In conclusion of this section, let's give an explicit calculation of the propagator $K(x, t; x_0, t_0)$ for a freely moving particle and for an oscillator. For the first case, the Lagrange function equals

$$L(x, \dot{x}) = \frac{m}{2} \dot{x}^2.$$

The corresponding functional integral comes out of (21) if we set there $V(x_k) = 0$. Let's employ an elementary property of an integral

$$\begin{aligned} C^2(\Delta t) \int_{-\infty}^{+\infty} \exp \left\{ \frac{i}{\hbar} \frac{m}{2} \left[\frac{(x - x_1)^2}{\Delta t} + \frac{(x - x_0)^2}{\Delta t} \right] \right\} dx_1 = \\ = C(2\Delta t) \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{(x - x_0)^2}{2\Delta t} \right], \end{aligned}$$

where C is defined by (19). Applying this formula successively $(N - 1)$ times, we obtain

$$K(x, t; x_0, t_0) = \left(\frac{m}{2\pi i \hbar (t - t_0)} \right)^{1/2} \exp \left[\frac{i}{\hbar} \frac{m}{2} \frac{(x - x_0)^2}{t - t_0} \right]. \quad (24)$$

This result easily generalizes to a three-dimensional case

$$K(\mathbf{x}, t; \mathbf{x}_0, t_0) = \left(\frac{m}{2\pi i \hbar (t - t_0)} \right)^{3/2} \exp \left[\frac{i}{\hbar} \frac{m (\mathbf{x} - \mathbf{x}_0)^2}{2(t - t_0)} \right]. \quad (25)$$

As expected, the formula (25) coincides (up to the factor $-i/\hbar$)⁵ with the retarding Green function of a free Schrödinger equation.

In the case of a harmonic oscillator the Lagrange function is of the form

$$L(x, \dot{x}) = \frac{m}{2}(\dot{x}^2 - \omega_0 x^2),$$

where ω_0 is the oscillator eigen frequency. The direct calculation of a propagator K for that Lagrange function via the multiple approximation (formula (21)) is quite challenging. Instead, it is convenient to use the following approach. Let's change variables in the formula (22), setting

$$x(t) = x_{\text{cl}}(t) + y(t),$$

where $x_{\text{cl}}(t)$ is the classical trajectory, connecting initial x_a and final x_b points. Obviously, $y(t_a) = y(t_b) = 0$. If the Lagrangian is quadratic over coordinates and velocities, then the action S can be rendered in the following way

$$S[x(t)] = S_{\text{cl}}(x_a, x_b) + S'[y(t)],$$

where $S_{\text{cl}}(x_a, x_b) = S[x_{\text{cl}}(t)]$, and S' is an additional action depending only on $y(t)$ ⁶. Now let's represent $K(x_b, t_b; x_a, t_a)$ in the form

$$K(x_b, t_b; x_a, t_a) = \exp \left[\frac{i}{\hbar} S_{\text{cl}}(x_a, x_b) \right] \int d\{y(t)\} \exp \left[\frac{i}{\hbar} S'[y(t)] \right]. \quad (26)$$

Therefore, one manages to pick out explicitly the dependence of a propagator on coordinates of initial and final points (x_a and x_b). If now the Lagrangian of the system does not depend on time, then the remaining functional integral in the formula (26) depends functionally only on the time difference $t_b - t_a$. For a number of cases this functional dependence can be found without an explicit resorting to calculation of the integral over trajectories. For the harmonic oscillator $S_{\text{cl}}(x_a, x_b)$ is

$$S_{\text{cl}}(x_a, x_b) = \frac{m\omega_0}{2 \sin \omega_0 T} [(x_a^2 + x_b^2) \cos \omega_0 T - 2x_a x_b],$$

⁵ The factor $(-i/\hbar)$ stems from the different normalization of the propagator $K(x, t; x_0, t_0)$ and the Green function $g(\mathbf{x} - \mathbf{x}_0, t - t_0)$.

⁶ Terms containing products $x_{\text{cl}}(t)y(t)$ give zero contribution after the integration over time.

where $T = t_b - t_a$.

The expression for the propagator in that case can be written in the following way:

$$K(x_b, t_b; x_a, t_a) = F(t) \exp \left\{ \frac{im\omega_0}{2\hbar \sin \omega_0 T} [(x_a^2 + x_b^2) \cos \omega_0 T - 2x_a x_b] \right\}. \quad (27)$$

The function $F(T)$ can be found from the condition that the harmonic oscillator propagator (27) becomes a propagator for the freely moving particle when $\omega_0 \rightarrow 0$. The calculation shows that

$$F(t) = \left(\frac{m\omega_0}{2\pi i \hbar \sin \omega_0 T} \right)^{1/2}.$$

The propagator knowledge gives practically all information needed for the quantum description of a system. First and foremost, the propagator can be used to obtain transition probabilities between various systems states, as well as wave functions and an energy spectrum. All that issues will be not considered here because of the room limitation. A more detailed exposition can be found in the aforementioned book of R. Feynman and A. Hibbs.

In closing this brief introduction to Feynman's approach to quantum mechanics the following should be pointed out. Even though this method did not result in fundamentally new discoveries in quantum theory, its undisputable advantages are the physical clarity and more close connection to the classical description of physical phenomena.

§139. Some methodological problems. Wave functions and quantum ensembles

New physical ideas brought up by quantum mechanics lead in 1930s to rather serious and at times quite heated arguments between proponents of different philosophical concepts.

The discussions continued to a certain extent also after World War II. These discussions were not useless as they helped clarify more crisply many important aspects related to foundations of quantum mechanics and its consequences for the methodology of science.

Key debates focused around understanding of a wave function ψ . Does a wave function provide an objective and complete description of physical reality or is it only a "notebook" of an observer who records in it an available information? Does a wave function describe a state of an individual particle or an ensemble of particles?

Another variety of issues was related to the causality in quantum mechanics. The point here is that quantum mechanics is a statistical theory. In this regard there was a lot of views

on the nature of that “statisticality” and it was assumed by many that this statisticality needs a justification based on some fully deterministic mechanics.

The existence of different views stemmed partially from the lack of trust in quantum mechanics, and partially from an insufficiently sound analysis of some quantum mechanics consequences, appearing paradoxical.

Nowadays there are no whatsoever reasons to not to believe in quantum mechanics. The power of its methods was completely proved both in atomic and nuclear physics. By abandoning the description of particles as moving over trajectories, which during the centuries appeared as a scientific apex, we lost only some illusory hopes. Instead what opened to us was striking in its beauty a harmony of laws ruling the atomic world.

An exposition of old discussions here would be only of a historical interest⁷. Therefore, in what follows we would restrict ourselves only to the elucidation of problems – posed above – based on concepts of quantum ensembles which underlie the presentation of quantum mechanics in this course.

It should be noted that from the methodological viewpoint this concept differs from the more popular Copenhagen school concept in that it spares more modest role to the observer and emphasize everywhere an objective nature of quantum ensembles and laws they obey⁸.

The concept of quantum ensembles is quite close to the one of the Gibbs classical ensemble, well known in the statistical thermodynamics. In the Gibbs ensemble a microsystem is considered as interacting with the macroscopic thermostat \mathcal{M} , with the temperature θ . The probability $W_\theta(\mathcal{P}, \mathcal{Q})$ of one or another result of measuring dynamic variables $(\mathcal{P}, \mathcal{Q})$ refers to the ensemble, formed by endless repetitions of setups comprising a microsystem μ and thermostat \mathcal{M} ; in other words – via unlimited copying of the system μ in one and the same macroscopic environment, provided in this case by the thermostat with temperature θ . As a result of this, the probability $W_\theta(\mathcal{P}, \mathcal{Q})$ includes characteristics of the microsystem $(\mathcal{P}, \mathcal{Q})$, as well as characteristics of a macroscopic environment – the temperature θ .

In the full analogy with the Gibbs classical ensemble the quantum ensemble is formed via an unlimited reproduction of the situations formed by one and the same microsystem (but not the same sample!) submerged into one and the same macroscopic environment \mathcal{M} .

⁷ See, for instance, the previous, 4th edition of this book, D.I. Blokhintsev, Reidel, 1964.

⁸ See D.I. Blokhintsev “The philosophy of quantum mechanics”, , Reidel, 1968.

This way, in quantum mechanics the microsystem μ is considered in the connection with that macroscopic environment \mathcal{M} which encompasses it and prescribes it a “state” in a quantum mechanical sense.

However, that state is not described, as opposed to a classical ensemble, via some probability, but rather by a probability amplitude $\Psi_{\mathcal{M}}(\mathcal{Q})$, i.e. wave function, or, more generally, by density matrix $\rho_{\mathcal{M}}(\mathcal{Q}, \mathcal{Q}')$ (see §46 of main text). Here an index \mathcal{M} points out to a macroscopic environment determining a quantum ensemble. In most simple cases the index \mathcal{M} can be reduced to quantum numbers. For instance, for a sufficiently cold gas the thermostate temperature θ can be replaced by n_0 – the quantum number of the lowest atomic level E_0 , if the average thermal energy of atoms $3/2 k\theta$ is much smaller than the atomic excitation energy $\epsilon = E_1 - E_0$; index \mathcal{M} can be replaced by a linear momentum p – the momentum of a particle μ , if the macroscopic environment is such that it provides for the monochromatic de Broglie wave.

All predictions of quantum mechanics refer to the ensemble made up of copies of the macroscopic environment \mathcal{M} and the microscopic system μ submerged into it.

The question of whether the wave function belongs to only one single particle is as irrelevant as the question of whether the probability of winning is the attribute of the given lottery ticket.

A wave function (or density matrix) includes coordinates of the microsystem μ , e.g. coordinates (\mathcal{Q}) , as well as parameters of the environment \mathcal{M} that determines the state of this microsystem.

Therefore, while the wave function $\Psi_{\mathcal{M}}(\mathcal{Q})$, or density matrix $\rho_{\mathcal{M}}(\mathcal{Q}, \mathcal{Q}')$, indicates what is the specific quantum ensemble that the microsystem μ belongs to, the probability of a one or another measurement result for dynamic variables \mathcal{Q} is given by

$$dW_{\mathcal{M}}(\mathcal{Q}) = |\Psi_{\mathcal{M}}(\mathcal{Q})|^2 d\mathcal{Q} \quad \text{or} \quad dW_{\mathcal{M}}(\mathcal{Q}) = \rho_{\mathcal{M}}(\mathcal{Q}, \mathcal{Q}) d\mathcal{Q}.$$

The environment \mathcal{M} can arise either in the artificial set up, when one strives to prepare particles in a certain way, or by itself under the natural conditions.

In that sense a wave function $\Psi_{\mathcal{M}}(\mathcal{Q})$, or density matrix $\rho_{\mathcal{M}}(\mathcal{Q}, \mathcal{Q}')$, is an objective characteristics of a quantum ensemble and can be found, at least in principle, via measurements. However, neither $\Psi_{\mathcal{M}}$ nor $\rho_{\mathcal{M}}$ can be restored from the one single sample of a microsystem.

Newcomers to quantum mechanics usually ask questions about the physical essence of

the phenomenon of a wave function contraction (*Editorial Note* – “collapse” in a modern parlance), when an arbitrary wave function $\Psi(\mathcal{Q})$ converts into an operator L eigen wave function ψ_n after the measurement of a dynamical variable $L = L_n$,

$$\Psi_{\mathcal{M}}(\mathcal{Q}) = \sum_n c_n \psi_n(\mathcal{Q}), \quad (28)$$

if the measurement gave $L = L_n$. This way after the series of measurements an initially pure ensemble transforms into a mixed ensemble (compare with §83 in the main text).

Those ready to accept a pure informational view of that process, would then answer: as the result of a measurement an information available to the observer has changed, and in his notebook he crosses out $\Psi_{\mathcal{M}}(\mathcal{Q})$ function as an outdated and enters a new one ψ_n . Such an interpretation, being pragmatically quite satisfactory, faces difficulties when a quantum transition happens without an explicit presence of the observer. For instance, a radioactive atom can decay spontaneously and an initial wave function $\psi_0(r)$, concentrated in the nucleus, transforms into an outgoing wave e^{ikr}/r : the state $\psi_0(r)$ contracts into the state e^{ikr}/r , which is an eigen state for the momentum operator \hat{P}_r with an eigen value $p_r = \hbar k$. An answer to the question about the foregoing phenomenon can be given only on the basis of describing a microsystem and measuring instrument (an analyzer + detector) jointly. The key point here is that the coherence of initial states ψ_n prior to the measurement, is destroyed in the measurement process. The role of an analyzer performing a spectral decomposition here is insufficient in this regard, as beams separated by an analyzer, still remain coherent. That is, if we recombine these beams together via, say, a system of mirrors, they would reveal an interference picture.

A coherence of beams is destroyed by an activation of a macroscopic detector. All that is clarified on Fig. 2.

A macroscopic environment \mathcal{M} determines the state $\Psi_{\mathcal{M}}$ of a μ -microsystem. An analyzer A decomposes the initial ensemble wave function $\Psi_{\mathcal{M}}$ into the spectrum $c_1\psi_1, c_2\psi_2, \dots, c_n\psi_n, \dots$ over the attribute L characteristic of the given analyzer. Then the microsystem acts upon one of the channels $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots$ of the detector \mathcal{D} with the result that the particle reveals itself in one of the channels, say, channel n . After that we can say that the quantum transition from the state $\Psi_{\mathcal{M}}(x)$ into the state $\psi_n(x)$ took place. If now, after the detector activation, we collect particles with $L = L_1, L = L_2, \dots, L = L_n, \dots$ into groups, then the corresponding wave functions $\psi_1, \psi_2, \dots, \psi_n, \dots$, will be incoherent. Therefore, the most

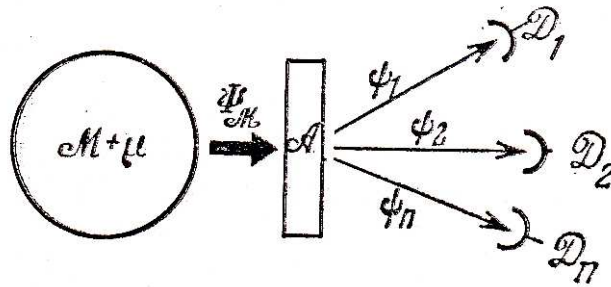


Figure 2: A scheme of quantum mechanical measurements: the circle $\mathcal{M} + \mu$ depicts a macroscopic environment arranging for a certain state $\Psi_{\mathcal{M}}$ of a microparticle μ .

A – an analyzer decomposing $\Psi_{\mathcal{M}}$ into a spectrum over values of the dynamic variable L being measured: $\psi_1, \psi_2, \dots, \psi_n, \dots$; $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots$ – different detector \mathcal{D} channels whose triggering is what establishes the measurement result.

important element of the wave function “contraction” $\Psi_{\mathcal{M}} \rightarrow \psi_n$ is a change in the state of the macroscopic system, i.e. the detector. This process can be considered via quantum mechanical methods if the detector is included into the quantum mechanical treatment. The incorporation of a macroscopic detector into the quantum mechanical representation requires the whole situation to be handled by a density matrix $\rho_{\mathcal{M}}$ method.

Consider now two idealized, but simple though, quantum mechanical measurements.

A. Let there be a diaphragm \mathcal{D} (Fig. 3) with two openings O_1 and O_2 of the diameter d .

An initial particle wave $\psi_0(x)$ hits the diaphragm. When passing through openings that wave creates two different beams $\psi_1(x)$ and $\psi_2(x)$ (it is assumed that the wavelength λ of the initial beam ψ_0 is of the order of the diameter d). In virtue of the coherence of waves $\psi_1(x)$ and $\psi_2(x)$, there emerges an interference pattern on the screen with the intensity distribution in it given by

$$I(x) = |\psi_1(x) + \psi_2(x)|^2 = |\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\text{Re} \psi_1^*(x)\psi_2(x).$$

The last term in this formula is caused by an interference of beams $\psi_1(x)$ and $\psi_2(x)$.

Let’s assume we would like to find out which one is the opening that the particle went through. The diaphragm serves as an analyzer of the particle position ($x \approx O_1$ or $x \approx O_2$). Besides, what is also needed is a detector. As detectors D_1 and D_2 we take two light beams L_1 and L_2 . The wavelength λ_0 of beams should be short enough so that these probing beams do not spread because of the diffraction. That means beams must obey the geometrical optics.

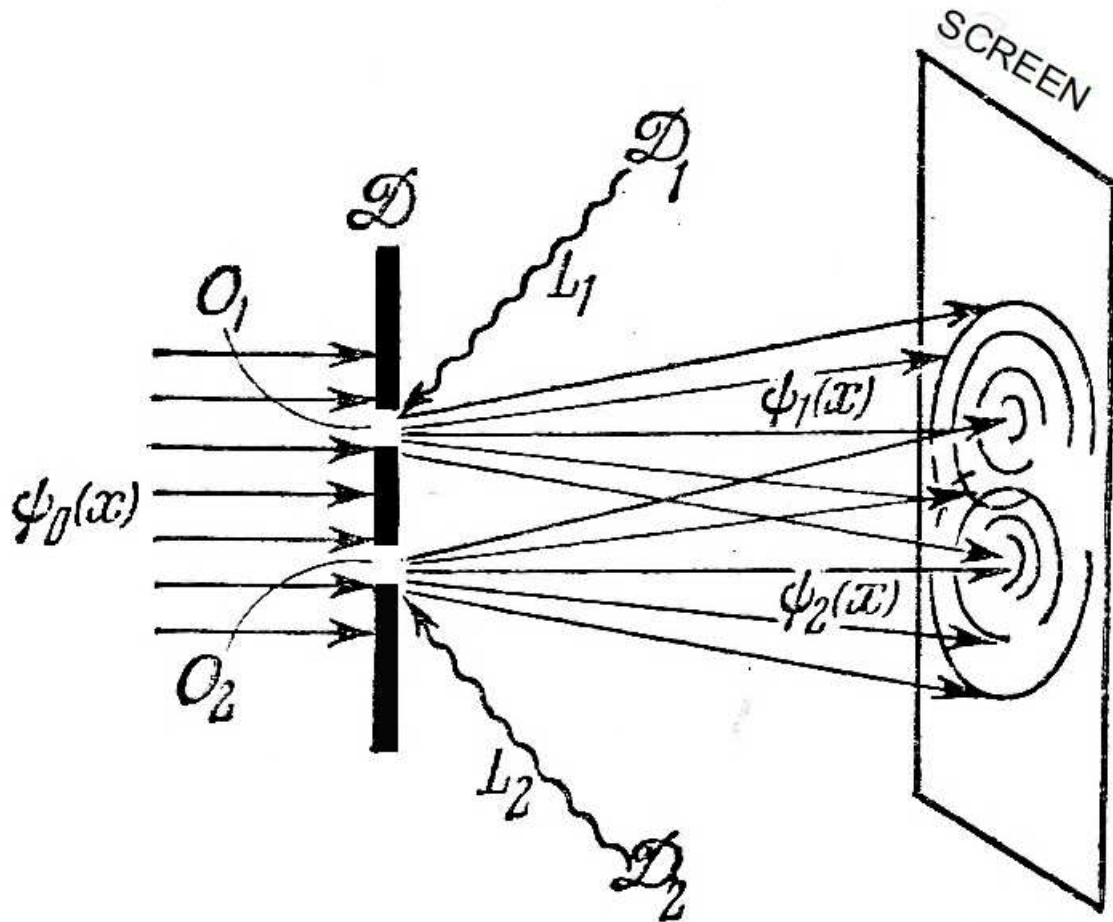


Figure 3: A wave $\psi_0(x)$ passes through two openings O_1 and O_2 in a diaphragm \mathcal{D} .

On the right side there emerges a field $\psi(x) = \psi_1(x) + \psi_2(x)$ which produces an interference on the screen.

\mathcal{D}_1 and \mathcal{D}_2 are probing beams identifying the location where a particle passes through a diaphragm.

Therefore, they are classical macroscopic beams. What's more, if the scattering happens for the beam L_1 , then it means that the particle passed through the opening O_1 and had a coordinate x nearby O_1 . If the scattering occurs for the beam L_2 , then it means that the particle passed through the opening O_2 and its coordinate x was close to O_2 .

After the beam scattering, the particle state will neither be wave $\psi_1(x)$ nor $\psi_2(x)$, but will become $\delta(x - x_1)$ or $\delta(x - x_2)$, ($x_1 \approx O_1$, $x_2 \approx O_2$), and one of the beams $\psi_1(x)$ or $\psi_2(x)$ will be destructed. Obviously, the coherency of beams will also be destroyed.

Measuring the particle coordinate by means of an intervention of the probing beam,

changes the environment for particles in the incident beam $\psi_0(x)$. There emerges a new quantum ensemble stemming from the new macroscopic environment. For this new environment an interference pattern on the screen no longer exists. It should be noted, by the way, that this example is a good illustration for the complementarity principle.

B. Consider another simplified measurement example⁹. Let microparticle μ to belong in an ensemble, in which its state is a standing wave

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}(e^{ikx} + e^{-ikx}) = \varphi^+(x) + \varphi^-(x).$$

Here x is a particle coordinate, k its linear momentum. As seen, $\varphi(x)$ is a coherent sum

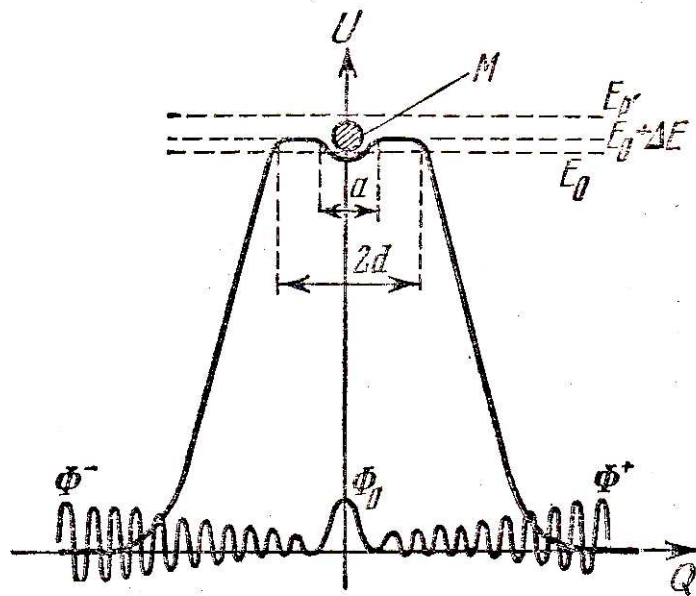


Figure 4: On the figure there is a scheme of a simplest measuring device.

On the ordinate axis there is a potential energy U of a ball, located on the cone top. On the abscissa axis is given its coordinate Q . On the same graph is depicted the ball wave function before (Φ_0) and after ($\Phi = \Phi^+ + \Phi^-$) the micro-particle scattering on it.

of two states $\varphi^\pm(x) = \frac{1}{\sqrt{2\pi}}e^{\pm ikx}$, of which one belongs to a momentum k , and the other to momentum $-k$. The measurement which is being planned should determine the sign of k , i.e. whether the particle is found in the state $\varphi^+(x)$ or $\varphi^-(x)$. As a detector which acts in

⁹ For more details and further examples see D.I. Blokhintsev The philosophy of quantum mechanics, Reidel, 1968.

this case as an analyzer as well serves here a macroscopic little ball \mathcal{M} placed on the top of the cone. To make it possible, imagine a little crater on the cone top, so that the ball in it is in a state very close to unstable. Such a cone can be described with a potential energy $U(Q)$ (Q is the ball centre of mass coordinate), depicted on the Fig. 4. The energy ΔE needed to push the ball off cone top is assumed very small, so that $\Delta E \ll \frac{(2p)^2}{2M}$. The latter value is the recoil energy received by the ball \mathcal{M} after the microparticle μ scattering on it. In view that \mathcal{M} is assumed large and μ – small, the momentum transfer in the scattering is $\pm 2p$. In virtue of the ball instability on the cone top, after the microparticle scattering, the ball will roll off the cone gaining in the process the kinetic energy $\frac{p'^2}{2M} = U_0$. This energy can be arbitrary large (if U_0 is large). Therefore, the physical phenomenon begins here at the microscopic quantum level (microparticle scattering), and converts into a macroscopic phenomenon – the motion of the heavy ball with a large speed. On Fig. 4 is given also, besides the potential energy $U(Q)$, the wave function of the ball initial state $\Phi_0(Q)$. With time this function transforms as a result of the microparticle scattering on the ball – into the function

$$\Phi(Q, t) = \Phi_0(Q) + \Phi^+(Q, t) + \Phi^-(Q, t),$$

where the second and third terms arise from the scattering on the ball of microparticle waves $\varphi^+(x)$ and $\varphi^-(x)$, respectively. For the ball density matrix $\rho_{\mathcal{M}}(Q, Q', t)$

$$\rho_{\mathcal{M}}(Q, Q', t) = \Phi^*(Q, t)\Phi(Q', t).$$

The first perturbation order estimate shows, that the diagonal term of that matrix $\rho_{\mathcal{M}}(Q, Q, t)$ for large t and $|Q| \gg a$ (a being a linear size of a crater on the cone top) reduces to two terms

$$\rho_{\mathcal{M}}(Q, Q, t) = |\Phi^+(Q, t)|^2 + |\Phi^-(Q, t)|^2.$$

with the first and second term being non-zero at $Q > a$ and $Q < -a$, respectively.

That means a) the absence of typical quantum interference terms in $\rho_{\mathcal{M}}(Q, Q, t)$, and b) for large t the ball will roll either to the right or left from the cone. Being though extremely simplified, this example illustrates quite a common feature of all quantum mechanical measurements: they all begin at a microscopical level and end up with a macroscopical phenomenon in an unstable system (the detector!). Therefore, all these measurements bear

signatures of an explosion inflicted by a micro-phenomenon¹⁰.

This paramountly important aspect of quantum measurements, being quite trivial by its very essence, remained though unemphasized for a long time. In particular, N. Bohr believed that the inclusion of the measuring device Π into the quantum mechanical description shifts the focus somewhere else, because – as Bohr thought – to study the system $(\mu + \Pi)$ there is a need for a new classical device Π' , and so on. However, what is missed in this consideration, is that because of a detector macroscopic instability the system $(\mu + \Pi)$ will – in virtue of quantum mechanical laws – pass on its own to a macroscopic level and a new device Π' will no longer “see” a micro-, but a macro-phenomenon. What’s more, the described situation may occur not only in a laboratory, but also on its own in the nature whenever the macroscopic phenomena are induced by microscopic events.

(*Editorial note:* The simple example B enables to determine the sign of a microparticle momentum in a superposition prior to the scattering on the ball. On a more general note that same example illustrates the concept of the “indirect quantum measurements” in superpositional states and ensuing interpretations.)

§140. Causality issues

The classical mechanics is a simplest example of a theory with an unlimited reign of determinism. We are indoctrinated that with laws of classical mechanics one can unconditionally forecast the future of a mechanical system if the initial conditions – coordinates and velocities (or momenta) – are known for parts making up a system.

In XVIII century Laplace, infatuated in a logical consistency and power of classical mechanics means, declared proudly: “Give me the initial data of all particles in the world, and I’ll predict the future”. However, now we are very far away from this hope of the mechanical era.

In fact, in the very concept of the classical mechanics there is already something that undecuts the power of strictly deterministic statements.

¹⁰ The above fragment (following the Fig. 4) is just a semi-heuristic exposition of rigorous calculations for this type of the measurement (via full wave functions of a complete (microparticle + macroscopic ball) system) and is found in D.I. Blokhintsev The philosophy of quantum mechanics, Reidel, 1968 , which we refer the reader to for all clarifications.

Clearly, the pre-setting of initial data of all particles in the Universe would require an unlimited time. Therefore, in reality one needs to be restricted to isolated mechanical systems. Predictions, stemming from the knowledge of initial data for such a system, are of a provisional nature. They are true only if there will be no violation of the system insulation in the future ¹¹

Similarly, to obtain certain conclusions about the future in the field theory, there is a need to know, besides initial data, the boundary conditions. The latter are given beforehand, into the future. Therefore, here the predictions bare the conditional nature as well. Everything will be as the field theory predicts unless something unexpected is happening on the boundaries.

Therefore, the determinism in classical physics is illusory to a certain extent. It contains assumptions about the future, stemming neither from mechanics nor field theory.

If we still try to avoid these difficulties by extending the system under consideration via introducing more and more secondary factors, we will reduce the very best determinism to the irreproducible randomness ¹².

L. Boltzmann, the great physicist and materialist, was one of the first to realize that using statistical methods we can understand laws of gases, which are totally incomprehensible in terms of mechanics of a system, comprising a large number of particles. In its famous *H*-theorem, Boltzmann showed that random interactions between gaseous particles result in the Maxwell distribution. It appears, there is no way to “derive” statistical laws from deterministic ones. At best they can be made compatible. In those systems, where the role of chance becomes significant, there is a need to make some special assumptions of a statistical nature to “deduce” ruling laws. Typically those assumptions are about an equilikelihood for one or another state of a mechanical system.

It should be acknowledged therefore, that chance is not inferior to determinism in creating law.

The founder of statistical mechanics, D. Gibbs, apparently was the first to realize, it is not necessary to seek a specific way the chance uses to bring one or another system to a certain – in a statistical scense – state. One can make some assumptions and then compare

¹¹ For instance, predictions of a space ship motion would be valid barring the collision with an asteroid. The latter showing up on the trajectory of a spaceship can be predicted only statistically.

¹² In this regard, see F.Engels, *Dialectics of Nature*, Moscow, 1969

them with experiments.

In various fields of modern science statistical methods are so prevalent and powerful, that we need to acknowledge – in the being of the Universe elements of games of chance can not be ignored: Chance is certainly favored by Law and cooks up events that are unexpected and unlikely. In quantum mechanics an element of randomness is laid into its very foundations – in the concept of a probability amplitude and wave function ψ .

Entering the field of quantum phenomena we need to abandon cozy illusions of determinism and acknowledge an existence of games of chance. Every time that a quantum transition occurs, there effectuates a selection from various potentialities (*Editorial note:* the term “potentiality” in the sense of a potential opportunity was repeatedly used by D. Bohm in his “Quantum theory”). While a likelihood of one or another selection is predicted by quantum mechanics, potentialities themselves are predetermined. In that respect quantum mechanics appears as an amazing blend of a statistical concept and a strict determinism.

In nonrelativistic quantum mechanics determinism is expressed via the condition that a wave function, comprehensively determining a quantum ensemble, follows the Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hat{H}(x,t)\psi(x,t). \quad (29)$$

From that equation it follows that a state $\psi(x, t + \Delta t)$ at time $t + \Delta t$, infinitesimally close to a preceding moment t , is determined by the equation (29)

$$\psi(x, t + \Delta t) = \psi(x, t) - \frac{i}{\hbar}\hat{H}(x, t)\psi(x, t)\Delta t,$$

i.e. the wave function value at the preceding time.

A more detailed idea of the causality in quantum mechanics can be obtained from Green functions. As is well known, a wave function $\psi(x, t)$ obeys the integral equation stemming from the Schrödinger equation

$$\psi(x, t) = \psi_0(x, t) + \int_{-\infty}^{+\infty} g(x - x', t - t')V(x', t')\psi(x', t')dx'dt'.$$

Here $\psi_0(x, t)$ is the initial value of a wave function before a potential $V(x, t)$ turns on, $g(x - x', t - t')$ is the retarded Green function of the Schrödinger equation for a free motion. A paramountly important property of that function is that it equals zero for $t' > t$.

Let's change a system state in the vicinity of x', t' . We'll express this change by means of giving the function $\psi(x, t)$ a variation in the neighborhood of x', t' . By taking the functional derivative of a $\psi(x, t)$ with respect to $\psi(x', t')$, we'll have

$$\frac{\delta\psi(x, t)}{\delta\psi(x', t')} = g(x - x', t - t')V(x', t').$$

From the basic Green function property it follows that the impact of a change made at x', t' on the state at x, t equals zero, if $t' > t$, i.e. if a change $\delta\psi(x', t')$ has happened later than a response $\delta\psi(x, t)$. This property becomes even more transparent in a relativistic quantum theory. While the exposition of that theory goes beyond the scope of this book, it might be still worthwhile to mention, that in the relativistic theory the Green function $g(x - x', t - t')$ is non-zero only in the area where

$$c(t - t') \geq |x - x'|. \tag{30}$$

Here c is the speed of light. In virtue of this condition the change at x', t' can cause a change at x, t only if these points can be connected to each other with a signal propagating with a speed $v = \frac{|x-x'|}{t-t'} \leq c$. The relativistic condition (30) becomes the non-relativistic one $t > t'$, if the speed of light is considered infinitely high.

Therefore, in quantum mechanics changes of a quantum system state are connected to each other via a simple condition of causality. Transitions, incompatible with the condition of causality, are impossible. In turn, quantum transitions compatible with the causality principle, are governed by probabilistic laws.