Multidimensional Numbers

Krishna Srinivasan, Rajaram Gana

Abstract. A multidimensional number will not be viewed as a single real scalar value, rather, as a set of scalar values, each associated with a dimension. This gives rise to variations of "complex numbers", and consequently, Euler's formula. The properties of complex numbers, such as the product of magnitudes being equal to the magnitude of the products, may also be applicable in the case of multidimensional numbers, depending on how they are constructed. Although similar ideas exist, such as a hypercomplex number, the differences will be discussed.

"I am interested in Mathematics only as a creative art." is a famous quote of the great English mathematician G. H. Hardy. Mathematics allows creative representations of numbers.

A multidimensional or an N-dimensional number, consists of

1. a set of real numbers, each present in one of, or a *combination* of the N dimensions,
2. and an associated $N \times N$ matrix, called the dimensions multiplication table, which contains the result of all the possible pairs of multiplication of the dimensions.

The idea is similar to hypercomplex numbers [**1**], a generalization of quaternions and octonions [**2**][**3**]. The difference is that a multidimensional number allows a real number to be associated with any combination of the N dimensions. For example, in the case of a 3D number, if the dimensions are represented as $\{\hat{i}_0, \hat{i}_1, \hat{i}_2\}$, the dimension of a real number could be $\hat{i}_1\hat{i}_2$, for example, or $\hat{i}_1\hat{i}_1$, which may not necessarily be the same as $\hat{i}_1$. In Section 8, it will be shown that this flexiblity may allow properties, such as the product of magnitudes being equal to the magnitude of the products, to also hold true in the case of multidimensional numbers.

Real numbers are "1D" numbers, represented on a number line, while complex numbers are "2D", viewed on a real-imaginary axis, represented as a vector on a 2D plane. Multidimensional numbers bridges the gap between real and imaginary numbers, showing that there is nothing "imaginary" about complex numbers, rather, complex numbers is a special case of 2D numbers. Furthermore, "complex numbers" can be extended to higher dimensions, and a 3D example will be discussed.

Multidimensional numbers can be viewed as vectors. There are differences, however. Multiplication and division of multidimensional numbers can be defined, but don't exist in the case of vectors.

1. GENERAL FORMULATION OF AN N-DIMENSIONAL (N-D) NUMBER

The following axioms will be assumed to hold true in N-D numbers:

1. The N dimensions will be represented as $\hat{i}_0$, $\hat{i}_1$, $\hat{i}_2$, ... $\hat{i}_{N-1}$. For example, in the 1D case, $3\,\hat{i}_0$ represents the scalar 3 in the first dimension, and the only dimen-

sion in the 1D case. In the case of 2D for example, $3\,\hat{i}_0 + 3.14\,\hat{i}_1$, represents 3 in the first dimension $\hat{i}_0$ and 3.14 in the second dimension $\hat{i}_1$.

2. Any number with no dimension, such as division in

$$\frac{2\,\hat{i}_1}{2\,\hat{i}_1} = 1 = 1\,\hat{i}_0, \tag{1}$$

by default, exists in the first dimension $\hat{i}_0$.

3. If the dimension of the denominator is only $\hat{i}_0$, this can be ignored, and the dimension of the number is the dimension of the numerator. For example,

$$\frac{2\,\hat{i}_1}{3\,\hat{i}_0} = \frac{2}{3}\,\hat{i}_1. \tag{2}$$

4. Addition and subtraction operations apply in the conventional sense as real numbers, to scalar values in the same dimension.
5. Multiplication and division result in numbers in new dimensions. For example,

$$2\,\hat{i}_0 \times 3\,\hat{i}_1 = 6\,\hat{i}_0\hat{i}_1, \tag{3}$$

and the result lies in the dimension $\hat{i}_0\hat{i}_1$. Similarly, the result of the division operation

$$2\,\hat{i}_0 \div 3\,\hat{i}_1, \tag{4}$$

lies in the dimension $\frac{\hat{i}_0}{\hat{i}_1}$. The dimensions can be simplified using the *dimensions multiplication table*, which will be discussed in Sections 2-3.

6. To accomodate the general case of a multidimensional number, the associative and commutative properties are not assumed to hold true. For example, $\hat{i}_3\hat{i}_4$ is not necessarily the same dimension as $\hat{i}_4\hat{i}_3$.
7. The distributive property holds true, when multiplying two multidimensional numbers. For example,

$$\left(2\,\hat{i}_0 + 5\,\hat{i}_1\right)\left(\hat{i}_0 + 3\,\hat{i}_1\right) = 2\,\hat{i}_0\hat{i}_0 + 6\,\hat{i}_0\hat{i}_1 + 5\,\hat{i}_1\hat{i}_0 + 15\,\hat{i}_1\hat{i}_1. \tag{5}$$

A general multidimensional number has been defined, which conforms to the above axioms. Similar to hypercomplex numbers, the dimensions can be simplified, by defining a dimensions multiplication table, which will be discussed next. There is complete flexibility in how this table can be defined, as illustrated by the examples in the following sections.

2. DIMENSIONS MULTIPLICATION TABLE IN THE MOST GENERAL FORM In a 2D number, for example, the possible dimensions arising from pairwise multiplication of the dimensions from the set $\{\hat{i}_0, \hat{i}_1\}$, are captured in the matrix

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 \end{bmatrix}. \tag{6}$$

In a 3D number, for example, given the dimensions $\{\hat{i}_0, \hat{i}_1, \hat{i}_2\}$, the possible pairwise products from the set are

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 & \hat{i}_0\hat{i}_2 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 & \hat{i}_1\hat{i}_2 \\ \hat{i}_2\hat{i}_0 & \hat{i}_2\hat{i}_1 & \hat{i}_2\hat{i}_2 \end{bmatrix}. \tag{7}$$

Similarly, the matrix can be generalized for N dimensions. Simplifications to the dimensions multiplication table can be made, as discussed next.

3. A REDUCED DIMENSIONS MULTIPLICATION TABLE A reduced dimensions multiplication table defines how the pairwise multiplications of the dimensions can be simplified. For example, here's an example in 2D:

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{i}_0 & \hat{i}_1 \\ \hat{i}_1 & -\hat{i}_0 \end{bmatrix} = A, \tag{8}$$

where the reduced dimensions matrix is labeled A. Starting with a number $5\,\hat{i}_0$ in the $\hat{i}_0$ dimension, multiplying by $\hat{i}_1$, would result in $5\,\hat{i}_0\hat{i}_1$. The dimensions can be simplified using the table. The entry in the table corresponding to the dimension $\hat{i}_0\hat{i}_1$ is the dimension $\hat{i}_1$. This would "relocate" this number to the $\hat{i}_1$ dimension. Therefore,

$$5\,\hat{i}_0\hat{i}_1\Big|_A = 5\,\hat{i}_1, \tag{9}$$

where A denotes the table used to simplify the dimensions. The commutative property holds true in the reduced table, since the off-diagonal entries are symmetric. In general, however, this need not be the case.

4. THE MAGNITUDE OF A MULTIDIMENSIONAL NUMBER Given a multidimensional number $\vec{r}$, which has been reduced using the dimensions multiplication table, with K unique dimensions, the magnitude of $\vec{r}$ is

$$|\vec{r}\,| = \sqrt{a_0^2 + a_1^2 + a_2^2 + ... + a_K^2}, \tag{10}$$

where $\{a_0, a_1, ... , a_K\}$ are the real scalar values corresponding to those unique dimensions.

5. 2D NUMBERS AND ITS RELATION TO COMPLEX NUMBERS The arithmetic operations of complex numbers can be viewed from a geometric or an algebraic perspective. From a geometric perspective, the addition and subtraction of complex numbers can be viewed as vector addition and subtraction. Multiplication and division can be viewed as a scaling and a rotation operation [**4**].

Alternately, the arithmetic operations of complex numbers can be viewed algebraically. The dimensions $\hat{i}_0$ and $\hat{i}_1$ in 2D numbers, represent the $real$ and the $imaginary$ values, respectively. Addition and subtraction are operations on numbers that belong to the same dimension. The distributive property, together with the dimensions multiplication table in Equation 8,

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{i}_0 & \hat{i}_1 \\ \hat{i}_1 & -\hat{i}_0 \end{bmatrix} = A, \tag{11}$$

is equivalent to the definition of multiplication, and consequently division, of complex numbers. In the more familiar form of a complex number, the dimensions multiplication table in Equation 8 is written as

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1\cdot 1 & 1\cdot i \\ i\cdot 1 & i\cdot i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \tag{12}$$

where the symbols 1 and i are used, instead of $\hat{i}_0$ and $\hat{i}_1$, to represent the complex numbers.

$$i^2 = -1, \tag{13}$$

in the entry of the second row and the second column of the matrix, or by definition

$$i = \sqrt{-1}. \tag{14}$$

Here's an example of division of complex numbers,

$$\frac{2\,\hat{i}_0 + 4\,\hat{i}_1}{3\,\hat{i}_0 - 2\,\hat{i}_1} = \left(\frac{2\,\hat{i}_0 + 4\,\hat{i}_1}{3\,\hat{i}_0 - 2\,\hat{i}_1}\right)\left(\frac{3\,\hat{i}_0 + 2\,\hat{i}_1}{3\,\hat{i}_0 + 2\,\hat{i}_1}\right), \tag{15}$$

where the numerator and the denominator are multiplied by the complex conjugate. Using the distributive property to multiply the numerator and the denominator, the resulting expression is

$$\frac{2\,\hat{i}_0 + 4\,\hat{i}_1}{3\,\hat{i}_0 - 2\,\hat{i}_1} = \frac{6\,\hat{i}_0\hat{i}_0 + 16\,\hat{i}_0\hat{i}_1 + 8\,\hat{i}_1\hat{i}_1}{9\,\hat{i}_0\hat{i}_0 - 4\,\hat{i}_1\hat{i}_1}, \tag{16}$$

From the reduced dimensions matrix in Equation 11, the denominator and numerator can be simplified as

$$\frac{2\,\hat{i}_0 + 4\,\hat{i}_1}{3\,\hat{i}_0 - 2\,\hat{i}_1} = \frac{-2\,\hat{i}_0 + 16\,\hat{i}_1}{13\,\hat{i}_0}, \tag{17}$$

Using Axiom 4, $\hat{i}_0$ in the denominator can be discarded, further simplifying the result as

$$\frac{2\,\hat{i}_0 + 4\,\hat{i}_1}{3\,\hat{i}_0 - 2\,\hat{i}_1} = -\frac{2}{13}\,\hat{i}_0 + \frac{16}{13}\,\hat{i}_1. \tag{18}$$

6. OTHER VARIATIONS OF "COMPLEX NUMBERS" If the dimensions multiplication table of complex numbers in Equation 11 is modified as

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{i}_0 & -\hat{i}_1 \\ -\hat{i}_1 & -\hat{i}_0 \end{bmatrix} = B, \tag{19}$$

the ramifications to complex numbers is studied next. From Axiom 3, the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ..., \tag{20}$$

is written as

$$e^x = \hat{i}_0 + x\,\hat{i}_0 + \frac{x^2}{2!}\,\hat{i}_0 + \frac{x^3}{3!}\,\hat{i}_0 + ... \tag{21}$$

Substituting

$$x = e^{\theta\,\hat{i}_1} \tag{22}$$

in Equation 21,

$$e^{\theta\,\hat{i}_1} = \hat{i}_0 + \theta\,\hat{i}_1\hat{i}_0 + \frac{\theta^2}{2!}\,\hat{i}_1\hat{i}_1\hat{i}_0 + \frac{\theta^3}{3!}\,\hat{i}_1\hat{i}_1\hat{i}_1\hat{i}_0 + ... \tag{23}$$

The order in which the dimension vectors are multiplied, when applying the dimensions multiplication table, is important, as this example illustrates. For example, in the term

$$\hat{i}_1\hat{i}_1\hat{i}_0, \tag{24}$$

if the dimension vectors are multiplied left to right, and from the dimensions multiplication table B in Equation 19, the resulting dimension is

$$-\hat{i}_0. \tag{25}$$

However, if the dimension vectors are multiplied right to left, the resulting dimension is

$$+\hat{i}_0. \tag{26}$$

To avoid this problem, the convention followed is that the dimension vectors are multiplied left to right. It follows that multiplication of numbers within parentheses

$$(a+d)(b)(c) \tag{27}$$

is also from left to right, unless parentheses are placed to override this rule, such as

$$a(bc), \tag{28}$$

where bc is multiplied first. The convention is arbitrary, and a multiplication rule of the dimension vectors from right to left could have been chosen as well.

Simplifying Equation 23 using the dimensions multiplication table B in Equation 19, and using the convention stated earlier,

$$e^{\theta\,\hat{i}_1}\Big|_B = \hat{i}_0\left[1 - \sum_{n=1}^{\infty}\frac{\theta^{2n}}{(2n)!}\right] - \hat{i}_1\sum_{n=1}^{\infty}\frac{\theta^{2n-1}}{(2n-1)!}. \tag{29}$$

The subscript B is used to denote that the dimensions multiplication table B has been used to simplify the dimensions. The above series can be simplified as

$$e^{\theta\,\hat{i}_1}\Big|_B = \hat{i}_0\,(2 - \cosh\theta) - \hat{i}_1\sinh\theta, \tag{30}$$

which is a different relation from Euler's formula,

$$e^{\theta\,\hat{i}_1}\Big|_A = \hat{i}_0\cos\theta + \hat{i}_1\sin\theta, \tag{31}$$

obtained by applying the dimension multiplication table A in Equation 11, to simplify the expression in Equation 23. In this example, given any two 2D numbers X and Y,

$$X = a\,\hat{i}_0 + b\,\hat{i}_1 \tag{32}$$

$$Y = c\,\hat{i}_0 + d\,\hat{i}_1, \tag{33}$$

and the dimensions multiplication table in Equation 19,

$$|XY| = |X||Y| \tag{34}$$

can be verified to be true, using the definition of the magnitude of a multidimensional number in Equation 10. In general, however, the above relation may not always hold, discussed in Sections 7-8.

7. 3D COMPLEX NUMBER In the case of a 2D complex number, the dimension vectors are 1 and i. Multiplication by i results in a counter-clockwise rotation of the second operand by $90°$. For example,

$$i \times 1 = i, \tag{35}$$

where 1 is rotated by $90°$ from the real to the imaginary axis, resulting in the answer i. Similarly,

$$i \times i = -1, \tag{36}$$

where i is rotated counter clockwise by $90°$ to the real axis, resulting in -1.

This idea is extended to create a 3D complex number, where multiplication by $\hat{i}_1$ and $\hat{i}_2$ rotates the second operand by $90°$, captured in Figure 1. Multiplication of $\hat{i}_0\hat{i}_1$

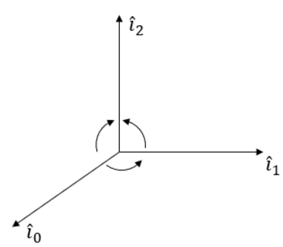

Figure 1. Rotation of dimensions in a 3D complex number.

results in the rotation of $\hat{i}_0$ to $\hat{i}_1$. Similarly, other rotations are shown in the figure, and captured in the dimensions multiplication matrix,

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 & \hat{i}_0\hat{i}_2 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 & \hat{i}_1\hat{i}_2 \\ \hat{i}_2\hat{i}_0 & \hat{i}_2\hat{i}_1 & \hat{i}_2\hat{i}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{i}_0 & \hat{i}_1 & \hat{i}_2 \\ \hat{i}_1 & -\hat{i}_0 & \hat{i}_2 \\ \hat{i}_2 & \hat{i}_2 & -\hat{i}_0 \end{bmatrix}. \tag{37}$$

This system, however, does not obey the relation $|XY| = |X||Y|$, where the magnitude of the product of two multidimensional numbers, is not the same as the product of the magnitudes, discussed next.

8. EULER'S FORMULA IN 3D COMPLEX NUMBERS

Substituting

$$x = e^{a\,\hat{i}_0},\ y = e^{b\,\hat{i}_1},\ \ z = e^{c\,\hat{i}_2}, \tag{38}$$

in the equation

$$e^{x+y+z} = e^x e^y e^z, \tag{39}$$

which has been proven in Reference [**5**], the resulting expression is

$$e^{a\,\hat{i}_0+b\,\hat{i}_1+c\,\hat{i}_2} = e^{a\,\hat{i}_0} e^{b\,\hat{i}_1} e^{c\,\hat{i}_2}. \tag{40}$$

Substituting $e^{a\,\hat{i}_0}$ in the power series in Equation 21, and using the dimensions multiplication table in Equation 37,

$$e^{a\,\hat{i}_0} = e^a\,\hat{i}_0. \tag{41}$$

Similarly,

$$e^{b\,\hat{i}_1} = \hat{i}_0 \cos b + \hat{i}_1 \sin b \tag{42}$$

$$e^{c\,\hat{i}_2} = \hat{i}_0 \cos c + \hat{i}_2 \sin c. \tag{43}$$

From the above equations,

$$e^{a\,\hat{i}_0+b\,\hat{i}_1+c\,\hat{i}_2} = \left(e^a\,\hat{i}_0\right)\left(\hat{i}_0 \cos b + \hat{i}_1 \sin b\right)\left(\hat{i}_0 \cos c + \hat{i}_2 \sin c\right). \tag{44}$$

Multiplying the terms using the distributive property, and simplifying the dimensions using the dimensions multiplication table in Equation 37,

$$\begin{aligned} e^{a\,\hat{i}_0+b\,\hat{i}_1+c\,\hat{i}_2} = e^a \Big[\hat{i}_0\ \cos b \cos c \\ + \hat{i}_1\ \sin b \cos c + \hat{i}_2 \left(\sin b \sin c + \cos b \sin c\right)\Big] \end{aligned} \tag{45}$$

From the above equations, note that

$$\left|e^{a\,\hat{i}_0}\right| \left|e^{b\,\hat{i}_1}\right| \left|e^{c\,\hat{i}_2}\right| \neq \left|e^{a\,\hat{i}_0+b\,\hat{i}_1+c\,\hat{i}_2}\right| \tag{46}$$

The dimensions multiplication table in Equation 37, in general, does not meet the relation

$$|XY| = |X||Y|, \tag{47}$$

where X and Y are two 3D numbers. Instead, however, if the dimensions multiplication table is modified as

$$\begin{bmatrix} \hat{i}_0\hat{i}_0 & \hat{i}_0\hat{i}_1 & \hat{i}_0\hat{i}_2 \\ \hat{i}_1\hat{i}_0 & \hat{i}_1\hat{i}_1 & \hat{i}_1\hat{i}_2 \\ \hat{i}_2\hat{i}_0 & \hat{i}_2\hat{i}_1 & \hat{i}_2\hat{i}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{i}_0 & \hat{i}_1 & \hat{i}_2 \\ \hat{i}_1 & -\hat{i}_0 & \hat{i}_1\hat{i}_2 \\ \hat{i}_2 & \hat{i}_1\hat{i}_2 & -\hat{i}_0 \end{bmatrix}, \tag{48}$$

repeating the same exercise,

$$e^{a\,\hat{i}_0+b\,\hat{i}_1+c\,\hat{i}_2} = e^a \Big[\hat{i}_0\,\cos b\cos c + \hat{i}_1\,\sin b\cos c + \hat{i}_2\,\cos b\sin c \\ + \hat{i}_1\hat{i}_2\,\sin b\sin c\Big]. \quad (49)$$

In this case, Equation 47 holds true. This can be verified by substituting Equation 41-43 and Equation 49 in

$$\left|e^{a\,\hat{i}_0}\right|\left|e^{b\,\hat{i}_1}\right|\left|e^{c\,\hat{i}_2}\right| = \left|e^{a\,\hat{i}_0+b\,\hat{i}_1+c\,\hat{i}_2}\right| \quad (50)$$

$$= e^a, \quad (51)$$

and verify that the product of the magnitudes in the left-hand side of the above equation, is equal to the magnitude of the products in the right-hand side. Using the dimensions multiplication table in Equation 48, which includes a combination of dimensions $\hat{i}_1\hat{i}_2$, instead of Equation 37, the above equation holds true.

9. ARITHMETIC OPERATIONS Addition and subtraction operations are similar to vector addition and subtraction, where numbers in the same dimensions can be added and subtracted. The multiplication operation follows from Axiom 5 and Axiom 7, and the dimensions can be simplified using the dimensions multiplication table. The division operation is done by a series of multiplication operations, to make the denominator purely $\hat{i}_0$.

Here's a division example, using the dimensions multiplication table in Equation 48, resulting in the peculiar expression on the right-hand side

$$\frac{\hat{i}_1}{2\,\hat{i}_1+\hat{i}_1\hat{i}_2+3\,\hat{i}_2} = -\frac{2}{13}\,\hat{i}_0 - \frac{7}{26}\,\hat{i}_2 + \frac{3}{13}\,\hat{i}_1 - \frac{9}{26}\,\hat{i}_1\hat{i}_2. \quad (52)$$

The above result can be verified by evaluating,

$$\left[-\frac{2}{13}\,\hat{i}_0 - \frac{7}{26}\,\hat{i}_2 + \frac{3}{13}\,\hat{i}_1 - \frac{9}{26}\,\hat{i}_1\hat{i}_2\right]\left[2\,\hat{i}_1+\hat{i}_1\hat{i}_2+3\,\hat{i}_2\right] = \hat{i}_1, \quad (53)$$

to show that the quotient multiplied by the divisor is the dividend. Division is done by multiplying the numerator and denominator of

$$\frac{\hat{i}_1}{2\,\hat{i}_1+\hat{i}_1\hat{i}_2+3\,\hat{i}_2}, \quad (54)$$

by the same number to make the denominator purely $\hat{i}_0$. This requires first multiplication by

$$\frac{\hat{i}_1}{\hat{i}_1}, \quad (55)$$

followed by

$$\frac{\hat{i}_2}{\hat{i}_2}, \quad (56)$$

and then

$$\frac{-2\,\hat{i}_2 + \hat{i}_0 + 3\,\hat{i}_1}{-2\,\hat{i}_2 + \hat{i}_0 + 3\,\hat{i}_1}, \tag{57}$$

and

$$\frac{6\,\hat{i}_0 + 4\,\hat{i}_2}{6\,\hat{i}_0 + 4\,\hat{i}_2}, \tag{58}$$

resulting in the result in the right-hand side of Equation 52.

REFERENCES

1. Kantor, I. L. and Solodovnikov, A. S., *Hypercomplex Numbers: An Elementary Introduction to Algebras*, New York: Springer-Verlag, 1989.
2. Conway, J. H. and Smith, D. A., *On Quaternions and Octonions*, A K Peters/CRC Press, 1st edition, Jan., 2003
3. Silviu Olariu, "Complex Numbers in n Dimensions", Available Online: https://arxiv.org/pdf/math/0011044.pdf
4. Paul J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$, Princeton, NJ: Princeton University Press, 1998.
5. "Prove $e^{x+y} = e^x e^y$ by using Exponential Series", Available Online: https://math.stackexchange.com/questions/414061/prove-exy-exey-by-using-exponential-series