Tiling the Plane with $k$-Gons

VOLKER WILHELM THÜREY
Bremen, Germany *

September 20, 2022

MSC-2020: 51
Keywords: Tiling; plane

Abstract

We present a way to tile the plane by $k$-gons for a fixed $k$. We use usual regular 6-gons by putting some in a row and fill them with $k$-gons. We use only one or two or four different $k$-gons.

1 Introduction

It is a widespread opinion that one can tile the plane $\mathbb{R}^2$ only with triangles, squares and regular 6-gons. This is wrong. A further possibility is to put regular 6-gons in a row. We think that it is useful to repeat the definition of a simple polygon.

A simple polygon with $k$ vertices consists of $k$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}), (x_k, y_k)$ called vertices, and the straight lines between the vertices, where $k > 2$. It is homeomorphic to a circle. We demand that there are no three consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ for $1 \leq i \leq k - 2$. Also we demand that the three points $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ and $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ are not collinear.

We call this just described simple polygon a $k$-gon.

Definition 1. Let $t$ be any natural number. We call a simple polygon a $t$ row 6 $-$ gon, if $t$ regular 6-gons are put in a row.

See the example in Figure 2. There we show a 5 row 6 $-$ gon.

Note that a 1 row 6 $-$ gon is just a regular 6-gon.

Proposition 1. One can tile the plane with $t$ row 6 $-$ gons for all fixed $t$.

Proof. Trivial.

Proposition 2. A $t$ row 6 $-$ gon has $2 + 4 \cdot t$ vertices.

Proof. Easy.
2 Tiling

Theorem 1. Let \( k \) be a natural number larger than 2. There exists for all \( k \) a tiling of \( \mathbb{R}^2 \) with \( k \)-gons.

Proof. For \( k = 3 \) and \( k = 4 \) and \( k = 6 \) the theorem is well-known. For \( k = 5 \) please see Figure 1. We also can take a regular 6-gon instead of a rectangle. We cut it into congruent halves. Now let \( k \) be a natural number larger than 6.

Lemma 1. It holds \( k - 2 \equiv p \mod 4 \), where \( p \in \{0, 1, 2, 3\} \).

Proof. Well-known. □

We discuss the four possibilities.

• Possibility 1: \( p = 0 \). In this easy case we take the polygon \( t \) row 6-gon as a \( k \)-gon.
  We get \( t \) from the equation \( k - 2 = 4 \cdot t \).
  The sequence of the numbers of \( k \) is 10, 14, 18, \ldots. By Proposition 2 the number of vertices of a \( t \) row 6-gon is \( 2 + 4 \cdot t \). This is \( k \).

• Possibility 2: \( p = 1 \). In this case we had to calculate. We take a \((4 \cdot t + 1)\) row 6-gon. It is filled with four \( k \)-gons. We use the vertices of the \((4 \cdot t + 1)\) row 6-gon as vertices of the four \( k \)-gons. See Figure 2. Note that the four \( k \)-gons have three edges in common. Therefore we have to subtract 6 from the number of the vertices.
  We get \( t \) from the equation \( k - 2 = 4 \cdot t + 1 \).
  The number of vertices both for a \((4 \cdot t + 1)\) row 6-gon and 4 \( k \)-gons \(- 6 \) is \( 16 \cdot t + 6 \).
  The sequence of the numbers of \( k \) is 7, 11, 15, 19, \ldots.

• Possibility 3: \( p = 2 \). We take a \((2 \cdot t + 1)\) row 6-gon. It is filled with two \( k \)-gons. We get \( t \) from \( k - 2 = 4 \cdot t + 2 \).
  The sequence of the numbers of \( k \) is 8, 12, 16, 20, \ldots. Two \( k \)-gons altogether have \( 8 + 8 \cdot t \) vertices. See Figure 3. Note that if two \( k \)-gons tile a polygon a pair of vertices is canceled, since the \( k \)-gons have a common edge. Therefore they have \( 6 + 8 \cdot t \) vertices. This is also the number of vertices of a \((2 \cdot t + 1)\) row 6-gon.

• Possibility 4: \( p = 3 \). We take a \((4 \cdot t + 3)\) row 6-gon. It is filled with four \( k \)-gons.
  We get \( t \) from \( k - 2 = 4 \cdot t + 3 \).
  The common number of vertices is \( 16 \cdot t + 14 \).
  The sequence of the numbers of \( k \) is 9, 13, 17, 21, \ldots.

The theorem is proved. □

It follows three figures.
Tiling the Plane with $k$-Gons

Figure 1:

Figure 2:

See below a 5 row 6 – gon, which is subdivided in four 7-gons.
We see also three edges. Each is a common edge of two 7-gons.

Figure 3:

On the right hand we see a 3 row 6 – gon.
It consists of two 8-gons.

Acknowledgement: We thank Gavin Crosby for informations