Improvement of the Siegel’s Theorem

JinHua Fei
ChangLing Company of Electronic Technology Baoji Shannxi P.R.China
E-mail: feijinhuayoujian@msn.com

Abstract. In this paper, we improved the Siegel’s theorem, we are going to turn a non-effective constant into an effective computable constant, only one modulu may be exceptional.

Keyword. Siegel’s theorem, Exceptional real zero, Dirichlet L function.
MR(2000) Subject Classification 11M20

In 1935, Siegel proved the following theorem.
For any $0 < \varepsilon < 1$, there exists a positive number $C(\varepsilon)$ such that, if $\chi_q$ is a real primitive character to the modulus $q$, then
\[ L(1, \chi_q) > C(\varepsilon)q^{-\varepsilon} \]

where the constant $C(\varepsilon)$ is a non-effective positive constant.

In this paper, we proved the following theorem,

Theorem. For any $0 < \varepsilon < \frac{1}{10}$, if $\chi_q$ is a real primitive character to the modulus $q$, $\beta_q$ is an exceptional real zero of the function $L(s, \chi_q)$, then
\[ \beta_q \leq 1 - \frac{\varepsilon q^{-\varepsilon}}{c\log q} \]

where the constant $c$ is an effectively computable positive constant. Only one modulu may be exceptional.

1. Some Lemma

Lemma 1. Let $\chi_D$ be any non-principal character modulo $D$, Then for any integers $M$ and $N$ with $N > 0$,
\[ \sum_{n=M+1}^{M+N} \chi_D(n) \ll D^{\frac{1}{2}} \log D \]

See the page 307 of references[1]

Lemma 2. For any real number $t$, we have
\[ \zeta\left(\frac{1}{2} + it\right) \ll \left(\mid t \mid + 1\right)^{\frac{1}{2}}, \quad L\left(\frac{1}{2} + it, \chi_D\right) \ll D^{\frac{1}{2}} \left(\mid t \mid + 1\right)^{\frac{1}{2}} \]
and when \(- \frac{1}{4} \leq \sigma \leq - \frac{1}{4}\), we have
\[
\Gamma(\sigma + it) \ll (|t| + 1)^{\sigma - \frac{1}{4}} e^{\frac{2\pi}{3}|t|}
\]

where \(\zeta(s)\) is the Riemann zeta function, \(L(s, \chi_D)\) is the Dirichlet \(L\) function and \(\Gamma(s)\) is the Euler \(\Gamma\) function.

See the page 27, page 350 of references [1] and the page 45 of references [2].

**Lemma 3.** For any \(0 \leq \eta \leq \frac{1}{10}\), we have
\[
L'(1 - \eta, \chi_D) \ll D^{\eta} \log^2 D
\]

where \(L'(s, \chi_D)\) is the first derivative function of the Dirichlet \(L\) function.

**Proof.** It is easy to see that
\[
L'(1 - \eta, \chi_D) = - \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n^{1-\eta}} \log n = - \sum_{1 \leq n \leq D \leq x} \frac{\chi_D(n)}{n^{1-\eta}} \log n - \sum_{D < n} \frac{\chi_D(n)}{n^{1-\eta}} \log n
\]

By lemma 1, we have
\[
- \sum_{D < n} \frac{\chi_D(n)}{n^{1-\eta}} \log n = - \sum_{n=1}^{\infty} \frac{\log u}{u^{\eta-\eta'}} d\left(\sum_{n \leq u} \chi_D(n)\right)
\]
\[
= \frac{\log D}{D^{\eta-\eta'}} \left(\sum_{n \leq D} \chi_D(n)\right) + \int_{D}^{\infty} \left(\frac{1}{u^{\eta-\eta'}} + (-1 + \eta) \frac{\log u}{u^{2-\eta'}}\right) \left(\sum_{n \leq u} \chi_D(n)\right) du
\]
\[
\ll \frac{\log^2 D}{D^{\eta-\eta'}} + D^{2} \log D \int_{D}^{\infty} \frac{\log u}{u^{2-\eta'}} du \ll \frac{\log^2 D}{D^{\eta-\eta'}}
\]

in addition
\[
- \sum_{1 \leq n \leq D} \frac{\chi_D(n)}{n^{1-\eta}} \log n \ll \sum_{1 \leq n \leq D} \frac{\log n}{n^{\eta-\eta'}} \ll D^{\eta} \sum_{1 \leq n \leq D} \frac{\log n}{n^{1-\eta}} \ll D^{\eta} \log^2 D
\]

This completes the proof of the lemma.

**Lemma 4.** Let \(\beta_D\) is the real zero of the function \(L(s, \chi_D)\) and \(0 < 1 - \beta_D \leq \frac{1}{10}\), then
\[
\frac{L(1, \chi_D)}{1 - \beta_D} \ll D^{1 - \beta_D} \log^2 D
\]

**Proof.** we write \(\delta = 1 - \beta_D\), then
\[
\int_0^\delta L'(1 - \sigma, \chi_D)d\sigma = -\int_0^\delta d L(1 - \sigma, \chi_D) = -L(1 - \delta, \chi_D) + L(1, \chi_D) = L(1, \chi_D)
\]

therefore
By lemma 3, we have

\[ L(1, \chi_D) \ll \delta D^\delta \log^2 D \]

therefore

\[ \frac{L(1, \chi_D)}{1 - \beta_D} \ll D^{1-\beta_D} \log^2 D \]

This completes the proof of the lemma.

**Lemma 5.** Let \( \chi_D \) and \( \chi_q \) is real character, when \( \text{Re } s > 1 \), we have

\[ \zeta(s) L(s, \chi_D) L(s, \chi_q) L(s, \chi_D \chi_q) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \]

where \( a(1) = 1 \) and \( a(n) \geq 0 \) for all \( n \). \( \zeta(s) \) is the Riemann zeta function. \( L(s, \chi_D) \), \( L(s, \chi_q) \) and \( L(s, \chi_D \chi_q) \) is the Dirichlet \( L \) function.

See the page 129 of references [3]

### 2. Proof of Theorem

Now we are always going to assume that \( \chi_D \) and \( \chi_q \) is the real primitive character.

For any \( 0 < \varepsilon \leq \frac{1}{10} \), If all the real zeros \( \beta_q \) satisfy \( \frac{\varepsilon}{6} < 1 - \beta_q \), then

\[ \beta_q < 1 - \frac{\varepsilon}{6} \leq 1 - \frac{\varepsilon q^{-\varepsilon}}{c \log^5 q} \]

obviously the theorem is true.

Let \( D \) be the smallest modulu, it satisfy \( L(\beta_D, \chi_D) = 0 \) and \( 0 < 1 - \beta_D \leq \frac{\varepsilon}{6} \).

From this assumption, when \( q < D \), the real zero of the function \( L(s, \chi_q) \) satisfy \( \frac{\varepsilon}{6} < 1 - \beta_q \), then, we have

\[ \beta_q < 1 - \frac{\varepsilon}{6} \leq 1 - \frac{\varepsilon q^{-\varepsilon}}{c \log^5 q} \]

obviously the theorem is true.

Now, let’s prove that the theorem is true when \( q > D \). We write \( \delta = 1 - \beta_D \), by our assumption, \( 0 < \delta \leq \frac{\varepsilon}{6} \). We write \( F(s) = \zeta(s) L(s, \chi_D) L(s, \chi_q) L(s, \chi_D \chi_q) \), by lemma 5 and \( F(1 - \delta) = 0 \),

\[ |L(1, \chi_D)| = \left| \int_0^\infty L'(1 - \sigma, \chi_D) d\sigma \right| \leq \int_0^\infty |L'(1 - \sigma, \chi_D)| d\sigma \]
we have
\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^{s+\delta}} e^{-\frac{x}{n}} = \frac{1}{2\pi i} \int_{(\delta)} F(s+1-\delta) \Gamma(s)x^s \, ds
\]
\[
= L(1, \chi_D) L(1, \chi_q) L(1, \chi_D \chi_q) \Gamma(\delta)x^\delta + \frac{1}{2\pi i} \int_{(\delta)} F(s+1-\delta) \Gamma(s)x^s \, ds
\]

By lemma 2, we have
\[
\frac{1}{2\pi i} \int_{(\delta)} F(s+1-\delta) \Gamma(s)x^s \, ds
\]
\[
\ll x^{-\delta} \int_{-\infty}^{\infty} |F(\frac{1}{2}+it)||\Gamma(-\frac{1}{2}+\delta+it)| \, dt \ll x^{-\frac{1}{2}+\delta} q^2
\]
we take \( x = q^5 \), and \( \Gamma(\delta) = \frac{\Gamma(1+\delta)}{\delta} \), we have
\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^{s+\delta}} e^{-\frac{x}{n}} = L(1, \chi_D) L(1, \chi_q) L(1, \chi_D \chi_q) \frac{\Gamma(1+\delta)}{\delta} q^{5\delta} + O \left( q^{-\frac{1}{2}+\delta} \right)
\]
By lemma 5, we have
\[
e^{-\frac{1}{5}} \leq L(1, \chi_D) L(1, \chi_q) L(1, \chi_D \chi_q) \frac{\Gamma(1+\delta)}{\delta} q^{5\delta} + O \left( q^{-\frac{1}{2}+\delta} \right)
\]
therefore
\[
\frac{1}{2} \leq 3 \frac{L(1, \chi_D)}{\delta} L(1, \chi_q) L(1, \chi_D \chi_q) q^{\frac{5\delta}{2}}
\]
By lemma 4, we have
\[
\frac{L(1, \chi_D)}{\delta} = \frac{L(1, \chi_D)}{1-\beta_D} \ll D^\delta \log^2 D \ll q^5 \log^2 q
\]
By lemma 4 and \( \beta_q \) is a exceptional real zero, then
\[
L(1, \chi_q) = (1-\beta_q) \frac{L(1, \chi_q)}{1-\beta_q} \ll (1-\beta_q) \log^2 q
\]
in addition
\[
L(1, \chi_D \chi_q) \ll \log D q \ll \log q
\]
therefore, we have
\[
1 \leq c(1-\beta_q) q^\delta \log^5 q
\]
\[
\beta_q \leq 1 - \frac{q^{-\varepsilon}}{c \log^5 q} \leq 1 - \frac{\varepsilon q^{-\varepsilon}}{c \log^5 q}
\]

As you can see from the proof above, the real zero \( \beta_D \) may not satisfy this inequality, however

\[
\beta_D \leq 1 - \frac{D^{\frac{1}{3}}}{c \log^2 D}
\]

This completes the proof of the theorem.

REFERENCES