A Problem on Sums of Powers

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1 Problem Statement

The motivation for the problem to be presented is based on Newton’s identity which states that

\[
\sum_{i=k-n}^{k} (-1)^{i-1} e_{k-i}(x_1, \ldots, x_n)p_i(x_1, \ldots, x_n) = 0,
\]
for all \( k > n \geq 1 \), where \( e_k(x_1, \ldots, x_n) \), for \( k \geq 0 \), is the sum of all distinct products of \( k \) distinct variables, and \( p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k \), where \( k \geq 1 \).

2 Problem

For \( n \geq 3 \), if

\[
\sum_{k=1}^{n-1} x_k = y_1 = 2n - 2,
\]
\[
\sum_{k=1}^{n-1} x_k^2 = y_2 = 3n - 2,
\]
\[
\sum_{k=1}^{n-1} x_k^3 = y_3 = 4n - 2,
\]
\[\vdots\]

\[
\sum_{k=1}^{n-1} x_k^{n-1} = y_{n-1} = n^2 - 2,
\]
then

\[
\sum_{k=1}^{n-1} x_k^n = y_n = n^2 + n - 2.
\]

Essentially, \( y_1, y_2, y_3 \ldots y_n \) are in arithmetic progression(A.P) whose first term is \( 2n - 2 \) and common difference is \( n \).
3 Special Cases

CASE n=3:

We see when \( n = 3 \) that
\[
\begin{align*}
  x_1 + x_2 &= 4, \\
  x_1^2 + x_2^2 &= 7.
\end{align*}
\]

Now, we need to show that \( x_1^3 + x_2^3 = 10 \).

We know that
\[
x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2),
\]
\[
(x_1^3 + x_2^3) = 64 - 12x_1x_2.
\]

(1)

Also, we know that
\[
(x_1 + x_2)^2 - (x_1^2 + x_2^2) = 2x_1x_2,
\]
\[
4^2 - 7 = 2x_1x_2,
\]
\[
x_1x_2 = \frac{9}{2}.
\]

(2)

Putting (2) in (1), we see that
\[
x_1^3 + x_2^3 = 10.
\]

And indeed,
\[
\left( \frac{4 + i\sqrt{2}}{2} \right)^3 + \left( \frac{4 - i\sqrt{2}}{2} \right)^3 = 10.
\]

CASE n=4:

We see when \( n = 4 \) that
\[
\begin{align*}
  x_1 + x_2 + x_3 &= 6, \\
  x_1^2 + x_2^2 + x_3^2 &= 10, \\
  x_1^3 + x_2^3 + x_3^3 &= 14.
\end{align*}
\]

Now, we need to show that \( x_1^4 + x_2^4 + x_3^4 = 18 \).

We know that
\[
x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_1x_3),
\]
\[
10 = 36 - 2(x_1x_2 + x_2x_3 + x_1x_3),
\]

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\[ x_1x_2 + x_2x_3 + x_1x_3 = 13, \]
\[ (x_1x_2 + x_2x_3 + x_1x_3)^2 = 169, \]
\[ x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 + 2x_1x_2x_3(x_1 + x_2 + x_3) = 169, \]
\[ x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 + 12x_1x_2x_3 = 169, \]
\[ \frac{(x_1^2 + x_2^2 + x_3^2)^2 - (x_1^4 + x_2^4 + x_3^4)}{2} + 12x_1x_2x_3 = 169. \hspace{1cm} (3) \]

Also, we know that
\[ x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1x_3), \]
we have
\[ 14 - 3x_1x_2x_3 = 6(10 - 13), \]
\[ x_1x_2x_3 = \frac{32}{3}. \hspace{1cm} (4) \]

Putting (4) into (3), we have
\[ \frac{10^2 - (x_1^4 + x_2^4 + x_3^4)}{2} + 12 \left( \frac{32}{3} \right) = 169, \]
\[ x_1^4 + x_2^4 + x_3^4 = 18. \]

And indeed, if
\[ \alpha = \sqrt[3]{\frac{1}{3} + \frac{2\sqrt{3}}{9}} + \sqrt[3]{\frac{1}{3} - \frac{2\sqrt{3}}{9}} + 2, \]
then
\[ \alpha^2 + \left( -3\alpha(\alpha - 6) + i\sqrt{384\alpha - (3\alpha(\alpha - 6))^2} \right) \frac{6\alpha}{1} = 6, \]
\[ \alpha^2 + \left( -3\alpha(\alpha - 6) + i\sqrt{384\alpha - (3\alpha(\alpha - 6))^2} \right) \frac{6\alpha}{2} = 10, \]
\[ \alpha^3 + \left( -3\alpha(\alpha - 6) + i\sqrt{384\alpha - (3\alpha(\alpha - 6))^2} \right) \frac{6\alpha}{3} = 14, \]
\[ \alpha^4 + \left( -3\alpha(\alpha - 6) + i\sqrt{384\alpha - (3\alpha(\alpha - 6))^2} \right) \frac{6\alpha}{4} = 18. \]
4 Conclusion

We have proven the special cases $n = 3$ and $n = 4$ of our claim. To prove higher cases of $n$ using the method used here (which is inefficient to prove all the cases of $n$) is long and complicated. We hope a profound technique will be used to prove every integer $n$.

References