A Formula of the Dirichlet Character Sum

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Abstract. In this paper, We use the Fourier series expansion of real variables function, We give a formula to calculate the Dirichlet character sum, and four special examples are given.

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The calculation of the Dirichlet character sum is very important in the number theory. This paper uses the Fourier series expansion of the functions, we give a general formula for calculating the Dirichlet character sum, Then, four examples are given to illustrate.

In this paper, \( \chi_q \) denote the Dirichlet primitive character of mod \( q \), If \( f(x) \) is a real function, we write

\[
\hat{f}(x) = \frac{f(x + 0) + f(x - 0)}{2}
\]

and

\[
G(n, \chi_q) = \sum_{k=1}^{q-1} \chi_q(k) e\left(\frac{k}{q}n\right)
\]

\[
\tau(\chi_q) = \sum_{k=1}^{q-1} \chi_q(k) e\left(\frac{k}{q}\right)
\]

where \( e(x) = e^{2\pi ix} \)

First, let’s give some lemmas.

**Lemma 1.** If \( \chi_q \) is the primitive character of module \( q \), then we have
\[ G(n, \chi_q) = \sum_{k=1}^{q-1} \chi_q(k)e\left(\frac{kn}{q}\right) = \bar{\chi}_q(n)\tau(\chi_q) \]

see page 287 of references[1].

**Lemma 2.** If \( \chi_q \) is the primitive real character of module \( q \), then we have

\[
\tau(\chi_q) = \begin{cases} 
\sqrt{q} & \text{if } \chi_q(-1) = 1 \\
i\sqrt{q} & \text{if } \chi_q(-1) = -1 
\end{cases}
\]

see page 167 of references[2].

**Lemma 3.** If the function \( f(x) \) is defined in the interval \([0, 1]\) and satisfies the Dirichlet condition, then we have

\[
f^*(x) = \int_0^1 f(t)dt + \sum_{n=-\infty}^{\infty} \exp(2\pi inx) \int_0^1 f(t) \exp(-2\pi int)dt
\]

see page 421 of references[3].

Now, we give the theorem of this paper.

**Theorem.** If the function \( f(x) \) is defined in the interval \([0, 1]\) and satisfies the Dirichlet condition, then, when \( \chi_q(-1) = 1 \), we have

\[
\sum_{k=1}^{q-1} \chi_q(k)f^*\left(\frac{k}{q}\right) = 2\tau(\chi_q) \sum_{n=1}^{\infty} \bar{\chi}_q(n) \int_0^1 f(t) \cos(2\pi nt)dt
\]

when \( \chi_q(-1) = -1 \), we have
\[
\sum_{k=1}^{q-1} \chi_q(k)f^*\left(\frac{k}{q}\right) = -2i\tau(\chi_q) \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 f(t) \sin(2\pi nt) dt
\]

**Proof.** By lemma 3, we have

\[
f^*(x) = \int_0^1 f(t) dt + \sum_{n=-\infty, n \neq 0}^{\infty} \exp(2\pi inx) \int_0^1 f(t) \exp(-2\pi int) dt
\]

we take \(x = \frac{k}{q}, 1 \leq k \leq q - 1\), then

\[
f^*\left(\frac{k}{q}\right) = \int_0^1 f(t) dt + \sum_{n=-\infty, n \neq 0}^{\infty} \exp\left(2\pi i \frac{nk}{q}\right) \int_0^1 f(t) \exp(-2\pi int) dt
\]

Multiply the above formula by \(\chi_q(k)\), then sum over \(k\), we have

\[
\sum_{k=1}^{q-1} \chi_q(k)f^*\left(\frac{k}{q}\right) = \sum_{k=1}^{q-1} \chi_q(k) \int_0^1 f(t) dt
\]

\[
+ \sum_{n=-\infty}^{\infty} \sum_{k=1}^{q-1} \chi_q(k) \exp\left(2\pi i \frac{nk}{q}\right) \int_0^1 f(t) \exp(-2\pi int) dt
\]

By Lemma 1,

\[
\sum_{k=1}^{q-1} \chi_q(k) \exp\left(2\pi i \frac{nk}{q}\right) = \overline{\chi}_q(n)\tau(\chi_q) \quad \text{and} \quad \sum_{k=1}^{q-1} \chi_q(k) = 0
\]
We have
\[ \sum_{k=1}^{q-1} \chi_q(k) f^*(\frac{k}{q}) = \tau(\chi_q) \sum_{n=-\infty}^\infty \overline{\chi}_q(n) \int_0^1 f(t) \exp(-2\pi int) dt \]
\[ = \tau(\chi_q) \sum_{n=1}^\infty \overline{\chi}_q(n) \int_0^1 f(t) \exp(-2\pi int) dt \]
\[ + \tau(\chi_q) \sum_{n=1}^\infty \overline{\chi}_q(-n) \int_0^1 f(t) \exp(2\pi int) dt \]
\[ = \tau(\chi_q) \sum_{n=1}^\infty \overline{\chi}_q(n) \left( \int_0^1 f(t) \exp(-2\pi int) dt \right) \]
\[ + \overline{\chi}_q(-1) \int_0^1 f(t) \exp(2\pi int) dt \]

therefore, when \( \chi_q(-1) = 1 \), we have
\[ \sum_{k=1}^{q-1} \chi_q(k) f^*(\frac{k}{q}) = 2\tau(\chi_q) \sum_{n=1}^\infty \overline{\chi}_q(n) \int_0^1 f(t) \cos(2\pi nt) dt \]

when \( \chi_q(-1) = -1 \), we have

\[ \sum_{k=1}^{q-1} \chi_q(k) f^*(\frac{k}{q}) = \tau(\chi_q) \sum_{n=1}^\infty \overline{\chi}_q(n) \int_0^1 f(t) \cos(2\pi nt) dt \]
\[
\sum_{k=1}^{q-1} \chi_q(k) f^* \left( \frac{k}{q} \right) = -2i\tau(\chi_q) \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 f(t) \sin(2\pi nt) dt
\]

This completes the proof of the theorem.

From this theorem, we can see that the calculation of the character sum becomes the calculation of integrals.

Below, we give a few special examples.

**The first example.**

Let \( \chi_q \) be the primitive real character and \( \chi_q(-1) = -1 \), then

\[
\sum_{k=1}^{q-1} \chi_q(k) \left( \frac{k}{q} \right)^2 = -\frac{\sqrt{q}}{\pi} L(1, \chi_q)
\]

By Theorem and Lemma 2, easily seen

\[
\sum_{k=1}^{q-1} \chi_q(k) \left( \frac{k}{q} \right)^2 = 2\sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 t^2 \sin(2\pi nt) dt
\]

We compute the integral as follows

\[
\int_0^1 t^2 \sin(2\pi nt) dt = -\frac{1}{2\pi n} \int_0^1 t^2 d\cos(2\pi nt)
\]

\[
= -\frac{1}{2\pi n} + \frac{2}{2\pi n} \int_0^1 t \cos(2\pi nt) dt = -\frac{1}{2\pi n} + \frac{2}{(2\pi n)^2} \int_0^1 t d\sin(2\pi nt)
\]
\[
= -\frac{1}{2\pi n} - \frac{2}{(2\pi n)^2} \int_0^1 \sin(2\pi nt) dt = -\frac{1}{2\pi n}
\]

therefore

\[
\sum_{k=1}^{q-1} \chi_q(k) \left( \frac{k}{q} \right)^2 = -\frac{\sqrt{q}}{\pi} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n} = -\frac{\sqrt{q}}{\pi} L(1, \chi_q)
\]

The second example.

Let \( \chi_q \) be the primitive real character and \( \chi_q(-1) = 1 \), then

\[
\sum_{k=1}^{q-1} \chi_q(k) \log k = -\frac{\sqrt{q}}{2} L(1, \chi_q) + c\sqrt{q}
\]

where \( c \) is a absolute constant.

By Theorem and Lemma 2, we have

\[
\sum_{k=1}^{q-1} \chi_q(k) \log k = 2\sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 \log t \cos(2\pi nt) dt
\]

Now, let's compute the integral

\[
\int_0^1 \log t \cos(2\pi nt) dt = \frac{1}{2\pi n} \int_0^1 \log t \, d(2\pi nt)
\]

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\[-\frac{1}{2\pi n} \int_0^1 t \sin(2\pi nt) dt = -\frac{1}{2\pi n} \int_0^n t \sin(2\pi t) dt\]

\[-\frac{1}{2\pi n} \int_0^\infty \frac{1}{t} \sin(2\pi t) dt = -\frac{1}{2\pi n} \int_0^\infty \frac{1}{t} \sin(t) dt = -\frac{1}{4n}\]

easily seen

\[-\frac{1}{2\pi n} \int_0^\infty \frac{1}{t} \sin(2\pi t) dt = -\frac{1}{2\pi n} \int_0^\infty \frac{1}{t} \sin(t) dt = -\frac{1}{4n}\]

because

\[\frac{1}{2\pi n} \int_n^\infty \frac{1}{t} \sin(2\pi t) dt = -\frac{1}{(2\pi)^2 n} \int_n^\infty \frac{1}{t} \cos(2\pi t) dt\]

\[= \frac{1}{(2\pi)^2} - \frac{1}{(2\pi)^2 n} \int_n^\infty \frac{1}{t^2} \cos(2\pi t) dt \ll \frac{1}{n^2} + \frac{1}{n} \int_n^\infty \frac{1}{t^2} dt \ll \frac{1}{n^2}\]

as well as

\[\sum_{k=1}^{q-1} \chi_q(k) \log \frac{k}{q} = \sum_{k=1}^{q-1} \chi_q(k) \log k - \log q \sum_{k=1}^{q-1} \chi_q(k) = \sum_{k=1}^{q-1} \chi_q(k) \log k\]
This completes the proof.

**The third example.**

Let $\chi_q$ be the primitive real character and $\chi_q(-1) = 1$, then

$$\sum_{k=1}^{q-1} \chi_q(k) e^{\frac{k}{q}} = 2(e - 1) \sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{1 + 4\pi^2 n^2}$$

Let $\chi_q$ be the primitive real character and $\chi_q(-1) = -1$, then

$$\sum_{k=1}^{q-1} \chi_q(k) e^{\frac{k}{q}} = -4\pi(e - 1) \sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n) n}{1 + 4\pi^2 n^2}$$

**Proof.** When $\chi_q(-1) = 1$, by Theorem and Lemma 2, we have

$$\sum_{k=1}^{q-1} \chi_q(k) e^{\frac{k}{q}} = 2 \sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_{0}^{1} e^t \cos(2\pi nt) dt$$

Because, we know

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

Therefore

$$\int_{0}^{1} e^t \cos(2\pi nt) \, dt = \frac{e - 1}{1 + 4\pi^2 n^2}$$
Therefore

\[ \sum_{k=1}^{q-1} \chi_q(k) e^{k\frac{1}{q}} = 2(e - 1) \sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{1 + 4\pi^2 n^2} \]

When \( \chi_q(-1) = -1 \), by Theorem and Lemma 2, we have

\[ \sum_{k=1}^{q-1} \chi_q(k) e^{k\frac{1}{q}} = 2 \sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 e^t \sin(2\pi nt) dt \]

Because, we know

\[ \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \]

Therefore

\[ \int_0^1 e^t \sin(2\pi nt) \, dt = -(e - 1) \frac{2\pi n}{1 + 4\pi^2 n^2} \]

Therefore

\[ \sum_{k=1}^{q-1} \chi_q(k) e^{k\frac{1}{q}} = -4\pi (e - 1) \sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n) n}{1 + 4\pi^2 n^2} \]

This completes the proof.
The fourth example.

This is a well-known formula.

We write

\[ F(y) = \sum_{1 \leq k \leq qy} \chi_q(k) \quad \text{and} \quad F^*(y) = \frac{F(y + 0) + F(y - 0)}{2} \]

When \( \chi_q(-1) = 1 \), we have

\[ F^*(y) = \frac{\tau(\chi_q)}{\pi} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n} \sin(2\pi ny) \]

When \( \chi_q(-1) = -1 \), we have

\[ F^*(y) = \frac{\tau(\chi_q)}{i\pi} L(1, \overline{\chi_q}) - \frac{\tau(\chi_q)}{i\pi} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n} \cos(2\pi ny) \]

Proof. Let \( 0 < y < 1 \), we define the function \( f(x) \) as follow

\[ f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq y \\ 0 & \text{if } y < x < 1 \end{cases} \]

By Theorem, when \( \chi_q(-1) = 1 \), we have

\[ \sum_{k=1}^{q-1} \chi_q(k) f^* \left( \frac{k}{q} \right) = 2\tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi_q(n)} \int_0^y \cos(2\pi nt)dt \]
Because

\[
\int_0^y \cos(2\pi nt) dt = \frac{1}{2\pi n} \int_0^y d \sin(2\pi nt) = \frac{\sin(2\pi ny)}{2\pi n}
\]

Therefore

\[
F^*(y) = \frac{\tau(\chi_q)}{\pi} \sum_{n=1}^{\infty} \frac{\overline{\chi_q}(n)}{n} \sin(2\pi ny)
\]

By Theorem, when \(\chi_q(-1) = -1\), we have

\[
\sum_{k=1}^{q-1} \chi_q(k) f^*(\frac{k}{q}) = -2i \tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi_q}(n) \int_0^y \sin(2\pi nt) dt
\]

Because

\[
\int_0^y \sin(2\pi nt) dt = -\frac{1}{2\pi n} \int_0^y d \cos(2\pi nt) = -\frac{1}{2\pi n} (\cos(2\pi ny) - 1)
\]

Therefore

\[
F^*(y) = \frac{\tau(\chi_q)}{i\pi} L(1, \overline{\chi_q}) - \frac{\tau(\chi_q)}{i\pi} \sum_{n=1}^{\infty} \frac{\overline{\chi_q}(n)}{n} \cos(2\pi ny)
\]
This completes the proof.

REFERENCES

