Equivalence Principle and Lorentz Covariant
Gravitation Contradiction

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Abstract

We show a Lorentz covariant gravitation does not satisfy the equivalence principle.

1 Introduction

We will restrict to a Lorentz covariant gravitation that has only constants $c$ and $G$ with dimension. General relativity [1] is an example. Units are chosen so that $c = G = 1$.

Let $m_A > 0$, $E_\gamma > 0$, and $0 \leq v < 1$ and define $M'_A$ and $E'_\gamma$ by

$$M'_A = \sqrt{1 - v^2} m_A \quad E'_\gamma = \sqrt{\frac{1 + v}{1 - v}} E_\gamma$$

Let $F$ be a frame of reference with coordinates $t$, $x$, $y$, $z$ and $F'$ be a frame of reference with coordinates $t'$, $x'$, $y$, $z'$. The coordinates of the frames being related by the Lorentz transformation

$$t = \frac{t' + vx'}{\sqrt{1 - v^2}} \quad x = \frac{x' + vt'}{\sqrt{1 - v^2}} \quad y = y' \quad z = z'$$

With respect to $F'$ let there be a zero rest mass particle $\gamma$ moving from positive $x'$ infinity towards the origin along the $x'$ axis. Let $t'_\gamma(x'_\gamma)$ be the path of $\gamma$. Also let there be a point mass $A$ such that when $\gamma$ is at at infinity $A$ is at rest at the origin. When $\gamma$ is at infinity let $M'_A$ be the mass of $A$ and $E'_\gamma$ be the energy of $\gamma$. When $\gamma$ is at infinity let $P'^\mu_A$ be the components of the energy-momentum four-vector of $A$ and $P'^\mu_\gamma$ be the components of the energy-momentum four-vector of $\gamma$. With respect $F'$ when $\gamma$ is at infinity

$$P'^0_A = M'_A \quad P'^1_A = P'^2_A = P'^3_A = 0$$

$$P'^0_\gamma = E'_\gamma \quad P'^1_\gamma = -E'_\gamma \quad P'^2_\gamma = P'^3_\gamma = 0$$

With respect to $F$ when $\gamma$ is at infinity the energy of $A$ is using (1) and (3) and the formula for transformation of energy

$$P^0_A = \frac{P'^0_A + vP'^1_A}{\sqrt{1 - v^2}} = \frac{M'_A}{\sqrt{1 - v^2}} = \sqrt{1 - v^2} m_A = m_A$$

and the energy of $\gamma$ is

$$P^0_\gamma = \frac{P'^0_\gamma + vP'^1_\gamma}{\sqrt{1 - v^2}} = \frac{E'_\gamma + v(-E'_\gamma)}{\sqrt{1 - v^2}} = \sqrt{\frac{1 - v}{1 + v}} E'_\gamma = \sqrt{\frac{1 - v}{1 + v}} \sqrt{\frac{1 + v}{1 - v}} E_\gamma = E_\gamma$$

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2 Energy and momentum functions

With respect $\mathcal{F}'$ let the functions $p^0_{\gamma}(x'_\gamma)$ be the components of the energy-momentum four-vector of $\gamma$. The values of $M'_{\gamma}$, $E'_{\gamma}$, and $x'_\gamma$ completely determines the system with respect to $\mathcal{F}'$. A component of $p^0_{\gamma}(x'_\gamma)$ is then a function of $M'_{\gamma}$, $E'_{\gamma}$, and $x'_\gamma$ and no other variables. Since we are considering Lorentz covariant gravitation with only $c$ and $G$ as constants with dimension we have $p^0_{\gamma}(x'_\gamma)/E'_{\gamma}$ will be a dimensionless function of the dimensionless variables $M'_{\gamma}/x'_\gamma$ and $E'_{\gamma}/x'_\gamma$. Note $M'_{\gamma}/E'_{\gamma} = (M'_{\gamma}/x'_\gamma)(1/(E'_{\gamma}/x'_\gamma))$. There is then a dimensionless function $C$ of $M'_{\gamma}/x'_\gamma$ and $E'_{\gamma}/x'_\gamma$ such that [2]

$$p^0_{\gamma}(x'_\gamma) = E'_{\gamma} + \frac{M'_{\gamma}E'_{\gamma}}{x'_\gamma}C\left(\frac{M'_{\gamma}}{x'_\gamma}, \frac{E'_{\gamma}}{x'_\gamma}\right) \tag{6}$$

Similarly for the $x'$ component of momentum there is a dimensionless function $D$ such that

$$p^1_{\gamma}(x'_\gamma) = -E'_{\gamma} + \frac{M'_{\gamma}E'_{\gamma}}{x'_\gamma}D\left(\frac{M'_{\gamma}}{x'_\gamma}, \frac{E'_{\gamma}}{x'_\gamma}\right) \tag{7}$$

With respect to $\mathcal{F}'$ if $\gamma$ is at the point $(t'_\gamma, x'_\gamma, 0, 0)$ where $t'_\gamma(x'_\gamma)$ then with respect to $\mathcal{F}$ it is at the point $(t_\gamma, x_\gamma, 0, 0)$ where

$$t_\gamma = \frac{t'_\gamma + vx'_\gamma}{\sqrt{1-v^2}} \quad x_\gamma = \frac{x'_\gamma + vt'_\gamma}{\sqrt{1-v^2}} \tag{8}$$

and $t_\gamma(x_\gamma)$. With respect to $\mathcal{F}$ let the functions $p^0_{\gamma}(x_\gamma)$ be the components of the energy-momentum four-vector of $\gamma$ at $x_\gamma$. The energy of $\gamma$ at time $t_\gamma$ is using (1), (6)-(8)

$$p^0_{\gamma}(x_\gamma) = p^0_{\gamma}(x'_\gamma) + v p^1_{\gamma}(x'_\gamma) = \frac{E'_{\gamma}}{\sqrt{1-v^2}} + \frac{M'_{\gamma}E'_{\gamma}}{x'_\gamma}\left[1 - v + \frac{M'_{\gamma}}{x'_\gamma}\left(C\left(\frac{M'_{\gamma}}{x'_\gamma}, \frac{E'_{\gamma}}{x'_\gamma}\right) + vD\left(\frac{M'_{\gamma}}{x'_\gamma}, \frac{E'_{\gamma}}{x'_\gamma}\right)\right)\right]$$

$$= \frac{E_{\gamma}}{\sqrt{1-v^2}} + (1 + v)m_{\gamma}E_{\gamma}\left[C\left(\frac{(1-v^2)m_{\gamma}(1+v)E_{\gamma}}{x_\gamma - vt_\gamma}, \frac{(1-v^2)m_{\gamma}(1+v)E_{\gamma}}{x_\gamma - vt_\gamma}\right) + vD\left(\frac{(1-v^2)m_{\gamma}(1+v)E_{\gamma}}{x_\gamma - vt_\gamma}, \frac{(1-v^2)m_{\gamma}(1+v)E_{\gamma}}{x_\gamma - vt_\gamma}\right)\right] \tag{9}$$

The $v \to 1$ limit of (9) is

$$p^0_{\gamma}(x_\gamma) = E_{\gamma} + \frac{2m_{\gamma}E_{\gamma}}{x_\gamma - t_\gamma}\left[C\left(0, \frac{2E_{\gamma}}{x_\gamma - t_\gamma}\right) + D\left(0, \frac{2E_{\gamma}}{x_\gamma - t_\gamma}\right)\right] \tag{10}$$

Similarly for the $x$ component of momentum the $v \to 1$ limit is

$$p^1_{\gamma}(x_\gamma) = -E_{\gamma} + \frac{2m_{\gamma}E_{\gamma}}{x_\gamma - t_\gamma}\left[C\left(0, \frac{2E_{\gamma}}{x_\gamma - t_\gamma}\right) + D\left(0, \frac{2E_{\gamma}}{x_\gamma - t_\gamma}\right)\right] \tag{11}$$

Subtracting (10) and (11) gives

$$p^0_{\gamma}(x_\gamma) - p^1_{\gamma}(x_\gamma) = 2E_{\gamma} \tag{12}$$

3 Velocity of $\gamma$

We will assume, of a Lorentz covariant gravitation, that the velocity of $\gamma$ does not depend on its energy. Consequently with respect to $\mathcal{F}'$ the velocity $dx'_\gamma/dt'_\gamma$ of $\gamma$ will then be a function of $M'_{\gamma}$ and $x'_\gamma$ and not $E'_{\gamma}$. We then have $dx'_\gamma/dt'_\gamma$ will be a dimensionless function of the dimensionless variable $M'_{\gamma}/x'_\gamma$. There is then a dimensionless function $S$ such that

$$\frac{dx'_\gamma}{dt'_\gamma} = -1 + \frac{M'_{\gamma}}{x'_\gamma}S\left(\frac{M'_{\gamma}}{x'_\gamma}\right) \tag{13}$$
The speed of $\gamma$ decreases as $\gamma$ moves towards the origin hence $S(0) > 0$. With respect to $\mathcal{F}$ the velocity of $\gamma$ is using (1), (8), (13) and the velocity addition formula

$$\frac{dx_\gamma}{dt_\gamma} = \frac{dx_\gamma'}{dt_\gamma'} + v$$

$$= \frac{-1 + \frac{M_A}{x_\gamma} S\left(\frac{M_A}{x_\gamma}\right) + v}{1 + v \left[-1 + \frac{M_A}{x_\gamma} S\left(\frac{M_A}{x_\gamma}\right)\right]} = \frac{-1 + \frac{(1+v)m_A}{x_\gamma-vt_\gamma} S\left(\frac{(1-v^2)m_A}{x_\gamma-vt_\gamma}\right)}{1 + \frac{v(1+v)m_A}{x_\gamma-vt_\gamma} S\left(\frac{(1-v^2)m_A}{x_\gamma-vt_\gamma}\right)}$$

(14)

The $v \to 1$ limit of (14) is

$$\frac{dx_\gamma}{dt_\gamma} = \frac{-1 + \frac{2m_A}{x_\gamma-t_\gamma} S(0)}{1 + \frac{2m_A}{x_\gamma-t_\gamma} S(0)}$$

(15)

Solving this differential equation gives

$$(x_\gamma - t_\gamma) - (x_{\gamma 1} - t_{\gamma 1}) + 2m_A S(0) \ln \frac{x_\gamma - t_\gamma}{x_{\gamma 1} - t_{\gamma 1}} = 2(t_\gamma - t_{\gamma 1})$$

(16)

where the point $(t_{\gamma 1}, x_{\gamma 1}, 0, 0)$ with $x_{\gamma 1} - t_{\gamma 1} > 0$ is on the path of $\gamma$. There is no point $(t_{\gamma 2}, x_{\gamma 2}, 0, 0)$ with $x_{\gamma 2} - t_{\gamma 2} = 0$ on the path of $\gamma$ hence all points on the path $x_\gamma - t_\gamma > 0$. From (16) then $x_\gamma - t_\gamma \to 0$ as $t_\gamma \to \infty$. Now

$$p^1_\gamma(x_\gamma) = \frac{dx_\gamma}{dt_\gamma}(x_\gamma)p^0_\gamma(x_\gamma)$$

(17)

By (12), (15), and (17) we have

$$p^0_\gamma(x_\gamma) = \left[1 + \frac{2m_A}{x_\gamma-t_\gamma} S(0)\right] E_\gamma$$

(18)

### 4 Contradiction

Let $T^\mu_\gamma$ be the $v \to 1$ limit of the energy-momentum tensor of $\gamma$. For points having $t-x$ approximately zero and $t$ large positive we have $T^\mu_\gamma$ will have approximately a $x-t$ functional dependence. By (15) and (17) we have $T^0_\gamma = T^0_{\gamma 1}$ as $t_\gamma \to \infty$. Also $T^{02} = T^{03} = 0$. Consequently $\partial_\mu T^\mu_\gamma \to 0$ as $t_\gamma \to \infty$. We can then conclude $p^0_\gamma$ is approximately constant in time as $t_\gamma \to \infty$ but (18) has $p^0_\gamma$ going to infinity as $t_\gamma \to \infty$. This is a contradiction. The velocity of $\gamma$ must then have a dependence on the energy of $\gamma$. By the equivalence principal the velocity of $\gamma$ does not depend on the energy of $\gamma$. We have a contradiction. A Lorentz covariant gravitation does not satisfy the equivalence principle.

### References
