On the Navier and Stokes equations

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Abstract

Here is proposed a solution to this three dimensions Navier-Stokes equations problem, described in cartesian coordinates. Are given the expression of a velocity defined as the initial one, conditions verifications, calculus description, and the proof of the existence of the pressure mathematical solution.

Introduction

The three dimensions Navier-Stokes equations problem concerns here an incompressible fluid motion. Respecting physical reasonable conditions of bounded energy and regularity, the motion can be described with the fluid velocity vector \( u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \) and its pressure \( p(x,t) \) defined for position \( x \in \mathbb{R}^3 \) and time \( t \geq 0 \). These two mathematical functions may be deducted from the following equations linking \( u_1, u_2, u_3, p \), and the fluid kinematic viscosity \( \nu \).

In cartesian coordinates, the Navier-Stokes equations are, if we take the force identically zero as proposed in the Clay Mathematical Institute official problem description (same as in the abstract), for \( t \geq 0 \):

\[
\text{Momentum equations, for each coordinate } i : \quad \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i + \frac{\partial p}{\partial x_i} = 0 \]

where \( \nu \) is the viscosity.

\[
\text{Incompressibility equation :} \quad \text{div } u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,
\]

and we define the initial velocity \( u(x,0) = u^0(x) \) as a given, \( C^\infty(\mathbb{R}^3) \) divergence-free vector field, which implies that \( \text{div } u^0(x) = 0 \).

For physically reasonable solutions, we need solutions to respect some conditions, for example, we want to make sure that \( u(x,t) \) does not grow large as \( |x| \to \infty \). So we restrict attention to initial velocities that satisfy \( |\partial_x^\alpha u^0(x)| \leq C_{\alpha K} (1 + |x|)^{-K} \) for \( x \in \mathbb{R}^3 \), for any \( \alpha \in \mathbb{N}^3 \) and any \( K \in \mathbb{R} \), \( C_{\alpha K} \) being a constant depending on \( \alpha \) and \( K \) (1). The pressure \( p \) and the velocity \( u \) have to be in \( C^\infty(\mathbb{R}^3 \times [0, \infty)) \), and the bounded energy condition is respected if there exists a real constant \( C \) such that \( \int_{\mathbb{R}^3} |u(x,t)|^2 \, dx < C \) for all \( t \geq 0 \).
Proposition

We define:

\[ \forall x \in \mathbb{R}^3 \forall t \geq 0 \ u(x, t) = u(x, 0) \]

For \( t \geq 0, \) \( \text{div} \ u = 0, \) because \( \text{div} \ u^o(x) = 0. \) The fluid is incompressible.

Let's verify now that \( \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx \) is upper-bounded. We have:

\[ \int_{\mathbb{R}^3} u_1(x, t)^2 + u_2(x, t)^2 + u_3(x, t)^2 \, dx. \]

For any \( \alpha \) in \( \mathbb{N}^3, \) any real \( K, \) \( |\partial_\alpha u^o(x)| \leq C_{\alpha K}(1 + |x|)^{-K}. \) Hence, because \( \forall x \in \mathbb{R}^3 (1 + |x|)^{-K} > 0, \) \( C_{\alpha K} \geq 0. \) And we have:

\[ |u^o(x)|^2 \leq C_{\alpha 0}^2(1 + |x|)^{4}. \]

\[ \Rightarrow \int_{\mathbb{R}^3} |u^o(x)|^2 \, dx \leq \int_{\mathbb{R}^3} \frac{C_{\alpha 0}^2}{(1 + |x|)^4} \, dx \]

\[ \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{C_{\alpha 0}^2}{(1 + |x|^2)^2} \, dx_1 \right) \, dx_2 \right) \, dx_3 \quad (\forall x \in \mathbb{R}^3 (1 + |x|^2) \geq (1 + |x|^2)^2 > 0) \]

\[ = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{\pi C_{\alpha 0}^2}{(1 + x_1^2 + x_2^2 + x_3^2)^2} \, dx_1 \right) \, dx_2 \right) \, dx_3 \]

\[ = \int_{\mathbb{R}} \pi C_{\alpha 0}^2 \frac{dx_3}{(1 + x_3^2)^2} \]

\[ = \pi^2 C_{\alpha 0}^2 \]

(Calculations helped by http://www.wolframalpha.com/)

So \( \int_{\mathbb{R}^3} |u^o(x)|^2 \, dx \leq \pi^2 C_{\alpha 0}^2. \) Consequently \( \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx \) is upper-bounded.

Bounded energy condition is also verified.

These three Navier-Stokes equations, for \( x \in \mathbb{R}^3, t \geq 0, i \in \{1, 2, 3\} : \)

\[ \frac{\partial p}{\partial x_i} = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu \Delta u_i \]

are linking \( C^\infty(\mathbb{R}^3 \times [0, \infty)) \) functions (for \( i \in \{1, 2, 3\}, -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu \Delta u_i \) is a \( C^\infty(\mathbb{R}^3 \times [0, \infty)) \) function because \( u_i \) is a \( C^\infty(\mathbb{R}^3 \times [0, \infty)) \) function), so \( p \) exists, and \( p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \)
Conclusion

For any smooth, divergence-free initial velocity vector field, satisfying condition (1), it has been verified that the presented fluid model satisfies all the conditions to solve this three dimensions Navier and Stokes equations problem, by proving the existence and smoothness of Navier-Stokes solutions on $\mathbb{R}^3$. Nevertheless, all solutions of the Navier-Stokes equations three dimensions problem have probably not been presented in this paper.