A simple method to compute sums of powers and polynomials

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Abstract

Several formulas are known to compute

$$\sum_{k=1}^{n} k^p$$

where $p \in \mathbb{Z}^+$, such as the Faulhaber's formula

$$\frac{1}{p+1}\sum_{k=1}^{p+1}(-1)^{\delta_{k\,p}}\binom{p+1}{k}\,B_{p+1-k}\,n^k$$

or the following double series

$$\sum_{i=1}^{p}\sum_{j=0}^{i-1}(-1)^j\,(i-j)^p\binom{n+p-i+1}{n-i}\binom{p+1}{j}$$

Consequently, the sum of a polynomial

$$\sum_{p=0}^{m-1} a_p k^p$$

where $m \in \mathbb{Z}^{++}$, $a_p \in \mathbb{R}$, i.e.

$$\sum_{k=1}^{n}\sum_{p=0}^{m-1} a_p\,k^p$$

would be computed as a linear combination with coefficients a_p's of the above formulas

$$\sum_{p=0}^{m-1} a_p\,(\sum_{k=1}^{n} k^p)$$

This paper introduces a simpler method to compute sums of powers and polynomials, which can be done by hand, and efficiently implemented in software.

keywords: sum of powers; polynomial; computation method

A summation identity

Central to the simple method is the following summation identity

$$\sum_{k=1}^{n}\prod_{p=0}^{m-1}(k+p) = \frac{\prod_{p=0}^{m}(n+p)}{m+1}$$

For example, for small values of m = 0, 1, 2, 3, 4, and 5,

$$\sum_{k=1}^{n} 1 = \frac{n}{1}$$

$$\sum_{k=1}^{n} k = \frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k\,(k+1) = \frac{n\,(n+1)(n+2)}{3}$$

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n\,(n+1)(n+2)(n+3)}{4}$$

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3) = \frac{n\,(n+1)(n+2)(n+3)(n+4)}{5}$$

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3)(k+4) = \frac{n\,(n+1)(n+2)(n+3)(n+4)(n+5)}{6}$$

The identity follows a simple pattern, and it is very easy to remember.

Coefficients Triangle

The coefficients of the right-hand expression $\frac{\prod_{p=0}^{m}(n+p)}{m+1}$ for each m can be constructed in a triangle, similar to how the Pascal triangle is constructed:

$$\frac{1}{1}$$

$$\frac{1}{2} \quad \frac{1}{2}$$

$$\frac{1}{3} \quad \frac{3}{3} \quad \frac{2}{3}$$

$$\frac{1}{4} \quad \frac{6}{4} \quad \frac{11}{4} \quad \frac{6}{4}$$

$$\frac{1}{5} \quad \frac{10}{5} \quad \frac{35}{5} \quad \frac{50}{5} \quad \frac{24}{5}$$

$$\frac{1}{6} \quad \frac{15}{6} \quad \frac{85}{6} \quad \frac{225}{6} \quad \frac{274}{6} \quad \frac{120}{6}$$

$$\frac{1}{7} \quad \frac{21}{7} \quad \frac{175}{7} \quad \frac{735}{7} \quad \frac{1624}{7} \quad \frac{1764}{7} \quad \frac{720}{7}$$

For example, the fourth row corresponds to the following:

$$\frac{n\,(n+1)(n+2)(n+3)}{4} = \frac{1}{4}n^4 + \frac{6}{4}n^3 + \frac{11}{4}n^2 + \frac{6}{4}n$$

For each row, the denominators are natural numbers 1, 2, 3, ….

Each numerator is the sum of the product of the numerator and denominator of the left side number above, and the numerator of the right-hand number above. For example, in row 5,

$$\frac{1}{5} = \frac{0 \times 5 + 1}{5}$$

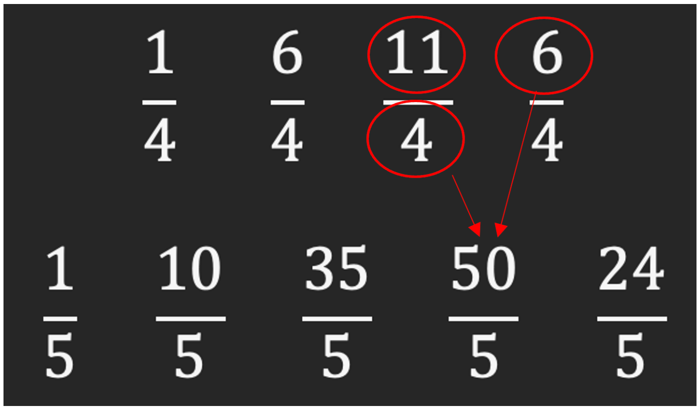
$$\frac{10}{5} = \frac{1 \times 4 + 6}{5}$$

$$\frac{35}{5} = \frac{6 \times 4 + 11}{5}$$

$$\frac{50}{5} = \frac{11 \times 4 + 6}{5}$$

$$\frac{24}{5} = \frac{6 \times 4 + 0}{5}$$

For example, for $\frac{50}{5}$,



The Computation Method

To compute the sum of powers and polynomials, we will write it as a linear combination of the left expression of the identity $\sum_{k=1}^{n}\prod_{p=0}^{m-1}(k+p)$, with the help of a triangle consisting of only the numerators:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 3 \quad 2 \\
1 \quad 6 \quad 11 \quad 6 \\
1 \quad 10 \quad 35 \quad 50 \quad 24 \\
1 \quad 15 \quad 85 \quad 225 \quad 274 \quad 120 \\
1 \quad 21 \quad 175 \quad 735 \quad 1624 \quad 1764 \quad 720
\end{array}
$$

For example, to evaluate

$$\sum_{k=1}^{n} k^5$$

we start with

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3)(k+4)$$

Using the coefficients from the triangle, it equals to

$$\sum_{k=1}^{n} (1k^5 + 10k^4 + 35k^3 + 50k^2 + 24k^1)$$

With $10k^4$, the next term will be multiplied by **-10**

$$-10\sum_{k=1}^{n} k(k+1)(k+2)(k+3)$$

Multiplying **-10** to the coefficients 1 6 11 6 from the triangle, it equals to

$$\sum_{k=1}^{n}(-10k^4 - 60k^3 - 110k^2 - 60k^1)$$

With $35k^3 - 60k^3 = -25k^3$, the next term will be multiplied by **25**

$$25\sum_{k=1}^{n} k(k+1)(k+2)$$

Multiplying **25** to the coefficients 1 3 2 from the triangle, it equals to

$$\sum_{k=1}^{n}(25k^3 + 75k^2 + 50k^1)$$

With $50k^2 - 110k^2 + 75k^2 = 15k^2$, the next term will be multiplied by **-15**

$$-15\sum_{k=1}^{n} k(k+1)$$

Multiplying **-15** to the coefficients 1 1 from the triangle, it equals to

$$\sum_{k=1}^{n}(-15k^2 - 15k^1)$$

With $24k^1 - 60k^1 + 50k^1 - 15k^1 = -k^1$, the next term will be multiplied by **1**

$$1\sum_{k=1}^{n} k$$

Practically, the above computations will be performed in a matrix of the coefficients.

Therefore, the sum is a linear combination of the right-hand expressions $\frac{\prod_{p=0}^{m}(n+p)}{m+1}$ multiplied by the same coefficients.

$$\sum_{k=1}^{n} k^5 = \frac{n\,(n+1)(n+2)(n+3)(n+4)(n+5)}{6} - 10\,\frac{n\,(n+1)(n+2)(n+3)(n+4)}{5} + 25\,\frac{n\,(n+1)(n+2)(n+3)}{4} - 15\,\frac{n\,(n+1)(n+2)}{3} + \frac{n\,(n+1)}{2}$$

With the help of the triangle of coefficients (with the denominators), the above expression can be easily calculated to the expected result of

$$\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

Again, practically, the computation should be performed in a matrix of the coefficients.

The method can easily handle sums of arbitrary polynomials as well as powers.

Conclusion

Given the ease of constructing the coefficient triangles and the simplicity of the summation identity, this method is simpler and practical to use than the alternatives.