Harmonic Theory of the Linear Representation of Partitions

Michalis Psimopoulos

on leave from Physics Dept. Imperial College, London, UK

Kanari 26, Pefki 15121, Athens, Greece

Email: m.psimopoulos@hotmail.com

Abstract

The number of partitions p_s of a positive integer s can be expressed in terms of $p_{s-1}, p_{s-2}, \ldots, p_1, p_0$ ($p_0=1$) by the linear equations

$$p_s = \frac{1}{s}(\lambda_1 p_{s-1} + \lambda_2 p_{s-2} + \cdots + \lambda_{s-1} p_1 + \lambda_s p_0) \quad ; \qquad s = 1, 2, 3, \ldots$$

where each coefficient λ_n ; $n=1, 2, 3, .., s$ represents the sum of divisors of n and has universal numerical values $\lambda_n = \{1, 3, 4, 7, 6, 12, 8, 15, 13, 18, \ldots\}$ independent of s.

In the present paper it is shown that λ_n can be obtained from a triangular algorithm where the columns are well defined harmonic sequences:

n	λ_n
1	$\lambda_1 = \ \ 1 = 1$
2	$\lambda_2 = \ \ 3 = 1 + 2$
3	$\lambda_3 = \ \ 4 = 1 + 0 + 3$
4	$\lambda_4 = \ \ 7 = 1 + 2 + 0 + 4$
5	$\lambda_5 = \ \ 6 = 1 + 0 + 0 + 0 + 5$
6	$\lambda_6 = 12 = 1 + 2 + 3 + 0 + 0 + 6$
7	$\lambda_7 = \ \ 8 = 1 + 0 + 0 + 0 + 0 + 0 + 7$
8	$\lambda_8 = 15 = 1 + 2 + 0 + 4 + 0 + 0 + 0 + 8$
9	$\lambda_9 = 13 = 1 + 0 + 3 + 0 + 0 + 0 + 0 + 0 + 9$
10	$\lambda_{10} = 18 = 1 + 2 + 0 + 0 + 5 + 0 + 0 + 0 + 0 + 10$

As a result λ_n is given exactly by the formula

$$\lambda_n = \sum_{\kappa=1}^{n} \sum_{\ell=0}^{\kappa-1} cos\left(2\pi \frac{n}{\kappa} \ell\right)$$

Inversing the linear equations it is also shown that the partitions p_s are given in terms of $\lambda_1, \lambda_2, \ldots, \lambda_s$ by an s^2-matrix establishing a new relation between partitions and harmonic functions.

1. Introduction

The study of partitions [1] is an old subject of number theory, still active today. The number of partitions p_s of a positive integer s is equal to the number of integer solutions of the equation

$$n_1 + 2n_2 + 3n_3 + \ldots + sn_s = s \tag{1}$$

where $n_1 \geq 0$, $n_2 \geq 0$, …. . $n_s \geq 0$. Therefore, partitions are also of importance in statistical mechanics [2] as they represent the number of states of macroscopic systems of N particles distributed among s discrete energy levels for $N \geq s$.

In two previous communications [3,4], Eq. (1) was used in order to express partitions by integrals over harmonic functions. The main result of this work is the exact formula

$$p_s = \frac{2}{\pi}\int_0^{\pi/2} \prod_{\kappa=1}^{s} \left\{\frac{sin[(s+\kappa)x]}{sin(\kappa x)}\right\} cos[(s^2-2s)x]\, dx \tag{2}$$

In the present paper, continuing on the same line of research, we establish a new matrix relation between partitions and harmonic sequences. The theory is based on a linear recursion formula of partitions connecting multiplicative with additive number theory, presented later in the text [Eq. (14)].

Euler, gave us the following recursion formula [1] of p_s :

$$p_s = p_{s-1} + p_{s-2} - p_{s-5} - p_{s-7} + p_{s-12} + p_{s-15} - p_{s-22} - p_{s-26} + \cdots \tag{3}$$

which can be written compactly in the form:

$$p_s = \sum_{\kappa=1}^{\infty} (-1)^{\kappa+1}\{p_{s-\omega(\kappa)} + p_{s-\omega(-\kappa)}\} \tag{4}$$

where $\omega(\kappa) = \frac{1}{2}\,(3\kappa^2 - \kappa)$ are the pentagonal numbers of Pythagoras $\omega(\kappa)=(1,5,12,28,\ldots)$.

Later, Theocharis [5] expressed also p_s by the recursion triangular algorithm:

$$
\begin{aligned}
&p_0 = 1 \;\}\; 1 \\
&p_1 = p_0 \;\}\; 1 \\
&\left.\begin{aligned} p_2 &= p_1 + p_0 \\ p_3 &= p_2 + p_1 \\ p_4 &= p_3 + p_2 \end{aligned}\right\} \; 3 \\
&\left.\begin{aligned} p_5 &= p_4 + p_3 - p_0 \\ p_6 &= p_5 + p_4 - p_1 \end{aligned}\right\} \; 2 \\
&\left.\begin{aligned} p_7 &= p_6 + p_5 - p_2 - p_0 \\ p_8 &= p_7 + p_6 - p_3 - p_1 \\ p_9 &= p_8 + p_7 - p_4 - p_2 \\ p_{10} &= p_9 + p_8 - p_5 - p_3 \\ p_{11} &= p_{10} + p_9 - p_6 - p_4 \end{aligned}\right\} \; 5
\end{aligned}
\qquad (5)
$$

where the summation of each line reproduces Eq.(3) and the steps of the algorithm are given by the symplectic sequence:

$$[1],\; 1,\; [3],\; 2,\; [5],\; 3,\; [7],\; 4,\; \ldots \qquad (6)$$

made up by the odd numbers in bracket and the positive integers. Summing up the above sequence we obtain the indices of Eq.(3):

$$
\begin{aligned}
&1=1\;; && 2=1+1\;; && 5=1+1+3\;; \\
&7=1+1+3+2\;; && 12=1+1+3+2+5\;; && 15=1+1+3+2+5+3\;; \ldots\ldots\ldots
\end{aligned}
\qquad (7)
$$

Clearly, the latter theories do not provide evidence that partitions depend on harmonic functions. However, we argue that such dependence can be manifested if we express p_s in terms of p_{s-1}, p_{s-2}, p_{s-3}, …, p_1, p_0, ($p_0=1$) by the linear representation

$$p_s = \varepsilon_1 p_{s-1} + \varepsilon_2 p_{s-2} + \varepsilon_3 p_{s-3} + \ldots + \varepsilon_{s-1} p_1 + \varepsilon_s p_0 \qquad (8)$$

where the coefficients ε_n ; $n=1,2,\ldots,s$ have *universal* numerical values independent of s and are consistent with Euler's expansion of Eq.(3).

$$
\begin{aligned}
\varepsilon_n = \{&[1], 1, 0, 0, [-1, 0], -1, 0, 0, 0, 0, [1, 0, 0], 1, 0, 0, 0, 0, 0, 0, [-1, 0, 0, 0], \\
&-1, 0, 0, 0, 0, 0, 0, 0, 0, \;\; [1, 0, 0, 0, 0], 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots\}
\end{aligned}
\qquad (9)
$$

Example:

$$
\begin{aligned}
p_{50} &= p_{49} + p_{48} - p_{45} - p_{43} + p_{38} + p_{35} - p_{28} - p_{24} + p_{15} + p_{10} \\
&= 173525 + 147273 - 89134 - 63261 + 26015 + 14883 - 3718 - 1575 \\
&\quad + 176 + 42 = 204226
\end{aligned}
\qquad (10)
$$

The structure of ε_n involves two separate groups of coefficients distinguished by brackets in sequence (9) and forming respectively two triangular algorithms:

$$\begin{array}{ccccc} 1 & & & & \\ -1 & 0 & & & \\ 1 & 0 & 0 & & \\ -1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{ccccccccccc} 1 & 0 & 0 & & & & & & & & \\ -1 & 0 & 0 & 0 & 0 & & & & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{11}$$

We observe that the first column of each algorithm is the harmonic sequence (1, -1, 1, -1,…) and the rest of columns all have zero terms. This shows that if Euler's recursion formula [Eq.(3)] is extended in the form of Eq.(8), then a certain harmonic behaviour of ε_n and subsequently of p_s is manifested.

Inversing the linear Eqs (8), we obtain p_s in terms of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s$ by the s^2-matrix:

$$p_s = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \varepsilon_1 \\ -\varepsilon_1 & 1 & 0 & \cdots & 0 & 0 & \varepsilon_2 \\ -\varepsilon_2 & -\varepsilon_1 & 1 & \cdots & 0 & 0 & \varepsilon_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\varepsilon_{s-3} & -\varepsilon_{s-4} & -\varepsilon_{s-5} & \cdots & 1 & 0 & \varepsilon_{s-2} \\ -\varepsilon_{s-2} & -\varepsilon_{s-3} & -\varepsilon_{s-4} & \cdots & -\varepsilon_1 & 1 & \varepsilon_{s-1} \\ -\varepsilon_{s-1} & -\varepsilon_{s-2} & -\varepsilon_{s-3} & \cdots & -\varepsilon_2 & -\varepsilon_1 & \varepsilon_s \end{vmatrix} \tag{12}$$

Example s=5:

$$p_5 = \begin{vmatrix} 1 & 0 & 0 & 0 & \varepsilon_1 \\ -\varepsilon_1 & 1 & 0 & 0 & \varepsilon_2 \\ -\varepsilon_2 & -\varepsilon_1 & 1 & 0 & \varepsilon_3 \\ -\varepsilon_3 & -\varepsilon_2 & -\varepsilon_1 & 1 & \varepsilon_4 \\ -\varepsilon_4 & -\varepsilon_3 & -\varepsilon_2 & -\varepsilon_1 & \varepsilon_5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{vmatrix} = 7 \tag{13}$$

In the present article, we study an alternative linear representation of the partitions p_s in terms of $p_{s-1}, p_{s-2}, p_{s-3}, \ldots, p_1, p_0$, ($p_0=1$) having the form [1] :

$$p_s = \frac{1}{s}\left(\lambda_1 p_{s-1} + \lambda_2 p_{s-2} + \cdots + \lambda_{s-1} p_1 + \lambda_s p_0\right) \quad ; \qquad s = 1, 2, 3 \ldots \tag{14}$$

where each coefficient λ_n represents the sum of divisors of the number n. For instance n=6 ; $\lambda_6 = 1+2+3+6 = 12$.

Therefore, the sequence λ_n has *universal* numerical values :

$$\lambda_n = \{1, 3, 4, 7, 6, 12, 8, 15, 13, 18, \ldots\} \quad ; \quad n=1, 2, 3,\ldots \tag{15}$$

independent of s. As mentioned in ref.[1], Eq.(14) is a remarkable relation connecting multiplicative with additive number theory.

Explicitly, Eq.(14) for s=1, 2, …, 10 reads:

$$p_1 = \frac{1}{1}\{p_0\}$$

$$p_2 = \frac{1}{2}\{p_1 + 3p_0\}$$

$$p_3 = \frac{1}{3}\{p_2 + 3p_1 + 4p_0\}$$

$$p_4 = \frac{1}{4}\{p_3 + 3p_2 + 4p_1 + 7p_0\}$$

$$p_5 = \frac{1}{5}\{p_4 + 3p_3 + 4p_2 + 7p_1 + 6p_0\}$$

$$p_6 = \frac{1}{6}\{p_5 + 3p_4 + 4p_3 + 7p_2 + 6p_1 + 12p_0\}$$

$$p_7 = \frac{1}{7}\{p_6 + 3p_5 + 4p_4 + 7p_3 + 6p_2 + 12p_1 + 8p_0\}$$

$$p_8 = \frac{1}{8}\{p_7 + 3p_6 + 4p_5 + 7p_4 + 6p_3 + 12p_2 + 8p_1 + 15p_0\}$$

$$p_9 = \frac{1}{9}\{p_8 + 3p_7 + 4p_6 + 7p_5 + 6p_4 + 12p_3 + 8p_2 + 15p_1 + 13p_0\}$$

$$p_{10} = \frac{1}{10}\{p_9 + 3p_8 + 4p_7 + 7p_6 + 6p_5 + 12p_4 + 8p_3 + 15p_2 + 13p_1 + 18p_0\} \tag{16}$$

In the second part of the article it is shown that the coefficients λ_n can be obtained from a well defined triangular algorithm based on harmonic sequences that are given by a simple formula. In the third part of the article, inversing the linear Eqs(14), the partitions p_s are expressed in terms of $\lambda_1, \lambda_2, \ldots, \lambda_s$ by an s^2 - matrix so that a new relation between partitions and harmonic functions is established.

2. Algorithm for the coefficients λ_n

Consider the harmonic sequences $h_\kappa(n)$; $n \geq \kappa$; $\kappa = 1,2,3, \ldots$

$h_1(n) = (1, 1, 1, 1, 1, 1, 1, 1, \ldots)$; $n=1,2,3,4,\ldots$

$h_2(n) = (2, 0, 2, 0, 2, 0, 2, 0, \ldots)$; $n=2,3,4,5,\ldots$

$h_3(n) = (3, 0, 0, 3, 0, 0, 3, 0, 0, \ldots)$; $n=3,4,5,6,\ldots$

$h_4(n) = (4, 0, 0, 0, 4, 0, 0, 0, \ldots)$; $n=4,5,6,7,\ldots$

. . . . (17)

The above sequences have the following important property:

$$h_\kappa(n) = \begin{cases} \kappa & if \ \ \kappa \ \ is \ a \ divisor \ of \ n \\ 0 & if \ \ \kappa \ \ is \ not \ a \ divisor \ of \ n \end{cases} \tag{18}$$

We construct next an algorithm in the form of a 2-D triangular matrix by using as columns the sequences $h_\kappa(n)$ of Eqs (17) where the sum of the terms of each row provides the coefficients λ_n of Eq.(14):

n	λ_n
1	$\lambda_1 = \ \ 1 = 1$
2	$\lambda_2 = \ \ 3 = 1 + 2$
3	$\lambda_3 = \ \ 4 = 1 + 0 + 3$
4	$\lambda_4 = \ \ 7 = 1 + 2 + 0 + 4$
5	$\lambda_5 = \ \ 6 = 1 + 0 + 0 + 0 + 5$
6	$\lambda_6 = 12 = 1 + 2 + 3 + 0 + 0 + 6$
7	$\lambda_7 = \ \ 8 = 1 + 0 + 0 + 0 + 0 + 0 + 7$
8	$\lambda_8 = 15 = 1 + 2 + 0 + 4 + 0 + 0 + 0 + 8$
9	$\lambda_9 = 13 = 1 + 0 + 3 + 0 + 0 + 0 + 0 + 0 + 9$
10	$\lambda_{10} = 18 = 1 + 2 + 0 + 0 + 5 + 0 + 0 + 0 + 0 + 10$
11	$\lambda_{11} = 12 = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 11$
12	$\lambda_{12} = 28 = 1 + 2 + 3 + 4 + 0 + 6 + 0 + 0 + 0 + 0 + 0 + 12$

. . . . (19)

The non-zero terms of the n^{th} row of algorithm (19) are all the divisors of n. This is due to the correct vertical alignment of the sequences $h_\kappa(n)$ forming the columns of the algorithm in accordance with property (18).

Therefore, the coefficients λ_n obtained from Eqs (19) coincide with the sequence of Eq.(15) and are given by

$$\lambda_n = \sum_{\kappa=1}^{n} h_\kappa(n) \tag{20}$$

The sequences $h_\kappa(n)$ introduced by Eqs (17) can be expressed for $\kappa=1,2,3,\ldots$ compactly as follows:

$$h_\kappa(n) = \sum_{\ell=0}^{\kappa-1} e^{2\pi i \frac{n}{\kappa}\ell} \quad ; \quad n = \kappa, \kappa+1, \kappa+2, \ldots \tag{21}$$

Example:

$\kappa=1$ ($\ell=0$) ; $h_1(n)=1$; $n=1, 2, 3,\ldots$ (22)

$\kappa=2$ ($\ell=0,1$) ; $h_2(n)=1+e^{i\pi n}$; $n=2, 3, 4, 5, \ldots$ (23)

In particular, the first four terms of sequence $h_2(n)$ read:

$$h_2(2) = 1 + e^{2\pi i} = 2 \quad ; \quad h_2(3) = 1 + e^{3\pi i} = 0$$
$$h_2(4) = 1 + e^{4\pi i} = 2 \quad ; \quad h_2(5) = 1 + e^{5\pi i} = 0 \tag{24}$$

Therefore, $h_2(n) = (2, 0, 2, 0, \ldots)$; $n=2, 3, 4, 5, \ldots$ (25)

$\kappa=3$ ($\ell=0,1,2$) ; $h_3(n) = 1+e^{\frac{2\pi}{3}in} + e^{\frac{4\pi}{3}in}$; $n=3, 4, 5, 6, \ldots$ (26)

In particular, the first six terms of sequence $h_3(n)$ read:

$$h_3(3) = 1 + e^{2\pi i} + e^{4\pi i} = 3$$
$$h_3(4) = 1 + e^{\frac{8\pi}{3}i} + e^{\frac{16\pi}{3}i} = \frac{e^{8\pi i} - 1}{e^{\frac{8\pi}{3}i} - 1} = 0$$
$$h_3(5) = 1 + e^{\frac{10\pi}{3}i} + e^{\frac{20\pi}{3}i} = \frac{e^{10\pi i} - 1}{e^{\frac{10\pi}{3}i} - 1} = 0$$
$$h_3(6) = 1 + e^{4\pi i} + e^{8\pi i} = 3$$
$$h_3(7) = 1 + e^{\frac{14\pi}{3}i} + e^{\frac{28\pi}{3}i} = \frac{e^{14\pi i} - 1}{e^{\frac{14\pi}{3}i} - 1} = 0$$
$$h_3(8) = 1 + e^{\frac{16\pi}{3}i} + e^{\frac{32\pi}{3}i} = \frac{e^{16\pi i} - 1}{e^{\frac{16\pi}{3}i} - 1} = 0 \tag{27}$$

Therefore $h_3(n)=(3, 0, 0, 3, 0, 0, \ldots)$; $n=3, 4, 5, 6, \ldots$ (28)

We prove that $h_\kappa(n)$ defined by Eq.(21) has property (18):

If κ is a divisor of n viz. $\frac{n}{\kappa}=m$; m=2, 3, 4, … we have

$$h_\kappa(n) = 1 + e^{2\pi i m} + e^{4\pi i m} + \cdots + e^{2(\kappa-1)\pi i m} = \kappa \tag{29a}$$

If κ is not a divisor of n we have

$$h_\kappa(n) = 1 + e^{2\pi i\frac{n}{\kappa}} + \left(e^{2\pi i\frac{n}{\kappa}}\right)^2 + \left(e^{2\pi i\frac{n}{\kappa}}\right)^3 + \cdots + \left(e^{2\pi i\frac{n}{\kappa}}\right)^{\kappa-1} = \frac{e^{2\pi i n} - 1}{e^{2\pi i\frac{n}{\kappa}} - 1} = 0 \tag{29b}$$

Taking the real part of each term of the ℓ-sum of Eq.(21), we can also obtain for κ=1,2,3,… another form of $h_\kappa(n)$:

$$h_\kappa(n) = \sum_{\ell=0}^{\kappa-1} cos\left(2\pi\frac{n}{\kappa}\ell\right) \quad ; \quad n = \kappa, \kappa+1, \kappa+2, \ldots \tag{30}$$

The previous examples of Eq. (21) are also derived for Eq.(30) as follows:

$\kappa = 1$ (ℓ=0) ; $h_1(n) = 1$; n=1, 2, 3, … (31)

$\kappa = 2$ (ℓ=0,1) ; $h_2(n) = 1+\cos(\pi n)$; n=2, 3, 4, 5,… (32)

In particular, the first four terms of sequence $h_2(n)$ read:

$$h_2(2) = 1 + \cos(2\pi) = 2 \quad ; \quad h_2(3) = 1 + \cos(3\pi) = 0$$
$$h_2(4) = 1 + \cos(4\pi) = 2 \quad ; \quad h_2(5) = 1 + \cos(5\pi) = 0 \tag{33}$$

Therefore $h_2(n) = (2, 0, 2, 0, \ldots)$; n=2, 3, 4, 5, … (34)

$\kappa = 3$ (ℓ=0,1,2) ; $h_3(n) = 1+\cos\left(\frac{2\pi}{3}n\right) + \cos\left(\frac{4\pi}{3}n\right)$; n=3, 4, 5, 6, … (35)

In particular, the first six terms of sequence $h_3(n)$ read:

$$h_3(3) = 1+\cos(2\pi)+\cos(4\pi) = 3$$
$$h_3(4) = 1+\cos\left(\frac{8\pi}{3}\right)+\cos\left(\frac{16\pi}{3}\right) = 0$$
$$h_3(5) = 1+\cos\left(\frac{10\pi}{3}\right)+\cos\left(\frac{20\pi}{3}\right) = 0$$
$$h_3(6) = 1+\cos(4\pi)+\cos(8\pi) = 3$$
$$h_3(7) = 1+\cos\left(\frac{14\pi}{3}\right)+\cos\left(\frac{28\pi}{3}\right) = 0$$
$$h_3(8) = 1+\cos\left(\frac{16\pi}{3}\right)+\cos\left(\frac{32\pi}{3}\right) = 0 \tag{36}$$

Therefore $h_3(n) = (3, 0, 0, 3, 0, 0, \ldots)$; $n=3, 4, 5, 6, \ldots$ (37)

We prove that $h_\kappa(n)$ defined by Eq.(30) has property (18):

If κ is a divisor of n viz. $\frac{n}{\kappa}=m$; $m=2, 3, 4, \ldots$ we have

$$h_\kappa(n) = 1 + \cos(2\pi m) + \cos(4\pi m) + \ldots + \cos[2(\kappa-1)\pi m] = \kappa \tag{38a}$$

If κ is not a divisor of n so that $\sin\left(\pi\frac{n}{\kappa}\right) \neq 0$ we have [6]

$$h_\kappa(n) = \frac{sin(\pi n)}{sin\left(\pi\frac{n}{\kappa}\right)} cos\left[(\kappa-1)\pi\frac{n}{\kappa}\right] = 0 \tag{38b}$$

Note that the imaginary part of the ℓ-sum in Eq.(21) is equal to zero:

If κ is a divisor of n viz. $\frac{n}{\kappa}=m$; $m=2, 3, 4, \ldots$ we have

$$\sum_{\ell=1}^{\kappa-1} sin(2\pi m\ell) = sin(2\pi m) + sin(4\pi m) + \cdots + sin[2(\kappa-1)\pi m] = 0 \tag{39a}$$

If κ is not a divisor of n so that $\sin\left(\pi\frac{n}{\kappa}\right) \neq 0$ we use the formula [6]

$$\sum_{\ell=1}^{\kappa-1} sin\left(2\pi\frac{n}{\kappa}\ell\right) = \frac{sin(\pi n)}{sin\left(\pi\frac{n}{\kappa}\right)} sin\left[(\kappa-1)\pi\frac{n}{\kappa}\right] = 0 \tag{39b}$$

Hence, for $\kappa=1, 2, 3, \ldots$ both Eqs(21,30) provide the harmonic sequences $h_\kappa(n)$ of Eqs (17) forming the columns of algorithm (19). Introducing Eqs (21,30) into Eq.(20), we obtain the coefficients λ_n of the linear representation of Eq.(14):

$$\lambda_n = \sum_{\kappa=1}^{n}\sum_{\ell=0}^{\kappa-1} e^{2\pi i\frac{n}{\kappa}\ell} = \sum_{\kappa=1}^{n}\sum_{\ell=0}^{\kappa-1} cos\left(2\pi\frac{n}{\kappa}\ell\right) \tag{40}$$

in terms of harmonic functions. Note also that Eq.(40) is an exact formula for the sum of the divisors of n.

Let us calculate explicitly $\lambda_1, \lambda_2, \ldots, \lambda_8$ from Eq.(40):

n=1

$$\lambda_1 = \sum_{\kappa=1}^{1}\sum_{\ell=0}^{\kappa-1} cos\left(2\pi\frac{\ell}{\kappa}\right) = cos(2\pi 0) = 1 \tag{41}$$

n=2

$$\lambda_2 = \sum_{\kappa=1}^{2} \sum_{\ell=0}^{\kappa-1} cos\left(4\pi \frac{\ell}{\kappa}\right) = cos(4\pi 0) + \{cos(2\pi 0) + cos(2\pi)\} = 3 \tag{42}$$

n=3

$$\lambda_3 = \sum_{\kappa=1}^{3} \sum_{\ell=0}^{\kappa-1} cos\left(6\pi \frac{\ell}{\kappa}\right)$$

$$\lambda_3 = cos(6\pi 0) + \{cos(3\pi 0) + cos(3\pi)\} + \{cos(2\pi 0) + cos(2\pi) + cos(4\pi)\} = 4 \tag{43}$$

n=4

$$\lambda_4 = \sum_{\kappa=1}^{4} \sum_{\ell=0}^{\kappa-1} cos\left(8\pi \frac{\ell}{\kappa}\right)$$

$$\lambda_4 = cos(8\pi 0) + \{cos(4\pi 0) + cos(4\pi)\} + \left\{cos\left(\tfrac{8\pi}{3}0\right) + cos\left(\tfrac{8\pi}{3}\right) + cos\left(\tfrac{16\pi}{3}\right)\right\}$$
$$+\{cos(2\pi 0) + cos(2\pi) + cos(4\pi) + cos(6\pi)\} = 7 \tag{44}$$

n=5

$$\lambda_5 = \sum_{\kappa=1}^{5} \sum_{\ell=0}^{\kappa-1} cos\left(10\pi \frac{\ell}{\kappa}\right)$$

$$\lambda_5 = cos(10\pi 0) + \{cos(5\pi 0) + cos(5\pi)\} + \left\{cos\left(\tfrac{10\pi}{3}0\right) + cos\left(\tfrac{10\pi}{3}\right) + cos\left(\tfrac{20\pi}{3}\right)\right\}$$
$$+\left\{cos\left(\tfrac{5\pi}{2}0\right) + cos\left(\tfrac{5\pi}{2}\right) + cos(5\pi) + cos\left(\tfrac{15\pi}{2}\right)\right\}$$
$$+\{cos(2\pi 0) + cos(2\pi) + cos(4\pi) + cos(6\pi) + cos(8\pi)\} = 6 \tag{45}$$

n=6

$$\lambda_6 = \sum_{\kappa=1}^{6} \sum_{\ell=0}^{\kappa-1} cos\left(12\pi \frac{\ell}{\kappa}\right)$$

$$\lambda_6 = cos(12\pi 0) + \{cos(6\pi 0) + cos(6\pi)\} + \{cos(4\pi 0) + cos(4\pi) + cos(8\pi)\}$$
$$+\{cos(3\pi 0) + cos(3\pi) + cos(6\pi) + cos(9\pi)\}$$
$$+\left\{cos\left(\tfrac{12\pi}{5}0\right) + cos\left(\tfrac{12\pi}{5}\right) + cos\left(\tfrac{24\pi}{5}\right) + cos\left(\tfrac{36\pi}{5}\right) + cos\left(\tfrac{48\pi}{5}\right)\right\}$$
$$+\{cos(2\pi 0) + cos(2\pi) + cos(4\pi) + cos(6\pi) + cos(8\pi) + cos(10\pi)\} = 12 \tag{46}$$

n=7

$$\lambda_7=\sum_{\kappa=1}^{7}\sum_{\ell=0}^{\kappa-1}cos\left(14\pi\frac{\ell}{\kappa}\right)$$

$$\lambda_7=cos(14\pi 0)+\{cos(7\pi 0)+cos(7\pi)\}+\left\{cos\left(\frac{14\pi}{3}0\right)+cos\left(\frac{14\pi}{3}\right)+cos\left(\frac{28\pi}{3}\right)\right\}$$
$$+\left\{cos\left(\frac{7\pi}{2}0\right)+cos\left(\frac{7\pi}{2}\right)+cos(7\pi)+cos\left(\frac{21\pi}{2}\right)\right\}$$
$$+\left\{cos\left(\frac{14\pi}{5}0\right)+cos\left(\frac{14\pi}{5}\right)+cos\left(\frac{28\pi}{5}\right)+cos\left(\frac{42\pi}{5}\right)+cos\left(\frac{56\pi}{5}\right)\right\}$$
$$+\left\{cos\left(\frac{7\pi}{3}0\right)+cos\left(\frac{7\pi}{3}\right)+cos\left(\frac{14\pi}{3}\right)+cos(7\pi)+cos\left(\frac{28\pi}{3}\right)+cos\left(\frac{35\pi}{3}\right)\right\}$$
$$+\{cos(2\pi 0)+cos(2\pi)+cos(4\pi)+cos(6\pi)+cos(8\pi)+cos(10\pi)+cos(12\pi)\}=8 \tag{47}$$

n=8

$$\lambda_8=\sum_{\kappa=1}^{8}\sum_{\ell=0}^{\kappa-1}cos\left(16\pi\frac{\ell}{\kappa}\right)$$

$$\lambda_8=cos(16\pi 0)+\{cos(8\pi 0)+cos(8\pi)\}+\left\{cos\left(\frac{16\pi}{3}0\right)+cos\left(\frac{16\pi}{3}\right)+cos\left(\frac{32\pi}{3}\right)\right\}$$
$$+\{cos(4\pi 0)+cos(4\pi)+cos(8\pi)+cos(12\pi)\}$$
$$+\left\{cos\left(\frac{16\pi}{5}0\right)+cos\left(\frac{16\pi}{5}\right)+cos\left(\frac{32\pi}{5}\right)+cos\left(\frac{48\pi}{5}\right)+cos\left(\frac{64\pi}{5}\right)\right\}$$
$$+\left\{cos\left(\frac{8\pi}{3}0\right)+cos\left(\frac{8\pi}{3}\right)+cos\left(\frac{16\pi}{3}\right)+cos(8\pi)+cos\left(\frac{32\pi}{3}\right)+cos\left(\frac{40\pi}{3}\right)\right\}$$
$$+\left\{cos\left(\frac{16\pi}{7}0\right)+cos\left(\frac{16\pi}{7}\right)+cos\left(\frac{32\pi}{7}\right)+cos\left(\frac{48\pi}{7}\right)+cos\left(\frac{64\pi}{7}\right)+cos\left(\frac{80\pi}{7}\right)+cos\left(\frac{96\pi}{7}\right)\right\}$$
$$+\{cos(2\pi 0)+cos(2\pi)+cos(4\pi)+cos(6\pi)+cos(8\pi)+cos(10\pi)$$
$$+cos(12\pi)+cos(24\pi)\}=15 \tag{48}$$

Extending Eqs(19) of the algorithm and developing Eqs(40), the sequence λ_n up to n=50 reads:

λ_n={1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, 20, 42, 32, 36, 24, 60, 31, 42, 40, 56, 30, 72, 32, 63, 48, 54, 48, 91, 38, 60, 56, 90, 42, 96, 44, 84, 78, 72, 48, 124, 57, 93} (49)

Replacing the above coefficients into Eq.(14) we get p_{50} as a sum of 50 terms:

$$p_{50}=\frac{1}{50}\left\{\frac{\lambda_1 p_{49}}{173525}+\frac{\lambda_2 p_{48}}{441819}+\frac{\lambda_3 p_{47}}{499016}+\frac{\lambda_4 p_{46}}{738906}+\frac{\lambda_5 p_{45}}{534804}+\frac{\lambda_6 p_{44}}{902100}+\frac{\lambda_7 p_{43}}{506088}\right.$$
$$+\frac{\lambda_8 p_{42}}{797610}+\frac{\lambda_9 p_{41}}{579579}+\frac{\lambda_{10} p_{40}}{672084}+\frac{\lambda_{11} p_{39}}{374220}+\frac{\lambda_{12} p_{38}}{728420}+\frac{\lambda_{13} p_{37}}{302918}+\frac{\lambda_{14} p_{36}}{431448}$$
$$+\frac{\lambda_{15} p_{35}}{357192}+\frac{\lambda_{16} p_{34}}{381610}+\frac{\lambda_{17} p_{33}}{182574}+\frac{\lambda_{18} p_{32}}{325611}+\frac{\lambda_{19} p_{31}}{136840}+\frac{\lambda_{20} p_{30}}{235368}+\frac{\lambda_{21} p_{29}}{146080}$$
$$+\frac{\lambda_{22} p_{28}}{133848}+\frac{\lambda_{23} p_{27}}{72240}+\frac{\lambda_{24} p_{26}}{146160}+\frac{\lambda_{25} p_{25}}{60698}+\frac{\lambda_{26} p_{24}}{66150}+\frac{\lambda_{27} p_{23}}{50200}+\frac{\lambda_{28} p_{22}}{56112}+\frac{\lambda_{29} p_{21}}{23760}$$
$$+\frac{\lambda_{30} p_{20}}{45144}+\frac{\lambda_{31} p_{19}}{15680}+\frac{\lambda_{32} p_{18}}{24255}+\frac{\lambda_{33} p_{17}}{14256}+\frac{\lambda_{34} p_{16}}{12474}+\frac{\lambda_{35} p_{15}}{8448}+\frac{\lambda_{36} p_{14}}{12285}+\frac{\lambda_{37} p_{13}}{3838}$$
$$+\frac{\lambda_{38} p_{12}}{4620}+\frac{\lambda_{39} p_{11}}{3136}+\frac{\lambda_{40} p_{10}}{3780}+\frac{\lambda_{41} p_{9}}{1260}+\frac{\lambda_{42} p_{8}}{2112}+\frac{\lambda_{43} p_{7}}{660}+\frac{\lambda_{44} p_{6}}{924}+\frac{\lambda_{45} p_{5}}{546}+\frac{\lambda_{46} p_{4}}{360}$$
$$\left.+\frac{\lambda_{47} p_{3}}{144}+\frac{\lambda_{48} p_{2}}{248}+\frac{\lambda_{49} p_{1}}{57}+\frac{\lambda_{50} p_{0}}{93}\right\}=\frac{10211300}{50}=204226 \quad (50)$$

3. Matrix representation of partitions

Inversing linear Eqs(14), we obtain the partitions p_s in terms of $\lambda_1, \lambda_2, \ldots, \lambda_s$ in the form of the determinant of an s^2-matrix. The method can be developed by the following steps:

For $s=1,2$ and $p_0=1$, Eqs(14) read

$$1p_1+0p_2=\lambda_1$$
$$-\lambda_1 p_1+2p_2=\lambda_2 \quad (51)$$

where

$$D_2=\begin{vmatrix}1 & 0\\ -\lambda_1 & 2\end{vmatrix}=2! \quad (52)$$

Solution

$$p_2=\frac{1}{D_2}\begin{vmatrix}1 & \lambda_1\\ -\lambda_1 & \lambda_2\end{vmatrix}=\frac{1}{2!}\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}=2 \quad (53)$$

For s=1,2,3 and $p_0=1$, Eqs(14) read

$$1p_1+0p_2+0p_3=\lambda_1$$
$$-\lambda_1 p_1+2p_2+0p_3=\lambda_2$$
$$-\lambda_2 p_1-\lambda_1 p_2+3p_3=\lambda_3 \quad (54)$$

where

$$D_3=\begin{vmatrix}1 & 0 & 0\\ -\lambda_1 & 2 & 0\\ -\lambda_2 & -\lambda_1 & 3\end{vmatrix}=3! \quad (55)$$

Solution

$$p_3=\frac{1}{D_3}\begin{vmatrix}1 & 0 & \lambda_1\\ -\lambda_1 & 2 & \lambda_2\\ -\lambda_2 & -\lambda_1 & \lambda_3\end{vmatrix}=\frac{1}{3!}\begin{vmatrix}1 & 0 & 1\\ -1 & 2 & 3\\ -3 & -1 & 4\end{vmatrix}=3 \quad (56)$$

For s=1,2,3,4 and $p_0=1$, Eqs(14) read

$$1p_1+0p_2+0p_3+0p_4=\lambda_1$$
$$-\lambda_1 p_1+2p_2+0p_3+0p_4=\lambda_2$$
$$-\lambda_2 p_1-\lambda_1 p_2+3p_3+0p_4=\lambda_3$$
$$-\lambda_3 p_1-\lambda_2 p_2-\lambda_1 p_3+4p_4=\lambda_4 \quad (57)$$

where

$$D_4=\begin{vmatrix}1 & 0 & 0 & 0\\ -\lambda_1 & 2 & 0 & 0\\ -\lambda_2 & -\lambda_1 & 3 & 0\\ -\lambda_3 & -\lambda_2 & -\lambda_1 & 4\end{vmatrix}=4! \quad (58)$$

Solution

$$p_4=\frac{1}{D_4}\begin{vmatrix}1 & 0 & 0 & \lambda_1\\ -\lambda_1 & 2 & 0 & \lambda_2\\ -\lambda_2 & -\lambda_1 & 3 & \lambda_3\\ -\lambda_3 & -\lambda_2 & -\lambda_1 & \lambda_4\end{vmatrix}=\frac{1}{4!}\begin{vmatrix}1 & 0 & 0 & 1\\ -1 & 2 & 0 & 3\\ -3 & -1 & 3 & 4\\ -4 & -3 & -1 & 7\end{vmatrix}=5 \quad (59)$$

For s=1,2,3,4,5 and $p_0=1$, Eqs(14) read

$$1p_1+0p_2+0p_3+0p_4+0p_5=\lambda_1$$
$$-\lambda_1 p_1+2p_2+0p_3+0p_4+0p_5=\lambda_2$$
$$-\lambda_2 p_1-\lambda_1 p_2+3p_3+0p_4+0p_5=\lambda_3$$
$$-\lambda_3 p_1-\lambda_2 p_2-\lambda_1 p_3+4p_4+0p_5=\lambda_4$$
$$-\lambda_4 p_1-\lambda_3 p_2-\lambda_2 p_3-\lambda_1 p_4+5p_5=\lambda_5 \quad (60)$$

where

$$D_5 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 2 & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & 3 & 0 & 0 \\ -\lambda_3 & -\lambda_2 & -\lambda_1 & 4 & 0 \\ -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & 5 \end{vmatrix} = 5! \tag{61}$$

Solution

$$p_5 = \frac{1}{D_5}\begin{vmatrix} 1 & 0 & 0 & 0 & \lambda_1 \\ -\lambda_1 & 2 & 0 & 0 & \lambda_2 \\ -\lambda_2 & -\lambda_1 & 3 & 0 & \lambda_3 \\ -\lambda_3 & -\lambda_2 & -\lambda_1 & 4 & \lambda_4 \\ -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & \lambda_5 \end{vmatrix} = \frac{1}{5!}\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 & 3 \\ -3 & -1 & 3 & 0 & 4 \\ -4 & -3 & -1 & 4 & 7 \\ -7 & -4 & -3 & -1 & 6 \end{vmatrix} = 7 \tag{62}$$

Clearly, the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_{s-1}$ provide for any s the determinant

$$D_s = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -\lambda_1 & 2 & 0 & \ldots & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & 3 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_{s-3} & -\lambda_{s-4} & -\lambda_{s-5} & \ldots & s-2 & 0 & 0 \\ -\lambda_{s-2} & -\lambda_{s-3} & -\lambda_{s-4} & \ldots & -\lambda_1 & s-1 & 0 \\ -\lambda_{s-1} & -\lambda_{s-2} & -\lambda_{s-3} & \ldots & -\lambda_2 & -\lambda_1 & s \end{vmatrix} = s! \tag{63}$$

and the general solution for the partition p_s reads

$$p_s = \frac{1}{s!}\begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \lambda_1 \\ -\lambda_1 & 2 & 0 & \ldots & 0 & 0 & \lambda_2 \\ -\lambda_2 & -\lambda_1 & 3 & \ldots & 0 & 0 & \lambda_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\lambda_{s-3} & -\lambda_{s-4} & -\lambda_{s-5} & \ldots & s-2 & 0 & \lambda_{s-2} \\ -\lambda_{s-2} & -\lambda_{s-3} & -\lambda_{s-4} & \ldots & -\lambda_1 & s-1 & \lambda_{s-1} \\ -\lambda_{s-1} & -\lambda_{s-2} & -\lambda_{s-3} & \ldots & -\lambda_2 & -\lambda_1 & \lambda_s \end{vmatrix} \tag{64}$$

Example: Using coefficients λ_n [Eq.(49)] up to $n=10$, Eq.(64) gives

$$p_{10} = \frac{1}{10!}\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 \\ -\lambda_1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \\ -\lambda_2 & -\lambda_1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 \\ -\lambda_3 & -\lambda_2 & -\lambda_1 & 4 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & 5 & 0 & 0 & 0 & 0 & \lambda_5 \\ -\lambda_5 & -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & 6 & 0 & 0 & 0 & \lambda_6 \\ -\lambda_6 & -\lambda_5 & -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & 7 & 0 & 0 & \lambda_7 \\ -\lambda_7 & -\lambda_6 & -\lambda_5 & -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & 8 & 0 & \lambda_8 \\ -\lambda_8 & -\lambda_7 & -\lambda_6 & -\lambda_5 & -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & 9 & \lambda_9 \\ -\lambda_9 & -\lambda_8 & -\lambda_7 & -\lambda_6 & -\lambda_5 & -\lambda_4 & -\lambda_3 & -\lambda_2 & -\lambda_1 & \lambda_{10} \end{vmatrix}$$

$$= \frac{1}{10!}\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ -3 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ -4 & -3 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 7 \\ -7 & -4 & -3 & -1 & 5 & 0 & 0 & 0 & 0 & 6 \\ -6 & -7 & -4 & -3 & -1 & 6 & 0 & 0 & 0 & 12 \\ -12 & -6 & -7 & -4 & -3 & -1 & 7 & 0 & 0 & 8 \\ -8 & -12 & -6 & -7 & -4 & -3 & -1 & 8 & 0 & 15 \\ -15 & -8 & -12 & -6 & -7 & -4 & -3 & -1 & 9 & 13 \\ -13 & -15 & -8 & -12 & -6 & -7 & -4 & -3 & -1 & 18 \end{vmatrix}$$

$$= \frac{152409600}{3628800} = 42 \qquad (65)$$

Since the coefficients λ_n have already been expressed in terms of harmonic sequences by Eqs (19,40), it is clear that the s^2-matrix representation of p_s in terms of λ_n [Eq.(64)] establishes a new relation between partitions and harmonic functions. Note that previous work [3,4] has already shown that partitions can be represented by harmonic integrals [Eq.(2)].

4. Conclusions

We study the linear representation of the partitions p_s [Eq.(14)] where each coefficient λ_n is the sum of divisors of the number n. It is shown that the coefficients λ_n are universal numbers [Eq.(15)] obtained by a well defined triangular algorithm [Eqs.(19)].

The columns of this algorithm are harmonic sequences $h_\kappa(n)$ defined by Eqs(17) and given explicitly by Eqs(21,30) so that λ_n can be expressed in terms of harmonic functions by Eqs(40). Inversing the linear Eqs(14), it is also shown that the partitions p_s depend on $\lambda_1, \lambda_2, \ldots, \lambda_s$ by an s^2-matrix [Eq.(64)], establishing a new relation between partitions and harmonic functions.

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