Set of Prime gaps and Subsets of Prime numbers are Countably arbitrarily large, and Origin point and Critical line are equivalent location for Nontrivial zeros of Riemann zeta function

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Abstract We prove that the cardinality of Set even prime gaps and corresponding Subsets odd prime numbers are all countably arbitrarily large in number. We prove that the countably infinitely large number of nontrivial zeros which are computed using Dirichlet eta function are uniquely located on its critical line whereby this function is the proxy function for Riemann zeta function. There are two major ingredients. For the former proof, there is zero probability that any particular prime gaps from eternal repeated groupings of small and/or large prime gaps that faithfully generate all the countably arbitrarily large number of odd primes will abruptly terminate or disappear. For the later proof, there is zero probability that any of the countably infinitely large number of nontrivial zeros can be located away from [geometric] Origin point, which is equivalent to [mathematical] critical line.

Keywords: Admissible Prime k-tuples, Gram’s Law, Inadmissible Prime k-tuples, Riemann zeta function, Rosser’s Rule

Mathematics Subject Classification (2010): Primary 00A05, Secondary 11M26 & 11A41

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A  Gram’s Law and Rosser Rule for Gram points

B  Freebasic programme to elucidate all possible patterns of Prime k-tuplets for k = 2 to 50

1. Introduction

Faithfully generated by relevant algorithms, prime and composite numbers form two complementary, mutually exclusive and dependent countably infinite sets of Incompletely Predictable numbers. The first two integer numbers 0 and 1 forming a separate countably finite set are neither prime nor composite. These three sets when combined together will constitute the countably infinite set of Completely Predictable integer numbers. The [alternating] harmonic series (equation) Dirichlet eta function is the proxy function for [non-alternating] harmonic series (equation) Riemann zeta function. Faithfully generated by Dirichlet eta function when parameter \( \sigma = \frac{1}{2} \), [mathematical] nontrivial zeros (or Gram[x=0,y=0] points) and closely-related two types of Gram points (viz, Gram[y=0] points) and Gram[x=0] points form [geometrical] Origin intercept points, x-axis intercept points and y-axis intercept points whereby they constitute three complementary, mutually exclusive and dependent countably infinite sets of Incompletely Predictable entities. Proofs for Riemann hypothesis regarding location of nontrivial zeros, and Polignac’s and Twin prime conjectures regarding cardinality of prime gaps and prime numbers can, respectively, be dubbed Equation-type and Algorithm-type proofs. When deriving these proofs, we crucially recognize the above-mentioned entities, with including all our designated Prime k-tuplets and Prime k-tuples mentioned below, must exist as Incompletely Predictable entities contained in well-defined sets, subsets, tuples and subtuples that manifest certain Incompletely Predictable properties. To this end, we employ simple mathematical tools such as basic arithmetic operations, modular arithmetic, set theory and probability theory.

We use the convenient Simplified Author–date referencing style in this research paper entitled ‘Set of Prime gaps and Subsets of Prime numbers are Countably arbitrarily large, and Origin point and Critical line are equivalent location for Nontrivial zeros of Riemann zeta function’. The main objective is to provide correct and complete mathematical arguments to fully validate this title. Our action will result in obtaining rigorous proofs for Riemann hypothesis, Polignac’s and Twin prime conjectures. In so doing, we have to crucially apply infinitesimal numbers in two places: Proposition 1.31 In the limit of never reaching a nonexisting zero [which is conceptually equivalent to the prevalence of odd prime numbers with various prime gaps never becoming zero] whereby the arbitrarily large number of different even prime gaps that accompany all odd prime numbers in totality will never stop recurring, and Proposition 1.32 In the limit of reaching an existing zero [which is conceptually equivalent to the trajectory of relevant Dirichlet eta function, as proxy function for Riemann zeta function, touching the zero-dimensional Origin point only when its parameter \( \sigma = \frac{1}{2} \)] whereby all [mathematical] nontrivial zeros located on \( \sigma = \frac{1}{2} \)-critical line will uniquely present themselves in totality as [geometrical] Origin intercept points. In addition, we outline vital concepts present in Completely Predictable and Incompletely Predictable entities, and classify countably infinite sets in Lemma 1.1 as accelerating, linear or decelerating subtypes that, respectively, manifest acceleratingly reaching an infinity value, linearly reaching an infinity value or deceleratingly reaching an arbitrarily large number value. As seen later on, this new classification allows optimal description of countably infinite sets such as those mentioned in the preceding paragraph. We also conveniently classify categories of irrational numbers to exist in either Isolated countably finitely-sized group or Connected countably infinitely-sized group.

1.1 Completely and Incompletely Predictable entities, Cardinality of Sets and Subsets

Legend: CP = Completely Predictable, IP = Incompletely Predictable.

The word number [singular noun] or numbers [plural noun] used in reference to CP even and odd numbers, IP prime and composite numbers, IP nontrivial zeros and two types of Gram points can interchangeably be replaced with the word entity [singular noun] or entities [plural noun]. Gram points refer to Gram[x=0,y=0] points or nontrivial zeros, Gram[y=0] points and Gram[x=0] points when \( \sigma = \frac{1}{2} \). Virtual Gram points refer to virtual Gram[y=0] points and virtual Gram[x=0] points when \( \sigma \neq \frac{1}{2} \) whereby virtual Gram[x=0,y=0] points do not exist. For \( i = \) all integers \( \geq 0 \) or \( i = \) all integers \( \geq 1 \); the \( i^{th} \) position of \( i^{th} \) CP numbers and \( i^{th} \) IP numbers is simply given by \( i \). Apart from the very first Gram[y=0] point and the very first virtual Gram[y=0] point both occurring at \( t = 0 \), we note all other types of Gram points and virtual Gram points will consist of \( t \)-valued transcendental numbers whose positions are IP with the infinitely many digits after the decimal point in each transcendental number again being IP. IP simple equation or algorithm generates CP numbers. A generated CP number is locationally defined as a number whose \( i^{th} \) position is independently determined by simple calculations without needing to know related positions of all preceding numbers. We supply the following example.

**E-O Pairing:** For \( i = 1, 2, 3, \ldots, \infty \); let \( i^{th} \) Even and \( i^{th} \) Odd numbers = \( E_i \) and \( O_i \), and \( i^{th} \) even and \( i^{th} \) odd number gaps = \( eGap_i \) and \( oGap_i \). We ignore \( E_0 = 0 \). We show positions of \( E_i \) and \( O_i \) are CP and their independence from each other.

<table>
<thead>
<tr>
<th>( i )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_i )</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>...</td>
</tr>
<tr>
<td>( eGap_i )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
We can precisely, easily and independently calculate $E_5 = (2 \times 5) = 10$ and $O_5 = (2 \times 5) - 1 = 9$.

<table>
<thead>
<tr>
<th>$O_i$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>....</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{\text{Gap}_i}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

IP complex equation or algorithm generates IP numbers. A generated IP number is **locationally defined** as a number whose $i^{th}$ position is determined by complex calculations with needing to know related positions of all preceding numbers. We supply the following example.

**P-C Pairing:** For $i = 1, 2, 3, ..., \infty$; let $i^{th}$ Prime and $i^{th}$ Composite numbers = $P_i$ and $C_i$, and $i^{th}$ prime and $i^{th}$ composite number gaps = $p_{\text{Gap}_i}$ and $c_{\text{Gap}_i}$. We show positions of $P_i$ and $C_i$ are IP and their dependence on each other.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>....</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{\text{Gap}_i}$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

We can precisely, tediously and dependently compute $P_6 = 13$: 2 is $1^{st}$ prime number, 3 is $2^{nd}$ prime number, 4 is $1^{st}$ composite number, 5 is $3^{rd}$ prime number, 6 is $2^{nd}$ composite number, 7 is $4^{th}$ prime number, 8 is $3^{rd}$ composite number, 9 is $4^{th}$ composite number, 10 is $5^{th}$ composite number, 11 is $5^{th}$ prime number, 12 is $6^{th}$ composite number, and our desired 13 is $6^{th}$ prime number.

<table>
<thead>
<tr>
<th>$C_i$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>....</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{\text{Gap}_i}$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

In email correspondences between author of this paper on 22 January 2022 and Professor Maciej Radziejewski, Secretary of Acta Arithmetica on 10 February 2022; the following is the appropriate email letter reply from Radziejewski on Subject Heading Error on Page 250 in Acta Arithmetica 148 (2011), 225 – 256. This issue concerns the first exception to Rosser’s rule being stated by Professor Timothy S. Trudgian on page 250 of his paper On the success and failure of Gram’s Law and the Rosser Rule, Acta Arithmetica, vol. 148 (2011), pp. 225 – 256, DOI: http://dx.doi.org/10.4064/aa148-3-2 as [incorrect] 13,999,825\textsuperscript{th} Gram point. This should instead be [correct] 13,999,525\textsuperscript{th} Gram point as stated by Professor Richard P. Brent in Table 3 (page 1369) of his paper On the zeros of the Riemann zeta function in the critical strip, Math. Comp., vol. 33 (1979), pp. 1361 – 1372, DOI: https://doi.org/10.1090/S0025-5718-1979-0537983-2, and also by Charles R. Greathouse IV, author of OEIS A216700 dated 17 September 2012 Violations of Rosser’s rule: numbers $n$ such that the Gram block $[g(n), g(n+k))]$ contains fewer than $k$ points $t$ such that $Z(t) = 0$, where $Z(t)$ is the Riemann-Siegel $Z$-function.

Dear Professor Ting,

Thank you for your letter. I understand that you are confused by the fact that two research papers contradict each other by giving different values for the same quantity. If you look at the first three sentences of Section 7.3 in T. Trudgian’s paper, it is clear that the author is referring to known previous computations. He has not repeated these computations and he is not pointing to any error therein. Therefore the discrepancy that you mention is clearly due to a typo in T. Trudgian’s paper. For your information, I contacted the author and he confirms that.

While it is regrettable that the value in that paper is incorrect, the Editor does not feel that it should lead to misunderstanding, unless someone uses that value for something. However, this should not be done without consulting the source works, precisely because such a typo is quite possible. For this reason we are not going to ask the author for an erratum.

Yours sincerely,

Maciej Radziejewski
Secretary of Acta Arithmetica

We respectfully acknowledge the above reproduced email letter has usefully provided necessary incentive and inspiration for relevant parts of this research paper. From Appendix A, we note Gram’s law is the tendency for nontrivial zeros of Riemann-Siegel function $Z(t)$ to alternate with Gram[y=0] points when $\sigma = \frac{1}{2}$. The first violation (failure) of Gram’s law occurs at $n = 126$. Rosser’s rule states that every Gram block contains the expected number of roots as Gram[y=0] points when $\sigma = \frac{1}{2}$. The first violation (failure) of Rosser’s rule occurs at the much larger $n = 13999525$. With assistance from Greathouse IV on 24 January 2022, we found one nontrivial zero between $n = 13999825$ as Gram[y=0] point with t-value of 6820050.98496... and $n = 13999827$ as Gram[y=0] point with t-value of 6820051.8891147...; and two nontrivial zeros in the Gram block between $n = 13999824$ as Gram[y=0] point and $n = 13999826$ as Gram[y=0] point. In any event, $n = 13999825$ mentioned by Professor Trudgian in his paper is [serendipitously] a bad Gram[y=0] point and a Gram’s law violation. Thus, $n = 13999825$ is not a valid choice for a Rosser’s rule violation (failure).

A set is a collection of zero (viz, the empty set) or more elements (viz, a finite set with a finite number of elements or an infinite set with an infinite number of elements). A singleton refers to a finite set with a single element. A set can be any kind of mathematical objects: numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other
sets whereby these [mutable] non-repeating elements are not arranged in an unique order. A subset can be a [smaller] finite set derived from its [larger] parent finite set or its [larger] parent infinite set. A subset can also be a [smaller] infinite set derived from its [larger] parent infinite set. A tuple, which can potentially be subdivided into sub-tuples, is a finite ordered list (sequence) of elements whereby these [immutable] non-repeating elements are arranged in an unique order. Thus, a tuple or a sub-tuple is usefully regarded as a special type of finite set with the extra imposed restriction.

**Lemma 1.1.** We can classify countably infinite sets as accelerating, linear or decelerating subtypes, and concisely define Prevalence of prime and composite numbers, and nontrivial zeros.

**Proof.** We logically provide the following required mathematical deductions.

**Cardinality:** With increasing size, arbitrary Set [or Subset] \(X\) can be countably finite set (CFS), countably infinite set (CIS) or uncountably infinite set (UIS). Cardinality of Set \(X\), \(|X|\), measures number of elements in Set \(X\). For example, Set negative Gram\([y=0]\) point as constituted by a [solitary] rational (\(\mathbb{Q}\)) t-value of 0 instead of a usual transcendental (\(\mathbb{R} - \mathbb{A}\)) t-value has CFS of negative Gram\([y=0]\) point with this particular negative Gram\([y=0]\) point \(= \mathbb{N}_0\). Set even Prime number \((\mathbb{P})\) has CFS of solitary even \(\mathbb{P} 2\) with even \(\mathbb{P} = 1\). Set Natural numbers \((\mathbb{N})\) has CIS of \(\mathbb{N}\) with \(|\mathbb{N}| = \mathbb{N}_0\), and Set Real numbers \((\mathbb{R})\) has UIS of \(\mathbb{R}\) with \(|\mathbb{R}| = \mathbb{C}\) (cardinality of the continuum). Then with |CIS| = \(\mathbb{N}_0\) = [countably] infinitely many elements; we provide a novel definitive classification for CIS based on its number of elements (cardinality) manifesting linear, accelerating or decelerating phenomena thus constituting the following three subtypes of CIS.

**CIS-IM-accelerating:** CIS with its cardinality given by |CIS-IM-accelerating| = \(\mathbb{N}_0\)-accelerating = [countably] infinitely many elements that will (overall) aceleratingly reach an infinity value. Examples: CP integers 0, 1, 4, 9, 16,... generated by simple equation \(y = x^2\) for \(x = 0, 1, 2, 3, 4,...\); and IP composite numbers 4, 6, 8, 9, 10... faithfully generated by complex Complement-Sieve-of-Eratosthenes algorithm [which is simply discarding 0, 1, and all generated prime numbers via Sieve-of-Eratosthenes algorithm from the set of integers 0, 1, 2, 3, 4, 5, ...].

**CIS-IM-linear:** CIS with its cardinality given by |CIS-IM-linear| = \(\mathbb{N}_0\)-linear = [countably] infinitely many elements that will (overall) linearly reach an infinity value. Examples: CP entities 0, 1, 2, 3, 4, 5,... [representing all positive integer numbers] generated by simple equation \(y = x\) for \(x = 0, 1, 2, 3, 4,...\); CP entities 0, 2, 4, 6, 8, 10... [representing all positive even numbers] generated by simple equation \(y = 2x\) for \(x = 0, 1, 2, 3, 4,...\); CP entities 1, 3, 5, 7, 9, 11... [representing all positive odd numbers] generated by simple equation \(y = 2x - 1\) for \(x = 1, 2, 3, 4, 5,...\); and IP nontrivial zeros, Gram\([y=0]\) points and Gram\([x=0]\) points (all given as \(\mathbb{R} - \mathbb{A}\) t-values) generated from complex equation Riemann zeta function via its proxy Dirichlet eta function. These IP entities will inevitably manifest IP perpetual repeating violations (failures) in Gram’s Law occuring infinitely many times resulting in Set negative Gram\([y=0]\) points. Thus the CIS of negative Gram\([y=0]\) points is constituted by \(\mathbb{R} - \mathbb{A}\) t-values and is classified as having negative Gram\([y=0]\) points = |CIS-IM-linear| = \(\mathbb{N}_0\)-linear.

**CIS-ALN-decelerating:** CIS with its cardinality given by |CIS-ALN-decelerating| = \(\mathbb{N}_0\)-decelerating = [countably] arbitrarily large number of elements that will (overall) deceleratingly reach an arbitrarily large number value. Examples: CP entities 0, 1, \(\sqrt{2}\), \(\sqrt{3}\), 2, \(\sqrt{5}\)... generated by simple equation \(y = \sqrt{x}\) for \(x = 0, 1, 2, 3, 4,...\); and IP prime numbers 2, 3, 5, 7, 11... faithfully generated by complex Sieve-of-Eratosthenes algorithm.

We analyze the data of all CIS-IM-accelerated computed composite numbers when extrapolated out over a wide range of \(x \geq 4\) integer values. We define the composite counting function Composite-\(\pi(x)\) = number of composites \(\leq x\) with \(x\) conveniently assigned to having odd number values of the form \(10^n - 1\) whereby \(n = 1, 2, 3, 4, 5,...\). Then, **Prevalence of composite numbers** = Composite-\(\pi(x)\) / \(x\) = Composite-\(\pi(x)\) / \(10^n - 1\) when \(x = 4\) to \(10^n - 1\). Composite gaps can only be constituted by CIS-IM-accelerating odd composite gap 1 and CIS-ALN-decelerating even composite gap 2.

We analyze the data of all CIS-ALN-decelerated computed prime numbers when extrapolated out over a wide range of \(x \geq 2\) integer values. We define the prime counting function Prime-\(\pi(x)\) = number of primes \(\leq x\) with \(x\) conveniently assigned to having odd number values of the form \(10^n - 1\) whereby \(n = 1, 2, 3, 4, 5,...\). Then, **Prevalence of prime numbers** = Prime-\(\pi(x)\) / \(x\) = Prime-\(\pi(x)\) / \(10^n - 2\) when \(x = 2\) to \(10^n - 1\). Prime gaps for all odd prime numbers can only be constituted by CIS-ALN-decelerating even prime gaps 2, 4, 6, 8, 10...

We analyze the data of all CIS-IM-linear computed nontrivial zeros (NTZ) when extrapolated out over a wide range of \(t \geq 0\) real number values. We can symbolically define the nontrivial zeros counting function NTZ-\(\pi(t)\) = number of NTZ \(\leq t\) with \(t\) conveniently assigned to having real number values of the [arbitrary] form \(10^n\) whereby \(n = 1, 2, 3, 4, 5,...\). Then, **Prevalence of nontrivial zeros** = NTZ-\(\pi(t)\) / \(t\) = NTZ-\(\pi(t)\) / \(10^n\) when \(t = 0\) to \(10^n\), whereby denominator \(t\) could also be [artificially] regarded as having integer number values. We can conceptually define all consecutive NTZ gaps as the \(i^{th}\) t-valued NTZ − (i-1)\(th\) t-valued NTZ. Thus there are CIS-IM-linear computed NTZ gaps.

*The proof is now complete for Lemma 1.1.\(\Box\)*

**Remark 1.** Incorporating the subtype classification of countably infinite sets, we apply simple and complex properties manifested by Complety Predictable and Incompletely Predictable entities. As an example of simple property, \(x\)-axis
is denoted here simply as this with the [less-than-100% accurate] approximate li(x) smooth-mathematical function an arbitrarily large number of > Prime-large number of times (Littlewood, 1914). This refute all prior numerical evidence that seem to suggest li(x) was always ≡ li(x) in the sense lim_{x\to\infty} \frac{\pi(x)}{ln x} = 1. Skewes’ number is any of several extremely large numbers used by South African mathematician Stanley Skewes as upper bounds for smallest natural number x for which li(x) < Prime-\pi(x). These bounds have since been improved by others: there is a crossing near e^{727.95133} but it is not known whether this is the smallest. John Edensor Littlewood who was Skewes’ research supervisor proved that there is such a [first] number; and found that sign of difference Prime-\pi(x) – li(x) changes an arbitrarily large number of times (Littlewood, 1914). This refute all prior numerical evidence that seem to suggest li(x) was always > Prime-\pi(x). The key point is the [100% accurate] perfect Prime-\pi(x) stepped-mathematical function being wrapped around by the [less-than-100% accurate] approximate li(x) smooth-mathematical function an arbitrarily large number of times via this sign of difference changes meant that li(x) is the most efficient approximate mathematical function. Contrast this with the crude [less-than-100% accurate] approximate \frac{x}{ln x} smooth-mathematical function whereby the studied values diverge away from Prime-\pi(x) at increasingly greater rate for larger range of prime numbers.

1.2 Admissible Prime k-tuples, Inadmissible Prime k-tuples and Nontrivial zeros

For k ≥ 2, a Prime k-tuple [that can be subdivided into available subtuples for sufficiently large k values] is a repeatable pattern of finite k consecutive primes \{p_1, p_2, ..., p_k\} [viz, a finite collection with p_1 < p_2 < ... < p_k] having diameter d defined as difference between its largest and smallest elements [viz, diameter d = p_k – p_1]. There are two main types of Prime k-tuples as per our proposed classification in Table 1 based on comparative same k values (which will then
insightfully depict the overlapping mathematical landscape of Prime k-tuples: (I) [repeating] Admissible Prime k-tuples as two subtypes with each subtype and relevant associated varieties deceleratingly reach an arbitrarily large number value, and (II) [non-repeating] Inadmissible Prime k-tuples as two subtypes with each subtype and relevant associated varieties deceleratingly reach an arbitrarily large number value.

<table>
<thead>
<tr>
<th>Admissible Prime k-tuples: Subtypes and Varieties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtype I Admissible Prime k-tuplets (Sub I Adm P k-tuples)</td>
</tr>
<tr>
<td>Subtype II Admissible Prime k-tuplets (Sub II Adm P k-tuples)</td>
</tr>
<tr>
<td>First variety of Subtype II Admissible Prime k-tuples (1st V Sub II Adm P k-tuples)</td>
</tr>
<tr>
<td>Second variety of Subtype II Admissible Prime k-tuples (2nd V Sub II Adm P k-tuples)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inadmissible Prime k-tuples: Subtypes and Varieties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtype I Inadmissible Prime k-tuplets (Sub I Inadm P k-tuples)</td>
</tr>
<tr>
<td>Subtype II Inadmissible Prime k-tuplets (Sub II Inadm P k-tuples)</td>
</tr>
<tr>
<td>First variety of Subtype II Inadmissible Prime k-tuples (1st V Sub II Inadm P k-tuples)</td>
</tr>
<tr>
<td>Second variety of Subtype II Inadmissible Prime k-tuples (2nd V Sub II Inadm P k-tuples)</td>
</tr>
</tbody>
</table>

Table 1. Classification of two main types of Prime k-tuples with their abbreviations. The [virtual] 1st V Sub II Inadm P k-tuples that do not mathematically exist are directly related to 1st V Sub II Adm P k-tuples which cater for large(r) prime numbers with large(r) prime gaps. The 2nd V Sub II Inadm P k-tuples that mathematically exist just once are directly related to 2nd V Sub II Adm P k-tuples which cater for small(er) prime numbers with small(er) prime gaps.

<table>
<thead>
<tr>
<th>k</th>
<th>p_k</th>
<th>k#</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
</tr>
<tr>
<td>2</td>
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</tr>
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</tbody>
</table>

Table 2. Tabulated data of k primorial for k = 1 to 10. Let p_k be the k^{th} prime with k = 1, 2, 3, 4, 5,... Then k primorial (k#) is the product of the first k primes whereby the [even] numbers in the third column are the product of the primes in the second column. It is an Incompletely Predictable function that will acceleratingly reach an infinity value.

The principles behind primorial are outlined in Table 2. As part of group theory with notation that read as \((Z/nZ)^*\), the concepts behind multiplicative group of integers modulo n are important for the theory of prime k-tuples or constellations. It contains a subset of the integers from 1 to n-1. The elements of \((Z/nZ)^*\) are the integers from 1 to n-1 that are relatively prime [or coprime] to n. If n is a prime number, then \((Z/nZ)^*\) contains all the integers from 1 to n-1. If n has many divisors, then \((Z/nZ)^*\) will contain fewer elements. To find Sub I P Adm k-tuplets, we need to consider the multiplicative group of integers mod k primorial. This group contains a subset of the integers less than k primorial that are relatively prime to k primorial with worked examples and their significances given below.

The multiplicative group mod 6 \([\#]\) has two elements; viz, \((Z/6Z)^* = \{1, 5\}\). Then all primes greater than 3 have the form 6*n ± 1. To search for the smaller of twin prime pairs [Sub I P Adm 2-tuplets], one should look at [odd] numbers of the form 6*n + 5. The multiplicative group mod 30 \([\#]\) has 8 elements; viz, \((Z/30Z)^* = \{1, 7, 11, 13, 17, 19, 23, 29\}\). By looking at the differences between adjacent elements in this set, we see Sub I P Adm 3-tuplets as pattern \((p, p+2, p+6)\) is found only in the expressions 30*n + 11 and 30*n + 17. The ordered set \((Z/30Z)^* = \{1, 7, 11, 13, 17, 19, 23, 29\}\) can be manipulated by taking the differences between adjacent elements; viz, d30 = \{6, 4, 2, 4, 2, 4, 6\} \(\Rightarrow\) the particular pattern \((p, p+2, p+6, p+8)\) which has differences \{2, 4, 2\} is found inside ordered set d30. Thus we see that Sub I P Adm 4-tuplets having the pattern \((p, p+2, p+6, p+8)\) must have the form 30*n + 11.

**Proposition 1.2.1.** Let consecutive primes \(p_1, p_3, \ldots, p_k\) represent Subtype I Admissible Prime k-tuples linked to Subtype I Inadmissible Prime k-tuples and/or first variety of Subtype II Admissible Prime k-tuples linked to [nonexisting] first variety of Subtype II Inadmissible Prime k-tuples and/or second variety of Subtype II Admissible Prime k-tuples linked to second variety of Subtype II Inadmissible Prime k-tuples. Then except for \(p_1 = 2\) case [having the empty set of Admissible Prime k-tuple] and for each \(p_1\) commencing from \(p_1 = 2, 3, 5, 7, 11, 13\;\ldots\;\), we can generate a finite number of these Admissible Prime k-tuples/k-tuples denoted by their k values and an associated arbitrarily large number of these Inadmissible Prime k-tuples denoted by their [larger and different complementary] k values whereby both types of tuples...
Suppose one is given a \( k \)-tuple \( H = (h_1, \ldots, h_k) \) of \( k \) distinct integers for some \( k_0 \geq 1 \), arranged in increasing order. We anticipate finding an arbitrarily large number of translates \( n + H = (n+h_1, \ldots, n+h_k) \) of \( H \) which consists entirely of consecutive primes could help prove Polignac’s and Twin prime conjectures. The case \( k_0 = 1 \) is just Euclid’s theorem on the infinitude of primes. The case \( k_0 = 2 \) [as subset] with \( H = (0, 2) \) correspond to twin prime conjecture that non-overlappingly deals with prime gap \( = 2 \). The arbitrarily large number of cases \( k_0 \geq 2 \) [as full set] in their entirety correspond to Polignac’s conjecture that [additionally] involve all other remaining cases such as \( k_0 = 3 \) with \( H = (0, 2, 6) \) as pattern-1 or \( (0, 4, 6) \) as pattern-2, \( k_0 = 4 \) with \( H = (0, 2, 6, 8) \) as solitary pattern, etc and will overlappingly deal with prime gaps \( = 2, 4, 6, 8, 10... \).

More generally, if there is a prime \( p_1 \) such that \( H \) meets each of the \( p_1 \) residue classes \( \mod p_1 \). \( 1 \mod p_1, \ldots, p_1-1 \mod p_1 \), then every translate of \( H \) contains at least one multiple of \( p_1 \). Since \( p_1 \) is the only multiple of \( p_1 \) that is prime, this shows that there are only finitely many translates of \( H \) that consist entirely of consecutive primes.

A \( k_0 \)-tuple \( H \) is admissible if it avoids at least one residue class \( \mod p \) for each prime \( p \). It is easy to check for admissibility in practice, since a \( k_0 \)-tuple is automatically admissible in every prime \( p \) larger than \( k_0 \), so one only needs to check a finite number of primes in order to decide on the admissibility of a given tuple. We can now succinctly state the first Hardy-Littlewood conjecture or Prime tuples conjecture in its qualitative form: If \( H \) is an admissible \( k_0 \)-tuple, then there exists an arbitrarily large number of translates of \( H \) that consist entirely of consecutive primes.

Our \( p_1 \) commencing values as constituted from the entire CIS-ALN-decelerating prime numbers \( 2, 3, 5, 7, 11, 13, \ldots \) will act as reference points to orderly include all possible subtypes and varieties of Admissible k-tuples/k-tuples and Inadmissible Prime k-tuples whereby these k-tuples and k-tuples are constituted by k consecutive prime numbers starting from \( p_1 \). We invoke multiplicative group of integers modulo \( p_1 \), that, via brute force algorithm, must result in a subset of consecutive integers as residues from \( 0 \) to \( p_1-2 \) and \( p_1-1 \) whereby some of these integers that represent corresponding residues will inevitably repeat more than once. For instance at \( p_1 \) commencing value \( = 11 \), the sequence of integers that mechanically represent corresponding residues from \( \mod p_1 \) as iteratively computed using all available prime gaps are \( 0, 2, 6, 8, 1, 7, 9, 4, 8, 10, 3, 9, 4, 6, 1 \) [Admissible] and \( 5 \) [Inadmissible] whereby [non-comprehensive] integers \( 1, 4, 6, 8 \) and 9 are overlappingly depicted more than once and two [uniquely nominated] integers 0 and 5 must always be non-overlappingly depicted just once but with the [solitary] integer 5 being (firstly) absent when the involved \( k \)-tuple is admissible and (secondly) present when the involved \( (k+1) \)-tuple is inadmissible. The \( p_1 \) commencing value \( = 11 \) has thus provided us with (i) Sub I Adm P 15-tuplet as consecutive primes \( (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67) \) that is mechanically equal to \( \equiv \) progressive prime gaps \( (0, 2, 4, 6, 2, 6, 4, 2, 6, 2, 6, 6, 2, 6, 6, 2, 6, 6, 2, 6, 6, 2, 6) \) \( \equiv \) cumulative prime gaps \( (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 56) \) and (ii) 2nd V Sub II Inadm P 16-tuplet as consecutive primes \( (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71) \) that is mechanically equal to \( \equiv \) progressive prime gaps \( (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 2, 6, 6, 2, 6, 4, 2) \) \( \equiv \) cumulative prime gaps \( (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60) \). We note all the involved [consecutive] prime gaps of \( 2, 4 \) and 6 are each overlappingly depicted more than once for both the involved admissible \( k \)-tuplet and inadmissible \( k \)-tuplet.

However, one can easily deduce and confirm there will be an arbitrarily large number of exceptions as scattered counter-examples of Admissible k-tuple/k-tuples and/or certain patterns of Admissible k-tuple/k-tuple and/or Inadmissible k-tuple being admissible whereby (i) the two uniquely nominated integers that represent corresponding residues do not repeat more than once e.g. at \( p_1 \) commencing value \( = 47 \) when all relevant integers \( 0, 1, 2, 3, \ldots, 46 \) [having cardinality of 47] for \( 2^nd \) V Sub II Inadm P \( 79 \)-tuple simply cannot be repeated more than once [because \( k = 79 \) is clearly not at least equal to \( (47–2)*2 + 2 = 92 \) with two uniquely nominated integers taken into account]; (ii) one of the prime gaps do not repeat more than once e.g. prime gap \( = 6 \) do not repeat in Sub I Adm P \( 7 \)-tuplet as pattern-1 \( (0, 2, 6, 8, 12, 18, 20) \) \( \equiv \) progressive prime gaps \( (0, 2, 4, 2, 4, 6, 2) \) and as pattern-2 \( (0, 2, 8, 12, 14, 18, 20) \) \( \equiv \) progressive prime gaps \( (0, 2, 6, 4, 2, 4, 2) \); and (iii) one of the prime gaps is totally missing [skipped] e.g. prime gap \( = 8 \) is missing in Sub I Adm P \( 25 \)-tuplet as pattern-9 \( (0, 6, 8, 14, 20, 24, 30, 36, 34, 45, 50, 54, 56, 66, 68, 78, 80, 84, 86, 90, 96, 98, 104, 108, 110) \) \( \equiv \) progressive prime gaps \( (0, 6, 2, 6, 4, 6, 2, 6, 6, 4, 2, 10, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2) \) whereby we easily observe there are a few other of the total 18 [abbreviated-without-comma] patterns listed below also with missing prime gap \( = 8 \).

\[
\begin{align*}
(0 & 2 6 8 12 18 20 26 30 32 36 42 48 50 56 62 68 72 78 86 90 96 98 102 110) \\
(0 & 2 6 8 12 20 26 30 36 38 42 48 56 66 68 72 78 80 86 90 92 96 98 108 110) \\
(0 & 2 6 8 12 20 26 30 36 38 42 48 50 56 66 68 72 78 80 86 90 92 98 108 110) \\
(0 & 2 6 8 12 20 26 30 36 38 42 50 56 66 68 72 78 80 86 90 92 96 98 108 110) \\
(0 & 2 6 8 12 20 26 30 36 38 42 50 56 62 66 68 72 78 80 86 90 92 96 108 110) \\
(0 & 2 6 8 12 20 26 30 36 38 42 48 56 62 66 68 72 78 80 86 90 92 96 108 110) \\
(0 & 2 6 8 12 18 20 26 30 32 36 42 50 56 62 68 72 78 86 90 92 96 98 102 110)
\end{align*}
\]
We need only consider the case of modulo $p_1$ for each $p_1$ commencing value since the very first result $p_1 \equiv 0 \pmod{p_1}$ is the solitary and unique result that can arise as part of the iterative computation to obtain the desired complete set of results $p_1 \equiv 0 \pmod{p_1}$, $p_1 \equiv 1 \pmod{p_1}$, $p_1 \equiv 2 \pmod{p_1}$, ..., $p_1 \equiv p_1 - 2 \pmod{p_1}$ [that will maximally conform to the admissibility criterion], $p_1 \equiv p_1 - 1 \pmod{p_1}$ [that will minimally conform to the inadmissibility criterion]. For each and every $p_1$ commencing value iteratively computed in Box 1 using first 15 $p_1$ commencing values [akin to a longitudinal study over the selected $p_1$ commencing values], we can always obtain the allowable set of $k$-valued Admissible Prime $k$-tuplets/$k$-tuples with its cardinality being a finite number and the associated allowable set of $k$-valued Inadmissible Prime $k$-tuplets with its cardinality being an arbitrarily large number. Progressively larger sets of finite $k$-valued Admissible Prime $k$-tuplets/$k$-tuples associated with the first 15 $p_1$ commencing values of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 have corresponding cardinality of 0, 1, 4, 5, 14, 20, 19, 34, 38, 58, 94, 72, 93, 76, 77.

**Box 1.** Admissible Prime $k$-tuplets/$k$-tuples and Inadmissible Prime $k$-tuplets for initial 15 $p_1$ commencing values.

$p_1$ **commencing value = 2.** Set Admissible Prime $k$-tuplets as $k$-value = 0 [empty set] with its cardinality $= |\text{CFS}| = 0$. Set Inadmissible Prime $k$-tuplets as $k$-value = 2 [having nadir diameter $d = 1$ and nadir average gap $= 1/2 = 0.5$], 3, 4, 5, 6... with its cardinality $= |\text{CIS-ALN-decelerating}| = N_0$-decelerating. At $p_1 = 2$, failure at mod 2 (term 3) first occur at $k = 2$ with minimum diameter $d = 1$.

$p_1$ **commencing value = 3.** Set Admissible Prime $k$-tuplets as $k$-value = 2 [having zenith diameter $d = 2$ and zenith average gap $= 2/1 = 2$] with its cardinality $= |\text{CFS}| = 1$. Set Inadmissible Prime $k$-tuplets as $k$-value = 3 [having nadir diameter $d = 4$ and nadir average gap $= 4/3 = 1.33$], 4, 5, 6... with its cardinality $= |\text{CIS-ALN-decelerating}| = \text{symbolically } N_0$. At $p_1 = 3$, failure at mod 3 (term 7) first occur at $k = 3$ with minimum diameter $d = 4$.

$p_1$ **commencing value = 5.** Set Admissible Prime $k$-tuplets/$k$-tuples as $k$-value = 2, 3, 4, 5 [having zenith diameter $d = 12$ and zenith average gap $= 12/5 = 2.4$] with its cardinality $= |\text{CFS}| = 4$. Set Inadmissible Prime $k$-tuplets as $k$-value = 6 [having nadir diameter $d = 14$ and nadir average gap $= 14/6 = 2.33$], 7, 8, 9, 10... with its cardinality $= |\text{CIS-ALN-decelerating}| = N_0$-decelerating. At $p_1 = 5$, failure at mod 5 (term 19) first occur at $k = 6$ with minimum diameter $d = 14$.

$p_1$ **commencing value = 7.** Set Admissible Prime $k$-tuplets/$k$-tuples as $k$-value = 2, 3, 4, 5, 6 [having zenith diameter $d = 16$ and zenith average gap $= 16/6 = 2.67$] with its cardinality $= |\text{CFS}| = 5$. Set Inadmissible Prime $k$-tuplets as $k$-value = 7 [having nadir diameter $d = 22$ and nadir average gap $= 22/7 = 3.14$], 8, 9, 10... with its cardinality $= |\text{CIS-ALN-decelerating}| = N_0$-decelerating. At $p_1 = 7$, failure at mod 7 (term 29) first occur at $k = 7$ with minimum diameter $d = 22$.

$p_1$ **commencing value = 11.** Set Admissible Prime $k$-tuplets/$k$-tuples as $k$-value = 2, 3, 4, 5,..., 15 [having zenith diameter $d = 56$ and zenith average gap $= 56/15 = 3.73$] with its cardinality $= |\text{CFS}| = 14$. Set Inadmissible Prime $k$-tuplets as $k$-value = 16 [having nadir diameter $d = 60$ and nadir average gap $= 60/16 = 3.75$], 17, 18, 19, 20... with its cardinality $= |\text{CIS-ALN-decelerating}| = N_0$-decelerating. At $p_1 = 11$, failure at mod 11 (term 71) first occur at $k = 16$ with minimum diameter $d = 60$.

$p_1$ **commencing value = 13.** Set Admissible Prime $k$-tuplets/$k$-tuples as $k$-value = 2, 3, 4, 5,..., 21 [having zenith diameter $d = 88$ and zenith average gap $= 88/21 = 4.19$] with its cardinality $= |\text{CFS}| = 20$. Set Inadmissible Prime $k$-tuplets as $k$-value = 22 [having nadir diameter $d = 90$ and nadir average gap $= 90/22 = 4.09$], 23, 24, 25, 26... with its cardinality $= |\text{CIS-ALN-decelerating}| = N_0$-decelerating. At $p_1 = 13$, failure at mod 13 (term 103) first occur at $k = 22$ with minimum diameter $d = 90$.

$p_1$ **commencing value = 17.** Set Admissible Prime $k$-tuplets/$k$-tuples as $k$-value = 2, 3, 4, 5,..., 20 [having zenith diameter $d = 84$ and zenith average gap $= 84/20 = 4.2$] with its cardinality $= |\text{CFS}| = 19$. Set Inadmissible Prime $k$-tuplets as $k$-value = 21 [having nadir diameter $d = 86$ and nadir average gap $= 86/21 = 4.10$], 22, 23, 24, 25... with its cardinality $= |\text{CIS-ALN-decelerating}| = N_0$-decelerating. At $p_1 = 17$, failure at mod 17 (term 103) first occur at $k = 21$ with minimum diameter $d = 86$.

$p_1$ **commencing value = 19.** Set Admissible Prime $k$-tuplets/$k$-tuples as $k$-value = 2, 3, 4, 5,..., 35 [having zenith diameter $d = 162$ and zenith average gap $= 162/35 = 4.63$] with its cardinality $= |\text{CFS}| = 34$. Set Inadmissible Prime $k$-tuplets as
k-value = 36 [having nadir diameter d = 172 and nadir average gap = 172/36 = 4.78], 37, 38, 39, 40... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 19\), failure at mod 19 (term 191) first occur at k = 36 with minimum diameter d = 172.

**p₁ commencing value = 23.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 39 [having zenith diameter d = 188 and zenith average gap = 188/39 = 4.82] with its cardinality = |CFS| = 38. Set Inadmissible Prime k-tuplets as k-value = 40 [having nadir diameter d = 200 and nadir average gap = 200/40 = 5], 41, 42, 43, 44... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 23\), failure at mod 23 (term 223) first occur at k = 40 with minimum diameter d = 200.

**p₁ commencing value = 29.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 59 [having zenith diameter d = 308 and zenith average gap = 308/59 = 5.22] with its cardinality = |CFS| = 58. Set Inadmissible Prime k-tuplets as k-value = 60 [having nadir diameter d = 318 and nadir average gap = 318/60 = 5.3], 61, 62, 63, 64... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 29\), failure at mod 29 (term 347) first occur at k = 60 with minimum diameter d = 318.

**p₁ commencing value = 31.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 95 [having zenith diameter d = 540 and zenith average gap = 540/95 = 5.68] with its cardinality = |CFS| = 94. Set Inadmissible Prime k-tuplets as k-value = 96 [having nadir diameter d = 546 and nadir average gap = 546/96 = 5.69], 97, 98, 99, 100... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 31\), failure at mod 31 (term 577) first occur at k = 96 with minimum diameter d = 546.

**p₁ commencing value = 37.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 73 [having zenith diameter d = 396 and zenith average gap = 396/73 = 5.24] with its cardinality = |CFS| = 72. Set Inadmissible Prime k-tuplets as k-value = 74 [having nadir diameter d = 402 and nadir average gap = 402/74 = 5.43], 75, 76, 77, 78... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 37\), failure at mod 37 (term 439) first occur at k = 74 with minimum diameter d = 402.

**p₁ commencing value = 41.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 94 [having zenith diameter d = 536 and zenith average gap = 536/94 = 5.70] with its cardinality = |CFS| = 93. Set Inadmissible Prime k-tuplets as k-value = 95 [having nadir diameter d = 546 and nadir average gap = 546/95 = 5.75], 96, 97, 98, 99... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 41\), failure at mod 41 (term 587) first occur at k = 95 with minimum diameter d = 546.

**p₁ commencing value = 43.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 77 [having zenith diameter d = 420 and zenith average gap = 420/77 = 5.45] with its cardinality = |CFS| = 76. Set Inadmissible Prime k-tuplets as k-value = 78 [having nadir diameter d = 424 and nadir average gap = 424/78 = 5.44], 79, 80, 81, 82... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 43\), failure at mod 43 (term 467) first occur at k = 78 with minimum diameter d = 424.

**p₁ commencing value = 47.** Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 78 [having zenith diameter d = 432 and zenith average gap = 432/78 = 5.54] with its cardinality = |CFS| = 77. Set Inadmissible Prime k-tuplets as k-value = 79 [having nadir diameter d = 440 and nadir average gap = 440/79 = 5.57], 80, 81, 82, 83... with its cardinality = |CIS-ALN-decelerating| = \(\aleph_0\)-decelerating. At \(p_1 = 47\), failure at mod 47 (term 487) first occur at k = 79 with minimum diameter d = 440.

In relation to all \(p_1\) commencing values, the cardinality of their associated Admissible Prime k-tuplets/k-tuples must be countably finite in number [except for \(p_1\) commencing value of 2 having nil Admissible Prime k-tuple]; and the cardinality of their associated Inadmissible Prime k-tuples must be countably arbitrarily large in number. Proposed to be predominantly applicable to selected Sub I Adm P k-tuplets and 2nd V Sub II Inadm P (k+1)-tuplets, there are generally three potential (pseudo-)default mechanisms: (i) all eligible residues should be self-replicating for every \(p_1\) commencing values [apart from the first-occurring Inadmissible Prime k-tuples with \(p_1\) commencing values of 2 as two consecutive primes (2, 3); of 3 as three consecutive primes (3, 5, 7); and of 7 as seven consecutive primes (7, 11, 13, 17, 19, 23, 29) whose residues were all not self-replicating], (ii) all eligible even prime gaps should be self-replicating for every \(p_1\) commencing values [apart from relevant Prime k-tuplets/k-tuples with \(p_1\) commencing values of 2 having odd prime gap = 1 which is not an even prime gap and will never self-replicate], and (iii) an eligible even prime gap should not be missing or skipped for every \(p_1\) commencing values [apart from Sub I Inadm P 2-tuple with \(p_1\) commencing value of 2 as consecutive primes (2, 3) having once-occurring odd prime gap 1 which is not an even prime gap and is permanently missing or skipped in all other Prime k-tuplets/k-tuples that do not contain even prime number 2].

**Remark 2.** All above three (pseudo-)default mechanisms are simply not ubiquitous mechanisms because they will not be applicable for an arbitrarily large number of Prime k-tuplets/k-tuples. Applying logical deductive reasoning to the last two (pseudo-)default mechanisms for relevant Admissible Prime k-tuplets/k-tuples and Inadmissible Prime k-tuplets, it is an observed *sine qua non* phenomenon that eligible even prime gaps that are [temporarily] not self-replicating and/or are [temporarily] missing or skipped in these tuples do not [permanently] persist over the entire sequence of consecutive odd
Named after him, Norman Luhn first noted on and multiplier 2541318803
the longest possible Admissible Prime k-tuple [and at least p1-consecutive integers representing residues 0, 1, 2, 3, ..., p1-1 that cater for the longest possible Admissible Prime k-tuple [and at least p1-consecutive integers representing residues 0, 1, 2, 3, ..., p1-1 that cater for the shortest possible Inadmissible Prime (k+1)-tuple]. Apart from the first four cardinality depicted above that are smaller than or equal to p1-1, all subsequent cardinality must not be smaller than their corresponding p1-1 with the important implication that we can always derive arbitrarily long Admissible Prime k-tuples with maximal k values that must be at least equal to [but are usually always larger than] p1-1. We incidentally recognize the Incompletely Predictable cyclical nature of the computed data in Box 1 but, in general, ever larger p1-commencing values are [overall] associated with ever larger k-valued Admissible Prime k-tuples/k-tuples that characteristically have ever larger zenith diameter d and zenith average gaps.

The proof is now complete for Proposition 1.2.\(\Box\).

For every appropriately paired Admissible Prime k-tuplet patterns endowed with same modulo number, there exists a counterpart. For instance, Sub I Adm Prime 7-tuplet pattern-1 (0, 2, 6, 8, 12, 18, 20) has its p1 congruent to 11 (modulo 210) and Sub I Adm Prime 7-tuplet pattern-2 (0, 2, 8, 12, 14, 18, 20) has its p1 congruent to 179 (modulo 210). We see that 11 + 179 (viz, the counterpart) + 20 (viz, the diameter d) = 210 (viz, the modulo number). The offset and multiplier containing variable n is closely related to \(p_1\) congruent to \(p (\text{modular } q)\) for Admissible Prime k-tuplets as explained using the following examples.

Example 1: For Sub I Adm P 7-tuplet with pattern-1 given as cumulative prime gaps (0, 2, 6, 8, 12, 18, 20) \(\equiv\) cumulative prime numbers (11, 13, 17, 19, 23, 29, 31) [as based on first-occurring \(p_1 = 11\)]; the \(p_1\) congruent to 11 (modulo 210) is equivalent to offset and multiplier 11 + 210*n. This is given by A022009 Initial members of prime septuplets \((p, p+2, p+6, p+8, p+12, p+18, p+20)\). (Roonguthai, n.d.) having values 11, 165701, 1068701, 1190051, 15760091, 18504371, 21036131, 25658441, 39431921, 45002591, 67816361, 86818211, 93625991, 124716071, 136261241, 140117051, 154635191, 162189101, 182403491, 187029371, 190514321, 198453371... which is cross linked to A182387 Numbers \(n\) such that \(210^*n+11, 13, 17, 19, 23, 29, 31\) are 7 consecutive primes. (Seidov, 2012) having values 0, 789, 5089, 56669, 75048, 88116, 100112, 122183, 187771, 214249, 322935, 413420, 445838, 593886, 648863, 676224, 736358, 772329, 868588, 888020, 890616, 907211, 945016, 1052954, 1078331, 110677, 1146724, 1223888, 1432230, 1452437, 1458355, 1509878, 1535216....

Example 2: For Sub I Adm P 7-tuplet with pattern-2 given as cumulative prime gaps (0, 2, 6, 8, 12, 14, 18, 20) \(\equiv\) cumulative prime numbers (11, 13, 17, 19, 23, 29, 31) [as based on first-occurring \(p_1 = 11\)]; the \(p_1\) congruent to 11 (modulo 210) is equivalent to offset and multiplier 179 + 210*n. This is given by A022010 Initial members of prime septuplets \((p, p+2, p+6, p+8, p+12, p+14, p+18, p+20)\). (Roonguthai, n.d.) having values 11, 13, 17, 19, 23, 29, 31 are 7 consecutive primes. (Seidov, 2012) having values 0, 789, 5089, 56669, 75048, 88116, 100112, 122183, 187771, 214249, 322935, 413420, 445838, 593886, 648863, 676224, 736358, 772329, 868588, 888020, 890616, 907211, 945016, 1052954, 1078331, 110677, 1146724, 1223888, 1432230, 1452437, 1458355, 1509878, 1535216....

Example 3: For Sub I Adm P 38-tuplet there are six possible patterns with pattern-4 given as cumulative prime gaps (0, 2, 6, 8, 12, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 60, 66, 74, 78, 80, 84, 86, 90, 104, 108, 114, 116, 126, 128, 134, 140, 144, 150, 156, 168, 170, 174, 176) \(\equiv\) cumulative prime numbers (23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199) [as based on first-occurring \(p_1 = 23\)]; the \(p_1\) congruent to 2541318803 (modulo 6469693230) which is equivalent to offset and multiplier 2541318803 + 6469693230*n is also applicable in a similar manner to previous two examples.

Named after him, Norman Luhn first noted on circa 9 February 1999 the prime number 229 belong to a special class of prime numbers 229, 239, 241, 257, 269, 271, 277, 281, 439, 443, 463, 467, 479... that is defined by A061783 Luhn primes: \(p\) such that \(p + (p \text{ reversed})\) is also a prime (Murthy, 2001). On 26 November 2022, Luhn generously supplied his tiny freeware program on patterns of Sub I Adm P k-tuplets that correctly work from \(k = 2\) to 50 – see Appendix B whereby A083409 Number of prime k-tuple constellations, i.e., patterns with minimal diameter A008407 (Eller mann, 2003) is relevant. Computed for \(k = 2, 3, 4, 5, 6...\); the number of possible patterns are 1, 2, 1, 2, 1, 2, 3, 4, 2, 2, 2, 6, 2, 4, 2, 4, 2, 4, 2, 4, 2, 4, 18, 2, 8, 10, 2, 2, 2, 4, 14, 20, 2, 2, 2, 6, 26, 28, 8, 2, 18, 4, 4, 4, 2, 2, 22, 22, 2, 26, 6, 6, 2, 4, 2, 2, 6, 2, 4, 2, 2, 18, 2, 20, 2, 2, 2, 10, 2, 14, 40, 2, 14, 16, 4, 2, 2, 60, 50, 2, 2, 16, 2, 18, 12....

Professor Neil J. A. Sloane is the Chairman, On-Line Encyclopedia of Integer Sequences Foundation. As chronologically alluded to by the author of this paper during email chain correspondences between himself, Sloane [7 January 2022 – 8 January 2022] and Luhn [24 November 2022 – 27 November 2022]; the former main type Admissible Prime k-tuples
when given in comparatively same k values consist of two subtypes (i) [repeating] Sub I Adm P k-tuples, often called Prime k-tuplets or Prime constellations, which are simply Admissible Prime k-tuples with their diameter d being the smallest possible diameter and (ii) [repeating] Sub II Adm P k-tuples, divided into two different varieties [viz, 1st V Sub II Adm P k-tuples and 2nd V Sub II Adm P k-tuples] as outlined in Part I Analysis and Part II Analysis below, with their diameter d being much larger than and slightly larger than the corresponding smallest possible diameter of Sub I Adm P k-tuplets. As per Part I Analysis which fully comply with Proposition 1.21, 1st V Sub II Adm P k-tuplets tend to generate large(r) prime numbers with large(r) prime gaps. As per Part II Analysis which fully comply with Proposition 1.21, 2nd V Sub II Adm P k-tuples tend to generate small(er) prime numbers with small(er) prime gaps.

Two subtypes of the later main type Inadmissible Prime k-tuples when given in comparatively same k values are fully discussed in Part I Analysis and Part II Analysis below: (i) [non-repeating] Sub I Inadm P k-tuples belong to the group with their diameter d being the smallest, and (ii) [non-repeating] Sub II Inadm P k-tuples belong to the other group which is again divided into two different varieties. As per Part I Analysis, the [nonexisting] 1st V Sub II Inadm P k-tuples with diameter d being much larger than this smallest possible diameter of Sub I Adm P k-tuplets are the Forbidden Inadmissible Prime k-tuples. As per Part II Analysis, the [existing-just-once] 2nd V Sub II Inadm P k-tuples have their varying diameter d being slightly smaller than, equal to or larger than the smallest possible diameter of Sub I Adm P k-tuplets.

A prime k-tuple is admissible in its sequence of consecutive primes \( \{p_1, p_2, \ldots, p_k\} \) such that for every prime \( q \leq k \), not all the residues modulo \( q \) are represented by \( p_1, p_2, \ldots, p_k \). Simplest Sub I Adm P k-tuples and Sub II Adm P k-tuples using \( k = 2 \) value include all twin primes as Sub I Adm P 2-tuplet with smallest possible diameter \( d \) (prime gap) \( = 2 \); all cousin primes as Sub II Adm P 2-tuplet with larger diameter \( d \) (prime gap) \( = 4 \); all sexy primes as Sub II Adm P 2-tuplet with larger diameter \( d \) (prime gap) \( = 6 \); etc. We note in next paragraph that Sub II Adm P 2-tuples could be [arbitrarily] classified as either 1st V Sub II Adm P 2-tuples or 2nd V Sub II Adm P k-tuples. The Sub I Adm P 3-tuplet pattern-1 \((p_1, p_2, p_3)\) and pattern-2 \((p_1, p_3, p_5)\) have their respective first occurrences at consecutive prime numbers \((5, 7, 11)\) and \((7, 11, 13)\). Then an example of 1st V Sub II Adm P 3-tuple pattern-1 \((p_1, p_2, p_3, p_4)\) is given by consecutive prime numbers \((5639, 5641, 5647, 5651)\). We duly note this particular 1st V Sub II Adm P 3-tuple is also a subtuple forming part of Sub I Adm P 7-tuplet pattern-2 \((p_1, p_2, p_4, p_6, p_8, p_{10}, p_{12})\) given by first occurrence consecutive prime numbers \((5639, 5641, 5647, 5651, 5653, 5657, 5659)\).

Both the Sub I Adm P 2-tuplets as two consecutive primes \((p_1, p_2)\) with diameter \( d \) or prime gap \( = p_2 - p_1 = 2 \) and the Sub II Adm P 2-tuplets as two consecutive primes \((p_1, p_2)\) with diameter \( d \) or prime gap \( = p_2 - p_1 \geq 4 \) can match an arbitrarily large number of positions in the sequence of prime numbers. The Sub II Adm P 2-tuples could arbitrarily be regarded as belonging to either the 1st V Sub II Adm P k-tuples conforming to criterion \( p_k - p_1 \geq p_1 - p_{k-2} \) or 2nd V Sub II Adm P k-tuples conforming to criterion \( p_k - p_1 > p_1 - p_{k-1} \). These two varieties of Sub II Adm P k-tuples could be arbitrarily large number of Sub II Adm P 2-tuples conforming to criterion \( p_k - p_1 = p_1 - p_{k-2} \) and manifesting as two identical consecutive prime gaps \((6n, 6n)\) \((6, 6)\), \((12, 12)\), \((18, 18)\), etc could arbitrarily be regarded as belonging to either 1st V or 2nd V Sub II Adm P k-tuples. They could in principle also form [bridging] smaller subtriples of steady primes in Sub I Adm P k-tuplets or 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples only match [non-repeating] smaller subtriples of accelerating primes in Sub I Adm P k-tuplets or 1st V and 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples or 2nd V Sub II Inadm P k-tuples when \( k \geq 3 \). For \( n = 1, 2, 3, 4, 5, \ldots; \) the rarely occurring but nevertheless arbitrarily large number of Sub II Adm P 2-tuples conforming to criterion \( p_k - p_1 = p_1 - p_{k-2} \) and manifesting as two identical consecutive prime gaps \((6n, 6n)\), \((6, 6)\), \((12, 12)\), \((18, 18)\), etc could arbitrarily be regarded as belonging to either 1st V or 2nd V Sub II Adm P k-tuples. They could in principle also form [bridging] smaller subtriples of steady primes in Sub I Adm P k-tuplets or 1st V and 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples or 2nd V Sub II Inadm P k-tuples when \( k \geq 3 \). The criterion \( p_k - p_1 < p_1 - p_{k-2} \) will be conformed to by an arbitrarily large number of 1st V or 2nd V Sub II Adm P 2-tuplets whereby they could in principle also form [bridging] smaller subtriples of decelerating primes in Sub I Adm P k-tuplets or 1st V and 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples or 2nd V Sub II Inadm P k-tuples when \( k \geq 3 \). Thus subtriples of accelerating, steady and decelerating primes [as further elaborated upon later on] essentially form formal repeated groupings of small and/or large prime numbers and gaps.

**Remark 3.** At ever larger range of \( x \geq 4 \) integer values manifesting progressively less prime numbers, we intuitively expect an overall slowly increasing prevalence of first variety of Subtype II Admissible Prime k-tuples that cater for large(r) prime numbers and gaps which is reciprocally and simultaneously associated with an overall slowly decreasing prevalence of second variety of Subtype II Admissible Prime k-tuples that cater for smaller(r) prime numbers and gaps. When Subtype I Admissible Prime 2-tuplets as two consecutive primes \((p_1, p_2)\) with diameter \( d \) or prime gap \( = p_2 - p_1 = 2 \) are combined with Subtype II Admissible Prime 2-tuplets as two consecutive primes \((p_1, p_3)\) with diameter \( d \) or prime gap \( = p_3 - p_1 \geq 4 \), they will be able to [uniquely] represent every known odd prime numbers in an non-overlapping manner.

A Prime k-tuple is inadmissible in its sequence of consecutive primes \((p_1, p_2, \ldots, p_k)\) such that for some of the prime \( q \leq k \) e.g. for one of the prime \( q \leq k \) when \( k \geq 3 \) or for two of the prime \( q \leq k \) if \( p_1 = 2 \) forms part of a Prime k-tuple when \( k \geq 4 \); all the residues modulo \( q \) are represented by \( p_1, p_2, \ldots, p_k \). All [non-repeating] Sub I Inadm P k-tuples only match one finite position in the sequence of prime numbers and are defined by their diameter \( d \) being the shortest. An arbitrarily large number of examples with one all-prime solution for this subtype include Prime 2-tuple \((p+0, p+1)\) as prime numbers.
(2, 3) with $d = 1$; Prime 3-tuple $(p+0, p+1, p+3)$ as prime numbers $(2, 3, 5)$ with $d = 3$; Prime 3-tuple $(p+0, p+2, p+4)$ as prime numbers $(3, 5, 7)$ with $d = 4$; Prime 4-tuple $(p+0, p+1, p+3, p+5)$ as prime numbers $(2, 3, 5, 7)$ with $d = 5$; Prime 4-tuple $(p+0, p+2, p+4, p+8)$ as prime numbers $(3, 5, 7, 11)$ with $d = 8$ etc.

Modular arithmetic: $a \equiv b \pmod{n}$ whereby $a = b$ is congruent, $n = \text{divisor}$ and $r = \text{remainder}$ [round up to the next integer]. Therefore, $a \equiv b \pmod{n} \equiv a - (r \times n)$. With abbreviation $n$ denoting numbers, we analyze the Completely Predictable even and odd $n$. For $i = 0, 1, 2, 3, 4, 5...\ldots$ congruence $n \equiv 0 \pmod{2}$ holds for even $n = E_i = 2^i i = 0, 2, 4, 6, 8, 10...$ and for $i = 1, 2, 3, 4, 5, 6...\ldots$ congruence $n \equiv 1 \pmod{2}$ holds for odd $n = O_i = (2^{i-1}) - 1 = 1, 3, 5, 7, 9, 11...$ whereby $0$ is the zeroth even number when we consider all (non-negative) positive even and odd $n$ obtained from whole $n = 0, 1, 2, 3, 4, 5...$. We analyze the Incompletely Predictable prime numbers collectively grouped as k-tuples. For the worked example of modular arithmetic applied to test for admissibility on Sub I Adm P 4-tuple $(p+0, p+1, p+3, p+5)$ as cumulative prime gaps $(0, 1, 3, 5)$ with earliest and only candidate as consecutive prime numbers $(2, 3, 5, 7)$ having progressive prime gaps $(0, 1, 2, 2)$; we can use either $[I]$ cumulative prime gaps: congruence $0, 1, 3, 5 \equiv 0, 1, 1, 1 \pmod{2}$ and congruence $0, 1, 3, 5 \equiv 0, 1, 0, 2 \pmod{3}$ or $[II]$ consecutive prime numbers: congruence $2, 3, 5, 7 \equiv 0, 1, 1, 1 \pmod{2}$ and congruence $2, 3, 5, 7 \equiv 2, 0, 2, 1 \pmod{3}$ (mod prime 3). There are two failures at [firstly] mod prime 2 on second term $= 1$ (as prime gap) or 3 (as prime number) and [secondly] mod prime 3 on last term $= 5$ (as prime gap) or 7 (as prime number) $\implies$ this Sub I Adm P 4-tuple is now confirmed to be inadmissible. Since twin prime $(3, 5)$ is a Sub I Adm P 2-tuple when first element $= 3$, we can redundantly generate a complete all-inclusive countably arbitrarily large number of $[non-repeating]$ Sub I Adm P k-tuples using progressively longer $k$ values, the countably arbitrarily large number of 1st V Sub II Adm P $k$-tuples are defined as having diameter $d$ which first appear as consecutive primes $(5, 7, 11)$. This Sub I Adm P 3-tuple (5, 7, 11) as prime numbers $(2, 3, 5, 7)$ with $d = 8$ is of the form $[single-digit]$ odd prime number 5 with its last and only digit also ending in odd number 5, all other larger $[multiple-digit]$ odd prime numbers cannot have their last digit ending in odd number 5 and, consequently, these forbidden numbers can never belong to any Prime k-tuples and Prime k-tuples. Otherwise, apart from the solitary odd prime number 5, it is an established mathematical fact that all odd prime numbers must have their last digit ending in odd number 5 and, consequently, these forbidden conditions must be conform to by all Prime k-tuples and Prime k-tuples including 1st V Sub II Inadm P k-tuples are now given below.

(1) Apart from the solitary $[single-digit]$ odd prime number 5 with its last and only digit also ending in odd number 5, all other larger $[multiple-digit]$ odd prime numbers cannot have their last digit ending in odd number 5 and, consequently, these forbidden numbers can never belong to any Prime k-tuples and Prime k-tuples. Otherwise, apart from the solitary odd prime number 5, it is an established mathematical fact that all odd prime numbers must have their last digit ending in odd number 1, 3, 7 or 9.

(2) The arbitrarily large number of Sub I Adm P 4-tuples $(p+0, p+2, p+6, p+8)$ with smallest possible diameter $d = 8$ is first given by consecutive primes $(5, 7, 11, 13)$ whereby this must be differentiated from the totally different $[solitary]$ Sub I Adm P 4-tuple $(p+0, p+2, p+4, p+8)$ given by consecutive primes $(3, 5, 7, 11)$ with $[same-valued]$ smallest diameter $d = 8$. All the arbitrarily large number of $\geq$ 2-digit primes in Sub I Adm P 4-tuples commencing sequentially as $(11, 13, 17, 19), (101, 103, 107, 109), (191, 193, 197, 199), (821, 823, 827, 829)...$ must always occur in the same ten-block. Hence it is an established mathematical fact that there must be exactly one with each of these unit digits 1, 3, 7 and 9 in all $\geq$ 2-digit primes from Sub I Adm P 4-tuples. Except for the first term $p_1 = 5$ in Sub I Adm P 4-tuple $(5, 7, 11, 13)$, all other terms are congruent to 11 (mod 30). Thus all Sub I Adm P 4-tuples except when first term $p_1 = 5$ are of the form $(15k-4, 15k-2, 15k+2, 15k+4)$ with $k \geq 1$, and so are centered on 15k.

Part I Analysis: Subtype I Admissible Prime k-tuples + First variety of Subtype II Admissible Prime k-tuples + First variety of Subtype II Inadmissible Prime k-tuples. For the comparative same $k$ value $[akin to a cross-sectional study at specific k values]$, the countably arbitrarily large number of 1st V Sub II Adm P k-tuples are defined as having diameter $d$ being $[much]$ larger than the corresponding diameter $d$ allocated for Sub I Adm P k-tuples.

As complements to the 1st V Sub II Adm P k-tuples, the Forbidden Inadmissible Prime k-tuples are equivalent to the $[nonexisting]$ 1st V Sub II Inadm P k-tuples with their varying diameter $d$ being much larger than the smallest possible diameter of $[repeating]$ Sub I Adm P k-tuples. All the countably arbitrarily large number of these literally forbidden Prime k-tuples are proposed to not match any position in the sequence of prime numbers.

For $k = 2, p = 2$ pattern $Sub I$ Inadm P 2-tuple $[smallest possible diameter d = 1]$ starts at $p = 2$ pattern with first and only occurrence as $(p+0, p+1) = (2, 3)$. There is no 1st V Sub II Adm P 2-tuple with $[consecutive]$ larger diameter $(prime gap) = 3$ as $(p+0, p+3), 5$ as $(p+0, p+5), 7$ as $(p+0, p+7), 9$ as $(p+0, p+9), 11$ as $(p+0, p+11)...$
For \( k = 2, p = 3 \) pattern Sub I Adm P 2-tuplet [smallest possible diameter \( d = 2 \)] starts at \( p = 3 \) pattern with first occurrence as \((p+0, p+2) = (3, 5)\). There are arbitrarily large number of 1st V Sub II Adm P 2-tuples with [consecutive] larger diameter (prime gap) = 4 as \((p+0, p+4), 6 \) as \((p+0, p+6), 8 \) as \((p+0, p+8), 10 \) as \((p+0, p+10), 12 \) as \((p+0, p+12)\)...

there is no existing Inadmissible Prime 2-tuple.

For \( k = 3, p = 5 \) pattern-1 Sub I Adm P 3-tuplet [smallest possible diameter \( d = 6 \)] starts at \( p = 5 \) pattern-1 with first occurrence as \((p+0, p+2, p+6) = (5, 7, 11)\). There are arbitrarily large number of 1st V Sub II Adm P 3-tuples with [non-consecutive] larger diameter = 8 as \((p+0, p+2, p+8), 12 \) as \((p+0, p+2, p+12), 14 \) as \((p+0, p+2, p+14), 18 \) as \((p+0, p+2, p+18), 20 \) as \((p+0, p+2, p+20)\)...

the non-existing [non-consecutive] larger diameter \( d = 10 \) as \((p+0, p+2, p+10), 16 \) as \((p+0, p+2, p+16), 22 \) as \((p+0, p+2, p+22), 28 \) as \((p+0, p+2, p+28)\)...

are Forbidden Inadmissible Prime 3-tuples all with failure at mod prime 3 (last term = 10, 16, 22, 28, 36) for \( n = 0, 1, 2, 3... \) in Prime 3-tuples of the one format \((p+0, p+2, p+4+6n)\), we notice that these are actually non-existing 1st V Sub II Inadm P 3-tuples apart from the solitary Sub I Adm P 3-tuple \((p+0, p+2, p+4)\) located at consecutive prime numbers \((3, 5, 7)\) when \( n = 0 \).

For \( k = 3, p = 7 \) pattern-2 Sub I Adm P 3-tuplet [smallest possible diameter \( d = 6 \)] starts at \( p = 7 \) pattern-2 with first occurrence as \((p+0, p+4, p+6) = (7, 11, 13)\). There are arbitrarily large number of 1st V Sub II Adm P 3-tuples with [non-consecutive] larger diameter = 10 as \((p+0, p+4, p+10), 12 \) as \((p+0, p+4, p+12), 16 \) as \((p+0, p+4, p+16), 18 \) as \((p+0, p+4, p+18), 22 \) as \((p+0, p+4, p+22), 24 \) as \((p+0, p+4, p+24)\)...

the non-existing [non-consecutive] larger diameter \( d = 8 \) as \((p+0, p+4, p+8), 14 \) as \((p+0, p+4, p+14), 20 \) as \((p+0, p+4, p+20)\)...

are Forbidden Inadmissible Prime 3-tuples all with failure at mod prime 3 (last term = 8, 14, 20, 26, 34, 38, 44, 54,...) for \( n = 0, 1, 2, 3... \) in Prime 3-tuples of the one format \((p+0, p+4, p+8+6n)\), we notice that these are actually non-existing 1st V Sub II Inadm P 3-tuples.

For \( k \geq 4, p = 5 \) pattern Sub I Adm P 4-tuplet [smallest possible diameter \( d = 8 \)] starts at \( p = 5 \) pattern with first occurrence as \((p+0, p+2, p+6, p+8) = (5, 7, 11, 13)\). There are arbitrarily large number of 1st V Sub II Adm P 4-tuples with [non-consecutive] larger diameter = 12 as \((p+0, p+2, p+6, p+12), 14 \) as \((p+0, p+2, p+6, p+14), 18 \) as \((p+0, p+2, p+6, p+18), 20 \) as \((p+0, p+2, p+6, p+20)\)...

the non-existing [non-consecutive] larger diameter \( d = 10 \) as \((p+0, p+2, p+6, p+10), 16 \) as \((p+0, p+2, p+6, p+16), 22 \) as \((p+0, p+2, p+6, p+22)\)...

are Forbidden Inadmissible Prime 4-tuples all with failure at mod prime 3 (last term = 10, 16, 22, 28, 36, 44, 54,...) for \( n = 0, 1, 2, 3... \) in Prime 4-tuples of the one format \((p+0, p+2, p+6, p+10+6n)\), we notice that these are actually non-existing 1st V Sub II Inadm P 4-tuples.

For \( k = 5, p = 5 \) pattern-1 Sub I Adm P 5-tuplet [smallest possible diameter \( d = 12 \)] starts at \( p = 5 \) pattern-1 with first occurrence as \((p+0, p+2, p+6, p+8, p+12) = (5, 7, 11, 13, 17)\). There are arbitrarily large number of 1st V Sub II Adm P 5-tuples with [non-consecutive] larger diameter = 18 as \((p+0, p+2, p+6, p+8, p+18), 20 \) as \((p+0, p+2, p+6, p+20)\)...

the non-existing [non-consecutive] larger diameter \( d = 14 \) as \((p+0, p+2, p+6, p+14), 16 \) as \((p+0, p+2, p+6, p+16), 22 \) as \((p+0, p+2, p+6, p+22), 24 \) as \((p+0, p+2, p+6, p+24)\)...

are Forbidden Inadmissible Prime 5-tuples all with failure at mod prime 3 (last term = 14, 20, 26, 30, 38, 44,...) for \( n = 0, 1, 2, 3... \) in Prime 5-tuples of the one format \((p+0, p+2, p+6, p+10+6n)\), we notice that these are actually non-existing 1st V Sub II Inadm P 5-tuples.

For \( k = 5, p = 7 \) pattern-2 Sub I Adm P 5-tuplet [smallest possible diameter \( d = 12 \)] starts at \( p = 7 \) pattern-2 with first occurrence as \((p+0, p+4, p+6, p+10, p+12) = (7, 11, 13, 17, 19)\). There are arbitrarily large number of 1st V Sub II Adm P 5-tuples with [non-consecutive] larger diameter = 16 as \((p+0, p+4, p+6, p+10, p+16), 18 \) as \((p+0, p+4, p+6, p+18), 22 \) as \((p+0, p+4, p+6, p+22), 24 \) as \((p+0, p+4, p+6, p+24)\)...

are Forbidden Inadmissible Prime 5-tuples all with failure at mod prime 3 (last term = 14, 20, 26, 30, 38, 44,...) for \( n = 0, 1, 2, 3... \) in Prime 5-tuples of the one format \((p+0, p+4, p+6, p+10+4n+20n)\), we notice that these are actually non-existing 1st V Sub II Inadm P 5-tuples.

For \( k = 6, p = 7 \) pattern Sub I Adm P 6-tuplet [smallest possible diameter \( d = 16 \)] starts at \( p = 7 \) pattern with first occurrence as \((p+0, p+4, p+6, p+10, p+12, p+16) = (7, 11, 13, 17, 19, 23)\). There are arbitrarily large number of 1st V Sub II Adm P 6-tuples with [non-consecutive] larger diameter = 22 as \((p+0, p+4, p+6, p+10, p+12, p+22), 24 \) as \((p+0, p+4, p+6, p+12, p+24), 30 \) as \((p+0, p+4, p+6, p+10, p+30), 34 \) as \((p+0, p+4, p+6, p+10, p+34), 36 \) as \((p+0, p+4, p+6, p+10, p+36), 40 \) as \((p+0, p+4, p+6, p+10, p+40)\)...

are Forbidden Inadmissible Prime 6-tuples all with failure at mod prime 3 (last term = 20, 26, 32, 38, 44, 50, 56, 62, 68,...) for \( n = 0, 1, 2, 3... \) in Prime 6-tuples of the two formats \((p+0, p+4, p+6, p+10+10n/20n)\) and \((p+0, p+4, p+6, p+10, p+12, p+20+6n)\), we notice that
these are actually non-existing 1st V Sub II Inadm P 6-tuples.

For $k = 7$, $p = 11$ pattern-1 Sub I Adm Prime 7-tuple [smallest possible diameter $d = 20$] starts at $p = 11$ pattern-1 with first occurrence as $(p+0, p+2, p+6, p+8, p+12, p+18, p+20) = (11, 13, 17, 19, 23, 29, 31)$. There are arbitrarily large number of 1st V Sub II Adm P 7-tuples with [non-consecutive] larger diameter = $26$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+24), 50$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+30), 32$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+32), 36$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+36), 42$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+42), 48$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+48)$. \[ \implies \] the non-existing [non-consecutive] larger diameter $d = 22$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+22), 24$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+24), 28$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+28), 34$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+34), 38$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+38)$. \[ \implies \] are Forbidden Inadmissible Prime 7-tuples all with failure at mod prime 3 (last term = 22, 28, 34, 40, 46, 52, 58...) or mod prime 5 (last term = 24, 44, 54, 74, 84, 104...) or mod prime 7 (last term = 38, 66, 80, 108, 122, 150...). For n = 0, 1, 2, 3... in Prime 7-tuples of the three formats $(p+0, p+2, p+6, p+8, p+12, p+18, p+20/10n)$(and $(p+0, p+2, p+6, p+8, p+12, p+18, p+20)$. Sub I Inadm P 2-tuple $(2, 3)$ starts at $p = 5639$ pattern-2 with first occurrence as $(p+0, p+2, p+6, p+8, p+12, p+18, p+22)$, $30$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+30), 32$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+32), 36$ as $(p+0, p+2, p+6, p+8, p+12, p+18, p+36). \[ \implies \] are Forbidden Inadmissible Prime 7-tuples all with failure at mod prime 3 (last term = 22, 28, 34, 40, 46, 52, 58...) or mod prime 5 (last term = 24, 44, 54, 74, 84, 104...) or mod prime 7 (last term = 38, 66, 80, 108, 122, 150...). For n = 0, 1, 2, 3... in Prime 7-tuples of the two formats $(p+0, p+2, p+6, p+8, p+12, p+18, p+20)$(and $(p+0, p+2, p+6, p+8, p+12, p+18, p+20/10n)$. We notice that these are actually non-existing 1st V Sub II Inadm P 7-tuples.

Remark 4. Reminiscent of Remark 1 on Gram’s law and its violations, we hereby tentatively propose *Gram’s Prime k-tuple law for $k \geq 6$ values* to uniquely indicate the tendency for first variety of Subtype II Inadmissible Prime k-tuples [as complement to first variety of Subtype II Admissible Prime k-tuples] to manifest failure at mod prime 5 on last terms that will alternately enlarge by +10 and +20; and at mod prime 7 on last terms that will alternately enlarge by +14 and +28. Requiring further research for answers, could similar doubling results affecting the last terms be obtained when relevant modular arithmetic is applied using other prime numbers 11, 13, 17, 19...; and could we ever encounter an arbitrarily large number of violations of this law whereby the alternately enlarging by +10 and +20 and/or by +14 and +20 phenomenon fails to intermittently appear?

Part II Analysis: Subtype I Admissible Prime k-tuples + Second variety of Subtype II Admissible Prime k-tuples + Subtype I Inadmissible Prime k-tuples + Second variety of Subtype II Inadmissible Prime k-tuples. For the comparative same k value [akin to a cross-sectional study at specific k values], the countably arbitrarily large number of 2nd V Sub II Adm k-tuples are defined as having diameter d being [slightly] larger than the corresponding diameter d allocated for Sub I Adm k-tuples.

In perspective, we must differentiate the nonexisting 1st V Sub II Inadm P k-tuples or Forbidden Inadmissible Prime k-tuples mentioned above in Part I Analysis from the manifestly different non-repeating and proposed-to-exist-only-once 2nd V Sub II Inadm P k-tuples mentioned here. All countably arbitrarily large number of [non-repeating] Sub I Inadm P k-tuples and [non-repeating] 2nd V Sub II Inadm P k-tuples that utilize progressively larger consecutive k values are proposed to only match one position [for each k value] in the sequence of prime numbers. The Sub I Inadm P k-tuples and 2nd V Sub II Inadm P k-tuples are selectively defined by their diameter d and first element p which are compared to the diameter d and first element p from the same $k \geq 2$ value that is assigned to the eligible Sub I Adm P k-tuple pattern. Let $p_i$ = first element p of the eligible Sub I Adm P k-tuple pattern for a given $k \geq 2$ value. Then for the same given $k \geq 2$ value, the first element p in the corresponding Sub I Inadm P k-tuple or 2nd V Sub II Inadm P k-tuple = $p_i$, with its varying diameter d manifesting either [slightly] smaller than, equal to or larger than the smallest possible diameter d value of the eligible Sub I Adm P k-tuple pattern. Examples for various k values arranged in increasing order are now provided whereby there are complex intertwined Incompletely Predictable properties arising from selectively comparing and contrasting Sub I Adm P k-tuples, Sub I Inadm P k-tuples, 2nd V Sub II Adm P k-tuples, and 2nd V Sub II Inadm P k-tuples. We will subsequently outline our proposed Rosser’s Prime k-tuple rule and its violations in Remark 5.

For $k = 2$, $p = 3$ pattern and $p_{i-1} = 2$ Sub I Adm P 2-tuple $(3, 5)$: $k = 2$, $p_i = 3$, $d = 2$. Sub I Inadm P 2-tuple $(2, 3)$ with failure at mod prime 2 (term 3): $k = 2$, $p_{i-1} = 2$, $d = 1$. By incorporating ever larger consecutive k values of 3, 4, 5, 6..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime $p = 2$. 

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For $k = 3$, $p_i = 5$ pattern-1 and $p_{i-1} = 3$ Sub I Adm P 3-tuple (5, 7, 11): $k = 3$, $p_i = 5$, $d = 6$. Sub I Inadm P 3-tuple (3, 5, 7) with failure at mod prime 3 (term 7): $k = 3$, $p_{i-1} = 3$, $d = 4$. By incorporating ever larger consecutive k values of 4, 5, 6, 7,..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime $p = 3$.

For $k = 3$, $p_i = 7$ pattern-2 and $p_{i-1} = 5$ Sub I Adm P 3-tuple (7, 11, 13): $k = 3$, $p_i = 7$, $d = 6$. There is no Sub I Inadm P 3-tuple for $k = 3$ when $p_{i-1} = 5$ since Prime 3-tuple (5, 7, 11) with $d = 6$ is just the previous Sub I Adm P 3-tuple. However, there is a Sub I Inadm P 3-tuple (2, 3, 5) with failure at mod prime 2 (term 3): $k = 3$, $p_{i-2} = 2$ [as special exception], $d = 3$. By incorporating ever larger consecutive k values of 4, 5, 6, 7,..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime $p = 2$.

For $k = 4$, $p_i = 5$ pattern and $p_{i-1} = 3$ Sub I Adm P 4-tuple (5, 7, 11, 13): $k = 4$, $p_i = 5$, $d = 8$. Sub I Inadm P 4-tuple (3, 5, 7, 11) with failure at mod prime 3 (term 7): $k = 4$, $p_{i-1} = 3$, $d = 8$. By incorporating ever larger consecutive k values of 5, 6, 7,..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime $p = 3$.

For $k = 5$, $p_i = 5$ pattern-1 and $p_{i-1} = 3$ Sub I Adm P 5-tuple (5, 7, 11, 13, 17): $k = 5$, $p_i = 5$, $d = 12$. Sub I Inadm P 5-tuple (3, 5, 7, 11, 13) with failure at mod prime 3 (term 7): $k = 5$, $p_{i-1} = 3$, $d = 10$. By incorporating ever larger consecutive k values of 6, 7, 8,..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime $p = 3$.

For $k = 5$, $p_i = 7$ pattern-2 and $p_{i-1} = 5$ Sub I Adm P 5-tuple (7, 11, 13, 17, 19): $k = 5$, $p_i = 7$, $d = 12$. There is no Sub I Inadm P 5-tuple for $k = 5$ when $p_{i-1} = 5$ since Prime 5-tuple (5, 7, 11, 13, 17) with $d = 12$ is a Sub I Adm P 5-tuple. However, there is Sub I Inadm P 5-tuple (3, 5, 7, 11, 13) with failure at mod prime 3 (term 7): $k = 5$, $p_{i-2} = 3$ [as special exception], $d = 10$. By incorporating ever larger consecutive k values of 6, 7, 8,..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime $p = 3$.

For $k = 6$, $p_i = 7$ pattern and $p_{i-1} = 5$ Sub I Adm P 6-tuple (7, 11, 13, 17, 19, 23): $k = 6$, $p_i = 7$, $d = 16$. 2nd V Sub II Inadm P 6-tuple (5, 7, 11, 13, 17, 19) with failure at mod prime 5 (term 19): $k = 6$, $p_{i-1} = 5$, $d = 14$. By incorporating ever larger consecutive k values of 7, 8, 9,..., there is a countably arbitrarily large number of 2nd V Sub II Inadm P k-tuples when first prime $p = 5$.

For $k = 7$, $p_i = 11$ pattern-1 and $p_{i-1} = 7$ Sub I Adm P 7-tuple (11, 13, 17, 19, 23, 29, 31): $k = 7$, $p_i = 11$, $d = 20$. 2nd V Sub II Inadm P 7-tuple (7, 11, 13, 17, 19, 23, 29) with failure at mod prime 7 (term 29): $k = 7$, $p_{i-1} = 7$, $d = 22$. By incorporating ever larger consecutive k values of 8, 9,..., there is a countably arbitrarily large number of 2nd V Sub II Inadm P k-tuples when first prime $p = 7$.

For $k = 7$, $p_i = 5639$ pattern-2 and $p_{i-1} = 5623$ Sub I Adm P 7-tuple (5639, 5641, 5647, 5651, 5653, 5657, 5659): $k = 7$, $p_i = 5639$, $d = 20$. There is no 2nd V Sub II Inadm P 7-tuple for $k = 7$ when $p_{i-1} = 5623$ with $d = 34$ or $p_{i-2} = 5591$ with $d = 62$ since these two Prime 7-tuples are 2nd V Sub II Adm P 7-tuples.

For $k = 8$, $p_i = 11$ pattern-1 and $p_{i-1} = 7$ Sub I Adm P 8-tuple (11, 13, 17, 19, 23, 29, 31, 37): $k = 8$, $p_i = 11$, $d = 26$. 2nd V Sub II Inadm P 8-tuple (7, 11, 13, 17, 19, 23, 29, 31) with failure at mod prime 7 (term 29): $k = 8$, $p_{i-1} = 7$, $d = 24$. By incorporating ever larger consecutive k values of 9, 10,..., there is a countably arbitrarily large number of 2nd V Sub II Inadm P k-tuples when first prime $p = 7$.

For $k = 8$, $p_i = 17$ pattern-2 and $p_{i-1} = 13$ Sub I Adm P 8-tuple (17, 19, 23, 29, 31, 37, 41, 43): $k = 8$, $p_i = 17$, $d = 26$. There is no 2nd V Sub II Inadm P 8-tuple for $k = 8$ when $p_{i-1} = 13$ with $d = 28$ or $p_{i-2} = 11$ with $d = 26$ since these two Prime 8-tuples are 2nd V Sub II Adm P 8-tuples.

For $k = 8$, $p_i = 88793$ pattern-3 and $p_{i-1} = 88789$ Sub I Adm P 8-tuple (88793, 88799, 88801, 88807, 88811, 88813, 88817, 88819): $k = 8$, $p_i = 88793$, $d = 26$. There is no 2nd V Sub II Inadm P 8-tuple for $k = 8$ when $p_{i-1} = 88789$ with $d = 28$ or $p_{i-2} = 88771$ with $d = 42$ since these two Prime 8-tuples are 2nd V Sub II Adm P 8-tuples.

<table>
<thead>
<tr>
<th>$k$</th>
<th>2-pat 1</th>
<th>3-pat 1</th>
<th>4-pat 1</th>
<th>5-pat 1</th>
<th>6-pat 1</th>
<th>7-pat 1</th>
<th>8-pat 1</th>
</tr>
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<tbody>
<tr>
<td>Adm. k-Tuplet’s $p_i/d$</td>
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<td>2/3</td>
<td>2/3</td>
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<td>2/3</td>
<td>2/3</td>
<td>2/3</td>
</tr>
<tr>
<td>Inadm. k-Tuple’s $p_{i-1}/d$</td>
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<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>Failure at $p_{i-1}$ mod q (term p)</td>
<td>2 mod 3</td>
<td>3 mod 3</td>
<td>3 mod 3</td>
<td>3 mod 3</td>
<td>3 mod 3</td>
<td>3 mod 3</td>
<td>3 mod 3</td>
</tr>
</tbody>
</table>

Table 2. Computed data of anticipated Failure at $p_{i-1}$ mod prime q on (term prime p) for Sub I Inadm P k-tuples that
occur when \( k = 2, 3, 4 \) and \( 5 \); and 2nd V Sub II Inadm P \( k \)-tuples that occur when \( k \geq 6 \). Sub I Adm P \( k \)-tuples as eligible pattern-1 are simultaneously listed. The \( p_i = 17 \) or 23 fail to [orderly] appear before \( p_i \geq 29 \) or 19 in relation to Sub I Adm P \( k \)-tuples pattern-1 when \( k = 20, 21, 22, 23, 24, \) etc. We include two randomly selected calculations using [ineligible] Prime \( k \)-tuple pattern-2 firstly, when \( k = 17 \) with \( p_i = 17 \) and secondly, when \( k = 22 \) with \( p_i = 23 \). These calculations simply result in their respective Subtype I Prime 17-tuple pattern-1 and Subtype I Prime 22-tuple pattern-1.

For \( k = 2, 3, 4 \) and 5, Sub I Inadm P \( 2 \)-tuple is associated with failure at mod prime 2 (term 3) when \( k = 2 \) & \( d = 1 \) [c.f. \( d = 2 \) for Prime 2-tuple] & \( p_{i-1} = 2 \) [c.f. \( p_i = 3 \) for Prime 2-tuple], Sub I Inadm P 3-tuple is associated with failure at mod prime 3 (term 7) when \( k = 3 \) & \( d = 4 \) [c.f. \( d = 6 \) for Prime 3-tuple] & \( p_{i-1} = 3 \) [c.f. \( p_i = 5 \) for Prime 3-tuple], Sub I Inadm P 4-tuple is associated with failure at mod prime 3 (term 7) when \( k = 4 \) & \( d = 8 \) [c.f. \( d = 8 \) for Prime 4-tuple] & \( p_{i-1} = 3 \) [c.f. \( p_i = 5 \) for Prime 4-tuple], and Sub I Inadm P 5-tuple is associated with failure at mod prime 3 (term 7) when \( k = 5 \) & \( d = 10 \) [c.f. \( d = 12 \) for Prime 5-tuple] & \( p_{i-1} = 3 \) [c.f. \( p_i = 5 \) for Prime 5-tuple].

For \( k \geq 6 \), 2nd V Sub II Inadm P 6-tuple is associated with failure at mod prime 5 (term 19) when \( k = 6 \) & \( d = 14 \) [c.f. \( d = 16 \) for Prime 6-tuple] & \( p_{i-1} = 5 \) [c.f. \( p_i = 7 \) for Prime 6-tuple], 2nd V Sub II Adm P 7-tuple is associated with failure at mod prime 5 (term 19) when \( k = 7 \) & \( d = 22 \) [c.f. \( d = 20 \) for Prime 7-tuple] & \( p_{i-1} = 7 \) [c.f. \( p_i = 11 \) for Prime 7-tuple], 2nd V Sub II Inadm P 8-tuple is associated with failure at mod prime 7 (term 29) when \( k = 8 \) & \( d = 24 \) [c.f. \( d = 26 \) for Prime 8-tuple] & \( p_{i-1} = 7 \) [c.f. \( p_i = 11 \) for Prime 8-tuple], etc.

The diameter \( d \) for Sub I Inadm P \( k \)-tuples is smaller than diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 2 \) and 3, equal to diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 4 \), smaller than diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 5 \). On the limited data presented on diameter \( d \) in Table 2 for 2nd V Sub II Adm P \( k \)-tuples, these diameter \( d \) are seen to be cyclical in nature [although this is not yet proven to be categorically true in a perpetual manner]; viz, it is smaller than diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 6 \), it is larger than diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 7 \), it is smaller than diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 8 \), it is equal to diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 9 \), it is larger than diameter \( d \) for Sub I Adm P \( k \)-tuple when \( k = 10 \), etc.

Violations of Rosser’s Prime \( k \)-tuple rule for \( k \geq 2 \) values in Table 2 are seen to initially occur consecutively at \( k = 20 \) with \( p_i = 29 \) and \( k = 21 \) with \( p_i = 29 \) resulting in the Missing Inadmissible Prime \( k \)-tuples. For \( k = 20 \) with \( p_i = 29 \), these are instead manifested as corresponding for 2nd V Sub II Adm P 20-tuples with slightly larger diameter \( d = 84 \) at \( p_{i-1} = 23 \), \( p_{i-2} = 19 \), \( p_{i-3} = 17 \) and \( p_{i-4} = 13 \). For \( k = 21 \) with \( p_i = 29 \), these are instead manifested as corresponding one 2nd V Sub II Adm P 21-tuple with slightly larger diameter \( d = 86 \) at \( p_{i-1} = 23 \), two 2nd V Sub II Adm P 21-tuples with slightly larger diameter \( d = 88 \) at \( p_{i-2} = 19 \) and \( p_{i-4} = 13 \). However, \( k = 21 \) with \( p_{i-3} = 17 \) having slightly smaller diameter \( d = 84 \) corresponds instead to a 2nd V Sub II Adm P 21-tuple due to failure at mod prime 17 (term 103).

**Remark 5.** Reminiscent of Remark 1 on Rosser’s rule and its violations, we hereby tentatively propose Rosser’s Prime \( k \)-tuple rule for \( k \geq 2 \) values to uniquely indicate each Subtype I Admissible Prime \( k \)-tuple with its prime \( p_i \) commencing value is usually associated with its corresponding (initially) Subtype I Inadmissible Prime \( k \)-tuple [for \( k = 2, 3, 4 \) and 5] and (subsequently) second variety of Subtype II Inadmissible Prime \( k \)-tuple [for \( k \geq 6 \)] that are manifested as failure at mod prime \( p_{i-1} \) (relevant term \( p \)). However, we observe in our tabulated data in Table 2 violations of this rule that is defined as Missing Inadmissible Prime \( k \)-tuples, which refers to the complete lack of association between certain Subtype I Admissible Prime \( k \)-tuples and their corresponding [absent] second variety of Subtype II Inadmissible Prime \( k \)-tuples. These [absent] second variety of Subtype II Inadmissible Prime \( k \)-tuples are instead replaced by the second variety of Subtype II Admissible Prime \( k \)-tuples. All of these findings are fully consistent with the iteratively computed data from Box 1. Further research is required to definitively determine whether these intermittent violations based on same modular arithmetic could recur arbitrarily many times for ever larger \( k \) values.

With needing to include diameter \( d = 2 \) when \( k = 2 \) [viz, \( s(2) = 2 \)]; Sub I Adm P \( k \)-tuples for \( k \geq 3 \) can be computed recursively using the following algorithm (Forbes, 1999, p. 1740) whereby the diameter \( d \) is denoted by \( s(k) \), gcd is abbreviation for greatest common divisor, and for \( p \) prime, the notation \( p# \) is product of all primes up to and including \( p \).

**Procedure \( s(k) \):**

1. Set \( U = q# \), the product of all the primes \( q \). Set \( D = \frac{U}{q} \) and \( h = H \). **Step 2.** Set \( B = \{i: i = 0, 2, \ldots, s, \gcd(h+i, U) = 1\} \). **Step 3.** If \( B \) does not contain both 0 and \( s \), go to step 8. **Step 4.** If \( B \) has less than \( k \) elements, go to step 8. **Step 5.** If \( B \) has more than \( k \) elements, do \( S(s, q', h) \), where \( q' \) is the next prime after \( q \). Then go to step 8. **Step 6.** If \( B \) has exactly \( k \) elements and if for each prime \( p, q < p \leq k \), all residues modulo \( p \) are represented by \( B \), go to step 8. **Step 7.** Indicate that \( B \) is an admissible set and report \( s(k) = s \). **Step 8.** Add \( D \) to \( h \). If \( h < H + U \), go to step 2. Otherwise return.

The above algorithm is related to A008407 Minimal difference \( s(n) \) between beginning and end of \( n \) consecutive large
primes \((n\text{-}tuplet)\) permitted by divisibility considerations. (Forbes, n.d.) having values 0 [symbolizing the nonexisting 1-tuple], 2, 6, 8, 12, 16, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60, 66, 70, 76, 80, 84, 90, 94, 100, 110, 114, 120, 126, 130, 136, 140, 146, 152, 156, 158, 162, 168, 176, 182, 186, 188, 196, 200, 210, 212, 216, 226, 236, 240, 246, 252, 254, 264, 270, 272, 278....

Remark 6. The first variety of Subtype II Admissible Prime \(k\)-tuples will cater more for existence of prime numbers with large(\(r\)) prime gaps that tend to occur at large(\(r\)) range of \(x\) integer values than the second variety of Subtype II Admissible Prime \(k\)-tuples which cater more for existence of prime numbers with small(\(r\)) prime gaps that tend to occur at small(\(r\)) range of \(x\) integer values. We hypothetically deduce the second variety of Subtype II Inadmissible Prime \(k\)-tuples usually appear with the first occurring or earliest known Subtype I Admissible Prime \(k\)-tuples pattern-1 derived with relevant \(k\) values except when violations of Rosser’s Prime \(k\)-tuple rule occur. When first variety and second variety of Subtype II Admissible Prime \(k\)-tuples are combined with Subtype I Admissible Prime \(k\)-tuples, they should, in principle, be able to represent every known odd prime numbers albeit in an overlapping manner.

We hereby provide a short summary: As previously alluded to in Part I Analysis, the 1st \(V\) Sub II Adm \(P\) \(k\)-tuples that are based on comparatively same \(k\) value present in the [reference] Sub I Adm \(P\) \(k\)-tuples will be associated with progressively much larger diameter \(d\). They are simply mediated via the [eligible] last prime number in the involved tuples being made progressively bigger. In contrast, the 2nd \(V\) Sub II Adm \(P\) \(k\)-tuples that are based on comparatively same \(k\) value present in the [reference] Sub I Adm \(P\) \(k\)-tuples will be associated with progressively slightly larger diameter \(d\). They are simply mediated via the [eligible] neighboring tuples being selectively nominated to represent them when appropriate.

Proposition 1.22. Both \(f(n)\) simplified Dirichlet eta function and \(F(n)\) Dirichlet Sigma-Power Law will manifest Principle of Equidistant for Multiplicative Inverse.

Proof. We use \(\eta(s)\) to denote \(f(n)\) Dirichlet eta function containing variable \(n\), and parameters \(t\) and \(\sigma\). Here, \(\eta(s)\) is the proxy function for Riemann zeta function, which can be denoted by \(\zeta(s)\). With also containing variable \(n\), and parameters \(t\) and \(\sigma\); the \(f(n)\) simplified Dirichlet eta function, denoted by \(\text{sim-}\eta(s)\), is essentially obtained by applying Euler formula to \(\eta(s)\) and the \(F(n)\) Dirichlet Sigma-Power Law, denoted by DSPL, refers to \(\int\text{sim-}\eta(s)dn\). Let variable \(\delta = \frac{1}{10}\). This will consistently generate in Figure 3 and Figure 4 the \(\delta\) induced shift of \([\text{infinitely many}]\) Varying Loops in reference to Origin; viz, the simple relationship of [more negative] left-shift given by \(\zeta(\frac{1}{2} - \delta + it)\) [Figure 3] < [neutral] nil-shift given by \(\zeta(\frac{1}{2} + it)\) [Figure 2] < [more positive] right-shift given by \(\zeta(\frac{1}{2} + \delta + it)\) [Figure 4].

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Figure 1. INPUT for \(\sigma = \frac{1}{2}, \frac{3}{2}, \text{and } \frac{3}{5}\). Riemann zeta function, \(\zeta(s)\), has countable infinite set of Completely Predictable trivial zeros located at \(\sigma = \text{all negative even numbers and [conjectured] countable infinite set of Incompletely Predictable nontrivial zeros located at } \sigma = \frac{1}{2} \text{ given by various } t \text{ values.}

Given \(\delta = \frac{1}{10}\), the \(\sigma = \frac{1}{2} - \delta = \frac{2}{5}\)-non-critical line (represented by Figure 3) and \(\sigma = \frac{1}{2} + \delta = \frac{2}{5}\)-non-critical line (represented by Figure 4) are \textit{equidistant} from \(\sigma = \frac{1}{2}\)-critical line (represented by Figure 2). The additive inverse operation of \(\sin(\delta) + \sin(-\delta) = 0\) indicating symmetry with respect to Origin [or \(\cos(\delta) - \cos(-\delta) = 0\) indicating symmetry with respect to \(y\)-axis] is not applicable to our complex single sine wave [or single cosine wave] since \((2n)\)- or \((2n-1)\)-complex term with transcendental functions consisting of sine, cosine, single sine wave, single cosine wave, natural logarithm are independent of parameter \(\sigma\).

However, \((2n)\)- or \((2n-1)\)-complex term with algebraic functions consisting of powers, fractional powers, root extraction [and scaled amplitude \(R\) on its (in)dependency on parameter \(t\)] are \textit{dependent on parameter } \(\sigma\). Let \(x = (2n)\) or \(\frac{1}{(2n)}\)
Figure 2. OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Schematically depicted polar graph of $\zeta\left(\frac{1}{2} + it\right)$ plotted along critical line for real values of $t$ running from 0 to 34, horizontal axis: $Re(\zeta(\frac{1}{2} + it))$, and vertical axis: $Im(\zeta(\frac{1}{2} + it))$. Total presence of all Origin intercept points.

Figure 3. OUTPUT for $\sigma = \frac{2}{5}$ as virtual Gram points. Varying Loops are shifted to left of Origin with horizontal axis: $Re(\zeta(\frac{2}{5} + it))$, and vertical axis: $Im(\zeta(\frac{2}{5} + it))$. Total absence of Origin intercept points.

Figure 4. OUTPUT for $\sigma = \frac{3}{5}$ as virtual Gram points with horizontal axis: $Re(\zeta(\frac{3}{5} + it))$, and vertical axis: $Im(\zeta(\frac{3}{5} + it))$. Varying Loops are shifted to right of Origin. Total absence of Origin intercept points.
we note by letting Principle of Equidistant for Multiplicative Inverse intrinsic presence of Multiplicative Inverse in sim-η(s) or DSPL for all σ values with this function or law rigidly obeying relevant trigonometric identity. We call this phenomenon Principle of Equidistant for Multiplicative Inverse. Finally, we note by letting δ = 0, we will always generate Figure 2 representing σ = 1/2-critical line.

The proof is now complete for Proposition 1.22.

1.3 Infinitesimal numbers applied to Prime numbers and Nontrivial zeros

A direct interpretation of Prime number theorem is that the average gap between primes increases as the natural logarithm of these primes, and therefore the ratio of the prime gap to the primes involved decreases (and is asymptotically zero). In analogy to Prime number theorem, the first Hardy-Littlewood conjecture, also called the Prime tuples conjecture as mentioned in Proposition 1.21, states that the asymptotic number of Admissible Prime k-tuplets or Prime constellations can be computed explicitly [and that every Admissible Prime k-tuplet matches an arbitrarily large number of positions in the sequence of prime numbers]. The second Hardy-Littlewood conjecture states that Prime-π(x + y) ≤ Prime-π(x) + Prime-π(y) for all x, y ≥ 2 whereby Prime-π(x) is the prime counting function; viz, the number of primes from x + 1 to x + y is always less than or equal to the number of primes from 1 to y. These two Hardy-Littlewood conjectures (Hardy & Littlewood, 1923) were subsequently proven to be incompatible with each other (Hensley & Richards, 1974) with an arbitrarily large number of violations. The first such violation is expected to likely occur for very large values of x; for example, an Admissible Prime k-tuplet of 447 primes [viz, Sub I Adm P 447-tuplet with smallest possible diameter = 3158] can be found in an interval of y = 3159 integers, while Prime-π(3159) = 446. Although unproven, the first Hardy–Littlewood conjecture is generally considered by most people to likely be true. If that is the case, it implies that the second Hardy–Littlewood conjecture, in contrast, is false. However, we do not need to prove [or disprove] the first Hardy–Littlewood conjecture per se for the purpose of this paper.

A required condition for Polignac’s and Twin prime conjectures to be considered true is that none of the arbitrarily large number of countably infinite subsets of odd prime numbers as generated by corresponding even prime gaps should ever become countably finite subsets. However, there are two somewhat anomalous situations.

(A) Prime numbers tend to be clustered around large or larger prime gaps occurring as multiples of 6; viz, prime gaps 6, 12, 18,... We deduce this observation do not prove or disprove Polignac’s and Twin prime conjectures, and can be logically explained as follow. Excepting the first two prime gaps, all prime gaps are between numbers that are either 1 or 5 modulo 6. Under the assumption that both cases are equally likely, half the prime gaps will be between numbers in the same class, and therefore of size 0 modulo 6, and the other half will be between numbers in different classes, which split up into sizes that are 2 and 4 modulo 6. Since each of the latter cases only gets one quarter of the total, it is clear that ignoring all other factors, gaps that are 2 or 4 modulo 6 are about half as likely to occur as gaps of the same approximate magnitude that are 0 modulo 6.

(B) Here is a simple proof for two consecutive prime gaps that are equal must be of the form (6n, 6n) for n = 1, 2, 3, 4, 5,: Suppose there were two consecutive gaps between 3 consecutive prime numbers that were equal, but not divisible by 6. Then the difference is 2k where k is not divisible by 3. Therefore the (supposed) prime numbers will be: p, p+2k, p+4k. But then p+4k = congruent modulo 3 to p+k. That makes the three numbers congruent modulo 3 to p, p+k, p+2k. One of those is divisible by 3 and so cannot be prime. So two consecutive gaps must be divisible by 3 and therefore (as they have to be even) by 6. We deduce this observation do not prove or disprove Polignac’s and Twin prime conjectures.

Riemann hypothesis propose all nontrivial zeros to be located on the σ = 1/2-critical line of Riemann zeta function. Previously confirming first 10,000,000,000,000 nontrivial zeros location on the critical line implies but does not prove Riemann hypothesis to be true. Hardy initially, and then with Littlewood, showed there are infinitely many nontrivial zeros lying on the critical line by considering moments of certain functions related to Riemann zeta function (Hardy, 1914; Hardy & Littlewood, 1921). This discovery cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of nontrivial zeros located away from the critical line when σ ≠ 1/2. Furthermore, it is literally a mathematical impossibility (mathematical impasse) to be able to computationally check [in a complete and successful manner] the locations of all infinitely many nontrivial zeros to correctly be the critical line.

An infinitesimal number is a quantity that is closer to zero than any standard real number, but that is not zero. The mathematical concept of infinity can be introduced using the symbol ∞. Then the reciprocal or inverse symbol 1/∞ is the symbolic representation of the mathematical concept of infinitesimal.

Proposition 1.31. With the prevalence of various selected odd prime numbers as the endpoint never becoming zero [which is conceptually defined as the nonexisting zero in this instance], we can apply infinitesimal numbers to rigorously show the prevalence of total odd prime numbers having all even prime gaps and the prevalence of subtotal odd prime numbers having corresponding even prime gaps will both never become zero.
Proof. We recall from Lemma 1.1 that all CIS-ALN-decelerating computed prime numbers are extrapolated out over a wide range of x \geq 2 integer values; the prime counting function \( \text{Prime-}\pi(x) = \text{number of primes} \leq x \) with x (conveniently) assigned to having odd number values of the form \( 10^n - 1 \) whereby \( n = 1, 2, 3, 4, 5,... \); and the Prevalence of prime numbers \( = \text{Prime-}\pi(x) / x = \text{Prime-}\pi(x) / (10^n - 2) \) when \( x = 2 \) to \( 10^n - 1 \). Note: The probability theory applied to n-digit primes and n-digit composites are given later on.

Apart from prime gaps of the form \( 6n \) with \( n = 1, 2, 3, 4, 5... \) and the [solitary] consecutive prime gaps \((2, 2)\) present in Sub 1 Inadin P 3-tuple with consecutive primes \((3, 5, 7)\), no other types of two consecutive prime gaps that are identical is possible. In reality, one could then rigorously argue from first principle alone there must be at least three even prime gaps that will perpetually reappear over the entire sequence of prime numbers because the alternatingly appearance of just two different prime gaps at extremely large x integer values simply cannot occur.

We recall from Proposition 1.21 concerning a given \( k_0\)-tuple \( \mathcal{H} = (h_1, \ldots, h_{k_0}) \) of \( k_0 \) distinct integers for some \( k_0 \geq 1 \), arranged in increasing order whereby one can, in principle, find an arbitrarily large number of translates \( n + \mathcal{H} = (n+h_1, \ldots, n+h_{k_0}) \) of \( \mathcal{H} \) which consists entirely of consecutive primes. The case \( k_0 = 1 \) is just Euclid’s theorem on the infinitude of primes. From this simple proven theorem, we can provide the following correct mathematical arguments.

The cardinality of all prime numbers or all odd prime numbers [when we validly ignore the only even prime number 2] is given by \(|\text{CIS-ALN-decelerating}| = N_0\)-decelerating when \( n \to \infty \) in \( x = 2 \) to \( 10^n - 1 \). Thus the cardinality of all \( x = 2 \) to \( \infty \) integer numbers is given by \(|\text{CIS-IM-linear}| = N_0\)-linear. As \( n \to \infty \), there are an arbitrarily large number (ALN) of deceleratingly-occurring prime numbers amongst the infinitely many linearly-occurring x integer numbers; viz, x integer numbers \( \gg \) prime numbers. Then Prevalence of prime numbers \( = \text{Prime-}\pi(x) / x = \text{ALN} / \infty = \) an infinitesimal number symbolized by \( \frac{1}{\infty} \) when \( x = 2 \) to \( \infty \). Since Euclid’s theorem [literally] holds for \( x = 2 \) to \( \infty \), then the Prevalence of prime numbers can be constituted by an infinitesimal number but can never become zero; viz, Prevalence of prime numbers conceptually have a nonexisting zero.

A substantial amount of previous materials refer to the proposal on subsets of odd prime numbers uniquely derived from corresponding arbitrarily large number of even prime gaps \( 2i \) with \( i = 1, 2, 3, 4, 5... \) in that all these subsets must also be arbitrarily large in number. Remark 2 from Proposition 1.21, in particular, rigorously support this proposal. Then there must also be full compliance with two required conditions: (i) Dimensional analysis homogeneity on relevant cardinality and (ii) even prime gaps will never terminate. All odd prime numbers having all even prime gaps = odd prime numbers having prime gap \( 2 \) + odd prime numbers having prime gap \( 4 \) + odd prime numbers having prime gap \( 6 \) +... + odd prime numbers having prime gap \( 2i \) \( \iff \) \( N_0\)-decelerating [all odd prime numbers] = \( N_0\)-decelerating [odd prime numbers having prime gap \( 2 \)] + \( N_0\)-decelerating [odd prime numbers having prime gap \( 4 \)] + \( N_0\)-decelerating [odd prime numbers having prime gap \( 6 \)] +... + \( N_0\)-decelerating [odd prime numbers having prime gap \( 2i \)]. Based on similar reasoning from previous paragraph, we logically deduce that for \( x = 2 \) to \( \infty \), the Prevalence of various odd prime numbers as specified by their corresponding even prime gaps \( 2i \) can similarly all be constituted by infinitesimal numbers symbolized by \( \frac{1}{\infty} \) but never become zero; viz, Prevalence of various odd prime numbers as specified by their corresponding even prime gaps \( 2i \) conceptually have a nonexisting zero.

The proof is now complete for Proposition 1.31.

Proposition 1.32. With Origin point or \( \sigma = \frac{1}{2} \)-critical line of Riemann zeta function regarded as the zero endpoint [which is conceptually defined as the existing zero in this instance], we can apply infinitesimal numbers to rigorously show the equivalent [geometric] Origin intercept points located at the zero-dimensional Origin point and [mathematical] nontrivial zeros located at the one-dimensional \( \sigma = \frac{1}{2} \)-critical line will uniquely appear only when parameter \( \sigma = \frac{1}{2} \).

Here, for simplicity, we use the term Riemann zeta function to also indicate Dirichlet \( \eta \)-function and Dirichlet Sigma-Power Law. We recall from Lemma 1.1 that all CIS-IM-linear computed nontrivial zeros are extrapolated out over a wide range of \( t \geq 0 \) real number values; and the nontrivial zeros gaps, counting function and prevalence can be defined. Although inevitably fluctuating, Prevalence [or Proportion] of nontrivial zeros must [linearly] be a fairly constant value over \( t = 0 \) to \( \infty \). This value is reasonably approximated by, for instance, using \( t = 0 \) to 100 range as \( 29/100 = 0.29 = 29\% \) since there are precisely 29 nontrivial zeros in this range.

We recall variable \( \delta \) with given value \( \frac{1}{100} \) [applied to Riemann zeta function] used in Proposition 1.22 to confirm Principle of Equidistant for Multiplicative Inverse that is applicable to Figure 3 representing \( \sigma = \frac{2}{3} \)-non-critical line and Figure 4 representing \( \sigma = \frac{1}{3} \)-non-critical line. We recognize the zero-dimensional Origin point in Figure 2 is synonymous with the one-dimensional \( \sigma = \frac{1}{2} \)-critical line, and this particular point or line is conceptually regarded as the existing zero. Then the Varying Loop trajectory in Figure 2 will only present its CIS-linear [geometrical] Origin intercept points that is precisely equivalent to the CIS-linear [mathematical] nontrivial zeros when \( \delta = 0 \) since the Origin point is a zero-dimensional point.
that can only be touched by the trajectory when \( \delta = 0 \) and \( \sigma = \frac{1}{2} \). We logically deduce variable \( \delta \) when constituted by an infinitesimal number symbolized by \( \frac{1}{\infty} \) will never become the existing zero since this equates to \( \sigma \approx \frac{1}{2} \) [or the trajectory is extremely close to zero-dimensional Origin point] but this is categorically still not the same as \( \sigma = \frac{1}{2} \) [or the trajectory touching zero-dimensional Origin point]. Thus variable \( \delta \) will instead only become the existing zero when both the \( \sigma = \frac{1}{2} \) and \( \delta = 0 \) conditions are simultaneously fully satisfied.

A worthy point of interest is the coincidental manifestation of Dimensional analysis (DA) homogeneity by parameter \( \sigma \) in Riemann zeta function. The *exact DA homogeneity* indicate calculated values of [exact] integer \(-1\) and 1 as derived from 
\[
\Sigma (\text{all fractional exponents}) = 2(\sigma) \text{ and } 2(1 - \sigma).
\]
Respectively, these act as surrogate markers in simplified Dirichlet eta function and Dirichlet Sigma-Power Law on the solitary unique \( \sigma = \frac{1}{2} \) situation. Otherwise, for the infinitely many non-unique \( \sigma \neq \frac{1}{2} \) corollary situations, calculated values of [inexact] fractional numbers \( \neq \) integer \(-1\) and \( \neq \) integer 1 are derived from 
\[
\Sigma (\text{all fractional exponents}) = 2(-\sigma) \text{ and } 2(1 - \sigma) \text{ to indicate inexact DA homogeneity.}
\]

The proof is now complete for Proposition 1.32\(\Box\).

2. Conjectures and Hypotheses involving Prime numbers and Nontrivial zeros

We hereby summarize using two theorems the convoluted but deceptively simple correct and complete mathematical arguments that rigorously form Algorithm-type proofs for Polignac’s and Twin prime conjectures, and Equation-type proof for Riemann hypothesis.

**Theorem 2.1 Modified Polignac-Twin-Prime Conjecture.** The 1849 Polignac’s conjecture involves studying all even prime gaps 2, 4, 6, 8, 10... and the 1846 Twin prime conjecture involves studying [subset] even prime gap 2. Traditionally, these two conjectures propose that the countably infinitely many elements in
\[\text{Set even prime gaps}\]
form a countably infinite set; and their corresponding countably infinitely many elements in
\[\text{Subsets odd prime numbers}\]
form countably infinite subsets. We innovatively propose the Modified Polignac-Twin-Prime Conjecture whereby elements forming these set and subsets are instead succinctly treated as countably arbitrarily large in number.

**Theorem 2.2 Riemann Hypothesis.** The 1859 Riemann hypothesis proposed that the countably infinitely many nontrivial zeros of Riemann zeta function are all located on its \( \sigma = \frac{1}{2} \) critical line. To generate countably infinitely many elements that will linearly reach an infinity value; we must use Dirichlet eta function, the proxy function for Riemann zeta function. We innovatively propose the one-dimensional [mathematical] critical line is precisely equivalent to the zero-dimensional [geometric] Origin point.

2.1 Outline of the Proofs for Open problems in Number theory

Let us start by stating Theorem 2.1 Modified Polignac-Twin-Prime Conjecture and Theorem 2.2 Riemann Hypothesis properly. In the Introduction we hinted that Theorem 2.1 will be a statement about the cardinality of Set even prime gaps and corresponding generated Subsets odd prime numbers being sub(sets) classified as countably arbitrarily large (sub)sets; and Theorem 2.2 will be a statement about \( \sigma = \frac{1}{2} \) critical line being the designated unique location for all countably infinitely large number of nontrivial zeros from Riemann zeta function. The “finitary” versions of the two theorems are as follows.

**Proposition 2.1 (Modified Polignac-Twin-Prime Conjecture).** To minimize complexity, we need not regurgitate the in-depth analysis previously performed on special properties in Admissible Prime k-tuples/k-tuples and Inadmissible Prime k-tuples as this (in)action do not invalidate our on-going proposition, conjecture and theorem on the cardinality of prime numbers and prime gaps.

As the sequence of prime numbers carries on, prime numbers with ever larger prime gaps from Set even prime gaps will appear, but not always in consecutive order e.g. prime gap 14 from prime number 113 first appear at position 30, which is earlier than prime gap 10 from prime number 139 that first appear at position 34 and prime gap 12 from prime number 199 that first appear at position 46. Let \[\text{Set even Prime gaps} = \text{|Set total odd Prime numbers| = countably arbitrarily large number in size.}\] Then there must be an countably arbitrarily many corresponding \[\text{Subsets odd Prime numbers} = \text{countably arbitrarily large number in size having the following property.}\] If the cardinality of sets and subsets are to be uniformly similar, then these intrinsically assigned cardinality must all be of countably arbitrarily large numbers in size.

Let \( P_i, P_{i+1}, P_{i+2} \) and \( P_{i+3} \) = four randomly selected consecutive prime numbers whereby \( P_{i+3} \geq P_{i+2} \geq P_{i+1} > P_i \). If this four primes are considered in total isolation, then there are only three possible prime gaps able to be computed: Prime gap, \( = P_{i+1} - P_i \), Prime gap\(i+1 = P_{i+2} - P_{i+1} \) and Prime gap\(i+2 = P_{i+3} - P_{i+2} \). In principle, we recognize these three prime gaps can be constituted by all possible combinations of small prime gaps 2 and 4 and/or large prime gaps \( \geq 6 \); viz, all three prime gaps are constituted by small prime gaps, all three prime gaps are constituted by large prime gaps, and the three prime gaps are constituted by a mixture of small and large prime gaps. Intuitively, every even prime gap 2, 4, 6, 8, 10...
and its correspondingly associated odd prime numbers must exist at least once; viz, occurring only one time, occurring a finite number of times, or occurring an arbitrarily large number of times. Proving the only correct possibility of both even prime gaps $2, 4, 6, 8, 10...$ and their correspondingly associated odd prime numbers will all occur an arbitrarily large number of times is equivalent to rigorously proving Modified Polignac-Twin-Prime Conjecture to be true. The term prime constellation in this paper is used to denote, and is synonymous with, Admissible Prime $k$-tuple having the smallest possible diameter. Otherwise, as explained before for some comparative $k$ values, there are also Admissible and Inadmissible Prime $k$-tuples with their various subtypes and varieties. Examples, the two Inadmissible Prime 3-tuples occurring only once at prime numbers $(p, p + 1, p + 3) = (2, 3, 5)$ with smallest diameter $= 3$ and at prime numbers $(p, p + 2, p + 4) = (3, 5, 7)$ with smallest diameter $= 4$, whereby these are closest possible [non-repeatable] grouping of three prime numbers since each of these three sequential odd numbers is a multiple of three, and hence not prime (except for itself).

We outline three possible trajectories of all prime gaps that are consistent with the existence of Prime-$\pi(x)$ as a stepped-mathematical function whereby we also use prime gaps 6n as common randomly chosen examples – in particular, for $n = 1, 2, 3...$ even prime gaps $= 6, 12, 18...$ [multiples of 6].

(a) **Accelerating primes:** Prime $gap_{p+2} > Prime gap_{p+1} > Prime gap_p$, occurring an arbitrarily large number of times e.g. Admissible Prime 3-tuple $(p, p+2, p+6)$ with smallest possible diameter $= 6$, Admissible Prime 3-tuple $(p+6, p+10, p+16) \equiv (p, p+4, p+10)$ with [not the smallest possible] diameter $= 10$ that is derived from Admissible Prime 18-tuple $(p, p+4, p+6, p+10, p+16, p+18, p+24, p+28, p+30, p+34, p+40, p+46, p+48, p+54, p+58, p+60, p+66, p+70)$ with smallest possible diameter $= 70$, and Admissible Prime 3-tuple $(p, p+6, p+18)$ from $(p-24, p-22, p-10, p, p+6, p+18, [p+42], [p+50])$ with [not the smallest possible] diameter $= 18$ occurring at consecutive prime numbers $(22391, 22397, 22409)$ with position of first $p = 2506$.

(b) **Decelerating primes:** Prime $gap_{p+2} < Prime gap_{p+1} < Prime gap_p$, occurring an arbitrarily large number of times e.g. Admissible Prime 3-tuple $(p, p+4, p+6)$ with smallest possible diameter $= 6$, Admissible Prime 3-tuple $(p+20, p+26, p+30) \equiv (p, p+6, p+10)$ with [not the smallest possible] diameter $= 10$ that is derived from Admissible Prime 9-tuple $(p, p+2, p+6, p+8, p+12, p+18, p+20, p+26, p+30)$ with smallest possible diameter $= 30$, and Admissible Prime 3-tuple $(p, p+18, p+30)$ from $(p-26, p-22, p-12, p, p+18, p+30, [p+50], [p+54])$ with [not the smallest possible] diameter $= 30$ occurring at consecutive prime numbers $(10193, 10211, 10223)$ with position of first $p = 1252$.

(c) **Steady primes:** Prime $gap_{p+2} = Prime gap_{p+1} = Prime gap_p$, that should occur an arbitrarily large number of times [albeit on extremely rare occasions] and can only involve prime gaps 6n. For instance, the Admissible Prime 3-tuple $(p, p+6, p+12)$ from $(p-2, p, p+6, p+12, [p+18], [p+28], [p+36])$ with [not the smallest possible] diameter $= 12$ occurring at consecutive prime numbers $(63691, 63697, 63703)$ with position of first $p = 6386$; and Admissible Prime 3-tuple $(p, p+18, p+36)$ from $(p-2, p, p+18, p+36, [p+54], [p+60])$ with [not the smallest possible] diameter $= 36$ occurring at consecutive prime numbers $(76543, 76561, 76579)$ with position of first $p = 7531$. An exception is the solitary Inadmissible Prime 3-tuple $(p, p+2, p+4)$ with smallest diameter $= 4$ occurring at consecutive prime numbers $(3, 5, 7) \equiv$ summative prime gaps $(0, 2, 4)$. We can explain using either $(3, 5, 7)$ tuple or $(0, 2, 4)$ tuple why this particular Prime 3-tuple is inadmissible, and we choose the former tuple. $k = 3$, prime $q \leq k \implies prime q = 2$ and 3 which are required for modular $q$. For modular 2: $3 \equiv 1 \pmod{2}$, $5 \equiv 1 \pmod{2}$, $7 \equiv 1 \pmod{2}$ \implies these three primes did not take on all two residue values 0 and 1 [considered as success]. However, for modular 3: $3 \equiv 0 \pmod{3}$, $5 \equiv 2 \pmod{3}$, $7 \equiv 1 \pmod{3}$ \implies these three primes did take on all three residue values 0, 1 and 2 [considered as failure]. By definition, this failure occurrence \implies the three primes are inadmissible since they would always include a multiple of 3 and therefore could not all be prime unless one of the numbers is 3 itself with finite one prime placement.

From above commentaries on possible trajectories of all prime gaps, we reach an important logical deduction. Even if the first Hardy-Littlewood conjecture (k-tuple conjecture) is theoretically proven to be false in the sense that every, or some, Admissible Prime $k$-tuples having smallest possible diameter do not match an arbitrarily large number of positions in the sequence of prime numbers; this finding will not, in principle, exclude Polignac’s and Twin prime conjectures to be true. This is because Admissible Prime $k$-tuples having even larger diameter [which is always greater than the smallest possible diameter of Admissible Prime $k$-tuples] can still be [conveniently] created that will match an arbitrarily large number of positions in the sequence of prime numbers. In particular, apart from the Admissible Prime 2-tuple of twin primes with diameter [prime gap] $= 2$ that can or will repeat itself an arbitrarily large number of times; the most extreme but simplest Admissible Prime 2-tuples having even larger diameter [prime gap] $= 4, 6, 8, 10, 12...$ can still be created whereby each Admissible Prime 2-tuple can or will repeat itself an arbitrarily large number of times. In effect, we do not need to involve the first Hardy-Littlewood conjecture to prove Polignac’s and Twin prime conjectures because apart from the only countably finite even prime number 2, all the countably arbitrarily large number of odd prime numbers $3, 5, 7, 11, 13...$ can be fully represented by the solitary Admissible Prime 2-tuple system that represent prime gap 2 and the
arbitrarily large number of Admissible Prime 2-tuples system that represent prime gaps 4, 6, 8, 10, 12,...

It is a breakthrough insight to correctly express the ultimate significance arising from observing the cardinality of (sub)sets of prime numbers and prime gaps reaching countably arbitrarily large number in size not in a infinite-scale smooth manner but instead in an infinite-scale stepwise manner. We deduce this infinite-scale stepwise phenomenon must reflect the above-mentioned repeated groupings of small and/or large prime gaps on an eternal basis resulting in inevitable presence of frequent repeating accelerating and decelerating primes [and also infrequent repeating steady primes]. Finding the correct dependence of \(|\text{Set even Prime gaps}| = |\text{Set total odd Prime numbers}| = |\text{Subsets odd Prime numbers}| = \text{countably arbitrarily large number in size on this infinite-scale stepwise phenomenon is the famous open problem of Modified Polignac-Twin-Prime Conjecture.}

**Conjecture 2.2 (Polignac’s and Twin prime conjectures on cardinality of odd prime numbers and even prime gaps to be countably arbitrarily large in numbers).** The countably arbitrarily large number of small and large prime gaps that form all even prime gaps are synonymous with, and must generate, all [corresponding] known countably arbitrarily large number of odd prime numbers [with all these entities deceleratingly reaching arbitrarily large number values]. Just as both small and large prime gaps must appear less frequently [conceptually complying with progressively diminishing non-zero probability that never become zero probability] at ever larger range of \(x\) integer values due to prime numbers overall also becoming progressively rarer [conceptually complying with progressively diminishing non-zero probability that never become zero probability] at ever larger range of \(x\) integer values, so must the repeated groupings of small and/or large prime gaps manifesting as infinite-scale stepwise phenomenon overall becomes progressively rarer [conceptually complying with progressively diminishing non-zero probability that never become zero probability] at ever larger range of \(x\) integer values.

Since this imply there is zero probability that any particular small and/or large prime gap(s) present in the countably arbitrarily large number of repeated groupings derived from these prime gaps will abruptly terminate or disappear, then we deduce both the even prime gaps and their associated corresponding odd prime numbers must, by default, also recur on an eternal basis to form countably arbitrarily large numbers of both elements [with all these entities deceleratingly reaching arbitrarily large number values]. *This encoded property would imply Theorem 2.1 Modified Polignac-Twin-Prime Conjecture.*

**Proposition 2.3 (Riemann Hypothesis).** To minimize complexity, we need not consider here the in-depth analysis on special properties of Gram’s Law and Rosser’s Rule as this (in)action do not invalidate our proposition, conjecture and theorem on location of nontrivial zeros.

Let there be three mutually exclusive but dependent **Set nontrivial zeros**, \(\text{Set Gram}\{x=0\}\) points generated by Dirichlet eta function when \(\sigma = \frac{1}{2}\). Let there be two mutually exclusive but dependent **Set virtual Gram\{y=0\}\) points and **Set virtual Gram\{x=0\}\) points** generated by Dirichlet eta function when \(\sigma \neq \frac{1}{2}\). Let \(|\text{Set nontrivial zeros}| = |\text{Set Gram}\{x=0\}| = |\text{Set Gram}\{y=0\}|\) be countably infinite large number in size that will linearly reach an infinity value. Let \(|\text{Set virtual Gram}\{x=0\}| = |\text{Set virtual Gram}\{y=0\}|\) be countably infinite large number in size that will linearly reach an infinity value.

As schematically depicted by Figure 1, the \(\sigma = \frac{1}{2}\) critical line is located in, and bisect, the \(0 < \sigma < \frac{1}{2}\) and \(\frac{1}{2} < \sigma < 1\). It is a breakthrough insight to correctly determine the ultimate significance arising from observing the solitary [mathematical] \(\sigma = \frac{1}{2}\) critical line [as 1-dimensional line] will precisely represent the solitary [geometric] Origin point [as 0-dimensional point], while the infinitely many [mathematical] \(\sigma \neq \frac{1}{2}\) non-critical lines that are located in \(0 < \sigma < \frac{1}{2}\) or \(\frac{1}{2} < \sigma < 1\) region will never represent the solitary [geometric] Origin point.

**Conjecture 2.4 (Riemann hypothesis on location of all nontrivial zeros to be the critical line).** The countably infinitely large number of nontrivial zeros that will linearly reach an infinity value are generated only when parameter \(\sigma = \frac{1}{2}\) in Dirichlet eta function, and not when parameter \(\sigma \neq \frac{1}{2}\) in Dirichlet eta function.

Since this imply there is zero probability that any particular parameter \(\sigma \neq \frac{1}{2}\) values that do occur in Dirichlet eta function will mathematically represent the \(\sigma = \frac{1}{2}\) critical line [or geometrically represent the analogous \(\sigma = \frac{1}{2}\) Origin point], then we deduce all countably infinitely large number of nontrivial zeros that linearly reach an infinity value as generated from Dirichlet eta function when parameter \(\sigma = \frac{1}{2}\) will, by default, also have to be located on the \(\sigma = \frac{1}{2}\) critical line. *This encoded property would imply Theorem 2.2 Riemann Hypothesis.*

### 2.2 The decimal system, rational and irrational numbers

The decimal system, or base-10 system with 10 as its base, will be used throughout this paper. Other base systems include binary (base-2), ternary (base-3), quaternary (base-4), quinary (base-5), senary (base-6), etc. Since the base is just a representation of the physical amount; integers such as prime and composite numbers, and transcendental numbers...
such as nontrivial zeros and closely related two types of Gram points are intuitively expected to generally have the same mathematical properties across any nominated base system. Based on the digits after decimal point, decimal numbers are divided into three types: (i) terminating, (ii) non-terminating but recurring, and (iii) non-terminating and non-recurring [with (i) and (ii) constituting rational numbers, and (iii) constituting algebraic and transcendental irrational numbers]. Note also that (i), (ii) and (iii) as numbers all represent zero-dimensional points whereby (i) and (ii) are able to do so with 100% accuracy but (iii) is only able to do so with accuracy that progressively increases with higher number of non-terminating and non-recurring digits after decimal point being supplied.

An algebraic irrational number [e.g. golden ratio $\phi$, $\sqrt{2}$, etc which together form a countably infinite set] is a number that is a root of a non-zero polynomial in one variable with integer (or, equivalently, rational) coefficients. A transcendental irrational number [e.g. nontrivial zeros, $\pi$, $e$, Liouville numbers, etc which together form an uncountably infinite set] is a number that is not an algebraic irrational number; viz, it is not the root of a non-zero polynomial of finite degree with rational coefficients. Thus, irrational numbers in totality will constitute an uncountably infinite set with both types of irrational numbers containing non-terminating and non-recurring digits. An important property of algebraic irrational numbers essentially states that these numbers cannot be well approximated by rational numbers. Since transcendental irrational numbers do not possess this property, they cannot be algebraic and must be transcendental. Irrational numbers can be further classified in a useful manner under Isolated countably finitely-sized group or Connected countably infinitely-sized group. For instance, the [isolated] countably finite $\pi$, $e$ and $\phi$ belong to the former group, and the [connected] countably infinite nontrivial zeros and closely related two types of Gram points, square roots, cube roots and Liouville numbers belong to the later group.

A Liouville number is a transcendental number which has very close rational number approximations. An irrational number $x$ is called a Liouville number if, for each positive integer $n$, there exist a pair of integers $(p, q)$ with $p>0$ and $q>1$ such that $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}$. For any integer $b \geq 2$ [viz. base $\geq 2$] and any sequence of integers $(a_1, a_2, a_3, ..., a_k)$ such that $a_k \in \{0, 1, 2, 3, ..., b-1\}$ for all $k$ and $a_k \neq 0$ for infinitely many $k$, we define the constructed Liouville number as $x = \sum_{k=1}^{\infty} \frac{a_k}{b^k}$.

In the special case when $b = 10$, and $a_k = 1$ for all $k$, the resulting number $x$ is called Liouville’s constant – a decimal fraction with a 1 in each decimal place corresponding to a factorial $n!$, and zeros everywhere else. Liouville numbers are precisely those transcendental irrational numbers that can be more closely approximated by rational numbers than any algebraic irrational number.

### 2.3 General notations

The following is a list of abbreviations used by this paper.

**CP entities:** Completely Predictable entities which will manifest CP independent properties.

**IP entities:** Incompletely Predictable entities which will manifest IP dependent properties.

$\zeta(s)$: $(n)$ Riemann zeta function containing variable $n$, and parameters $t$ and $\sigma$ will generate [via its proxy Dirichlet eta function] Zeros when $\sigma = \frac{1}{2}$ and virtual Zeros when $\sigma \neq \frac{1}{2}$.

$\eta(s)$: $(n)$ Dirichlet eta function, which represents the analytic continuation of $\zeta(s)$, containing variable $n$, and parameters $t$ and $\sigma$ will generate Zeros when $\sigma = \frac{1}{2}$ and virtual Zeros when $\sigma \neq \frac{1}{2}$.

$\text{sim-}\eta(s)$: $(n)$ simplified Dirichlet eta function, which is essentially derived by applying Euler formula to $\eta(s)$, containing variable $n$, and parameters $t$ and $\sigma$ will generate Zeros when $\sigma = \frac{1}{2}$ and virtual Zeros when $\sigma \neq \frac{1}{2}$.

**DSPL:** $F(n)$ Dirichlet Sigma-Power Law = $\int \text{sim - } \eta(s) \text{dn}$ containing variable $n$, and parameters $t$ and $\sigma$ will generate Pseudo-zeros when $\sigma = \frac{1}{2}$ and virtual Pseudo-zeros when $\sigma \neq \frac{1}{2}$.

**NTZ:** nontrivial zeros or $G[x=0,y=0]$: Gram[x=0,y=0] points = Origin intercept points located at the solitary-positioned zero-dimensional Origin point are generated by equation $G[x=0,y=0]P-\eta(s)$ [containing exponent $= \frac{1}{2}$] when $\sigma = \frac{1}{2}$.

**GP or G[y=0]P:** ‘usual’ (or ‘traditional’) Gram points = Gram[y=0] points = x-axis intercept points located at the multiple-positioned one-dimensional x-axis line are generated by equation $G[y=0]P-\eta(s)$ when $\sigma = \frac{1}{2}$. We deduce that Riemann hypothesis can also be usefully stated as none of the [additional] virtual $G[x=0]P$ generated by equation $G[x=0]P-\eta(s)$ when $\sigma \neq \frac{1}{2}$ – as demonstrated by Figure 7 for $\sigma = \frac{1}{2}$ can be constituted by t values of transcendental numbers that [incorrectly] coincide with $t$ values of transcendental numbers for NTZ when $\sigma = \frac{1}{2}$.

**G[x=0]P:** Gram[x=0] points = y-axis intercept points located at the multiple-positioned one-dimensional y-axis line are generated by equation $G[x=0]P-\eta(s)$ when $\sigma = \frac{1}{2}$.

**Virtual NTZ:** virtual nontrivial zeros or virtual $G[x=0,y=0]$: virtual Gram[x=0,y=0] points. These are virtual Origin intercept points located at the multiple-positioned virtual Origin points which are generated by equation $G[x=0,y=0]P-\eta(s)$ containing exponent values $\neq \frac{1}{2}$ when $\sigma \neq \frac{1}{2}$. We note that each virtual NTZ when $\sigma < \frac{1}{2}$ in Figure 3 equates to an [additional] negative virtual $G[y=0]P$ located at IP varying positions on horizontal axis, and each virtual NTZ when
2.4 Coprime numbers and Basic arithmetic operations on even and odd numbers

We ignore the negative integers. For \( i = 1, 2, 3, 4, 5, \ldots \), Set of integer numbers \( (\mathbb{Z}) = \{0, 1, 2, 3, 4, \ldots \} \) as CIS-IM-linear. For \( i = 0, 1, 2, 3, 4, \ldots \), Set of even numbers \( (\mathbb{E}) = \{0, 2, 4, 6, 8, \ldots \} \) as CIS-IM-linear complies with congruence \( \sigma \equiv 0 \) (mod 2) viz, “Any integer that can be divided exactly by 2 with last digit always being 0, 2, 4, 6 or 8” and with [extra] zeroth even number \( E_0 = 0 \). For \( i = 1 \), \( 2 \), \( 3 \), \( 4 \), \( 5 \), \( \ldots \), Set of odd numbers \( (\mathbb{O}) = \{1, 3, 5, 7, 9, \ldots \} \) as CIS-IM-linear complies with congruence \( \sigma \equiv 1 \) (mod 2) viz, “Any integer that cannot be divided exactly by 2 with last digit always being 1, 3, 5, 7 or 9”. Then, Set \( \mathbb{Z} = \mathbb{E} + \mathbb{O} \). The four basic arithmetic operations are addition, subtraction, multiplication, and division. We apply these operations to \( \mathbb{E} \) and \( \mathbb{O} \):

\[
\mathbb{E} + \mathbb{E} = \mathbb{E}; \quad \mathbb{O} + \mathbb{O} = \mathbb{O}; \quad \mathbb{E} + \mathbb{O} = \mathbb{E}; \quad \mathbb{O} + \mathbb{E} = \mathbb{E}; \quad \mathbb{O} + \mathbb{O} + \mathbb{O} = \mathbb{O}; \quad \mathbb{O} + \mathbb{O} + \mathbb{O} = \mathbb{O} = \mathbb{E}, \text{ etc.}
\]

We can ignore substraction since it is simply equivalent to addition of negative numbers. Addition of multiple \( \mathbb{O} \) with even number of elements will always give rise to an \( \mathbb{E} \), and odd number of elements will always give rise to an \( \mathbb{O} \). Thus, addition of various combinations of multiple \( \mathbb{E} \) and multiple \( \mathbb{O} \) that will result in either \( \mathbb{E} \) or \( \mathbb{O} \) will only depend on whether the number of elements for \( \mathbb{O} \) is even or odd.

\[
\mathbb{E} X \mathbb{E} = \mathbb{E}; \quad \mathbb{O} X \mathbb{O} = \mathbb{O}; \quad \mathbb{E} X \mathbb{O} = \mathbb{E}; \quad \mathbb{O} X \mathbb{E} = \mathbb{E}.
\]

Division is defined as dividend \( \div \) divisor = quotient whereby the dividend and divisor do not have common factors other than 1; and is undefined when the divisor = 0. Division is also identical to multiplying the dividend by the inverse of the divisor. The product of a fraction and its reciprocal is 1, hence the reciprocal is the multiplicative inverse of a fraction. Fraction = Numerator / Denominator / Denominator / [Divisor] which in its simplest form can have either terminating or non-terminating decimal representations. All terminating decimals can be expressed as \( \frac{a}{10^n} \). Fractions whose denominator does not include 5 and 2 as factors are non-terminating decimals. Fractions whose denominator is a power of 2 (e.g. 1/4 and 1/8) are terminating decimals. Fractions whose denominator is a power of 5 (e.g. 1/5, 1/25, 1/125) are terminating decimals. Fractions whose denominator includes 2 as a factor and includes a factor not equal to 5 are non-terminating decimals (e.g. 1/6 and 1/14). Fractions whose denominator includes 5 as a factor and includes a factor not equal to 2 (e.g. 1/15) are non-terminating decimals. From these observations, we can conjecture that a fraction is a terminating decimal if its factors are 2 or its powers, 5 or its powers, or both. If a fractions is in the form \( \frac{x}{y} \) is irreducible if \( \gcd(x, y) = 1 \), it is terminating.

Algorithm to decide whether a fraction will result in a terminating decimal include the following steps:

1. Reduce the fraction to its simplest form in which the numerator \( x \) and denominator \( y \) are integers that have no other common divisors than 1 (and –1, when negative numbers are considered). Thus, \( \frac{x}{y} \) is irreducible if \( x \) and \( y \) are coprime \( \implies \) no prime number divides both \( x \) and \( y \); viz, \( x \) and \( y \) have their greatest common divisor \( (\text{GCD}) = 1 \).
2. If denominator \( y \) ends in 0 or 5, divide \( x \) by 5. Repeat until \( y \) does not end in 0 or 5.
3. If denominator \( y \) ends in 0 or 2 or 4 or 6 or 8, divide \( y \) by 2. Repeat until \( y \) is an odd number.
4. If denominator \( y = 1 \), the fraction will result in a terminating decimal. Otherwise, the fraction will not result in a terminating decimal.

A set of integers \( S = \{a_1, a_2, a_3, \ldots, a_\ell \} \) is setwise coprime if \( \text{GCD} \) of all its elements = 1. If every pair in this set is coprime, then the set is pairwise coprime. Pairwise coprimality is a stronger condition than setwise coprimality. Every pairwise coprime finite set is also setwise coprime but the reverse is not true. It is possible for an infinite set of integers to be [completely] pairwise coprime with notable examples being set of all prime numbers, set of elements in Sylvester’s sequence, and set of all Fermat numbers. Then it is also possible for an infinite set of integers to not be [completely] pairwise coprime with simplest example being set of all composite numbers. In perspective, prime numbers are (smaller deceleratingly reaching) countably arbitrarily large in number whereas [complementary] composite number are (larger acceleratingly reaching) countably infinitely many in number.

The Chinese remainder theorem says we can uniquely solve every pair of, or larger congregation of, congruences having relatively prime (coprime) moduli. Let \( m \) and \( n \) be relatively prime (coprime) positive integers. For all integers \( a \) and \( b \), the pair of congruences \( x \equiv a \mod m, x \equiv b \mod n \) has a solution, and this solution is uniquely determined modulo \( mn \).
More generally, for \( r \geq 2 \), let \( m_1, m_2, \ldots, m_r \) be nonzero integers that are pairwise relatively prime (coprime): \((m_i, m_j) = 1\) for \( i \neq j \). Then, for all integers \( a_1, a_2, \ldots, a_r \), the system of congruences \( x \equiv a_i \mod m_i \), \( a_2 \mod m_2, \ldots, a_r \mod m_r \), has a solution, and this solution is uniquely determined modulo \( m_1 m_2 \ldots m_r \). With implications for previous paragraph, then employing Chinese remainder theorem meant that we can uniquely solve every pair of, or larger congregation of, congruences having pairwise, and not just setwise, relatively prime (coprime) moduli.

### 2.5 Probability theory applied to \( n \)-digit numbers

Probability = 100\% X Proportion. Probability and Proportion are literally equivalent to each other for our analysis on prime and composite numbers (and nontrivial zeros). If the probability [range between 0 or 0\% and 1 or 100\%] of an event occurring is \( Y \), then the probability [range between 0 or 0\% and 1 or 100\%] of the event not occurring is \( 1-Y \). The odds of an event represent the ratio of the (probability that the event will occur) / (probability that the event will not occur). This can be succinctly expressed as: Odds of event = \( Y / (1-Y) \).

\( P(\text{any number is divisible by a prime } p, \text{ or in fact any integer}) = 1/p \). Let there be \( k \) randomly chosen integers. When 
\( k = 2 \), \( P(\text{two numbers are both divisible by } p) = 1/p^2 \), and \( P(\text{at least one of the two numbers is not divisible by } p) = 1 - 1/p^2 \). Any finite collection of divisibility events associated to distinct primes is mutually independent. For example, in the case of two events, a number is divisible by primes \( p \) and \( q \) \iff it is divisible by \( pq \); the latter event has probability \( 1/pq \). We make the heuristic assumption that such reasoning can be extended to infinitely many divisibility events. Then, the fundamental theorem of arithmetic asserts that every nonzero integer can be written as a product of primes in a unique way, up to ordering and multiplication by units. Prime numbers are defined as *All integers apart from 0 and 1 that are evenly divisible by itself and by 1*. Composite numbers are defined as *All integers apart from 0 and 1 that are evenly divisible by numbers other than itself and 1*. The integer numbers \((\mathbb{Z}) = \{0, 1, 2, 3, 4, \ldots\}\), prime numbers \((\mathbb{P}) = \{2, 3, 5, 7, 11, \ldots\}\) and composite numbers \((\mathbb{C}) = \{4, 6, 8, 9, 10, \ldots\}\) can all be analyzed in terms of their corresponding unique \( n \)-digit numbers. The following two integer sequences \( A006879 \) and \( A006880 \) are directly related to our unique \( n \)-digit \( \mathbb{P} \) and \( n \)-digit \( \mathbb{C} \) groupings whereby \( n = 0, 1, 2, 3, 4, \ldots\) [to an arbitrarily large number]:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A006879 )</td>
<td>0</td>
<td>4</td>
<td>21</td>
<td>143</td>
<td>1061</td>
<td>8363</td>
<td>68906</td>
<td>586081</td>
<td>5096876</td>
<td>45086079</td>
<td>404204977</td>
</tr>
<tr>
<td>( A006880 )</td>
<td>0</td>
<td>4</td>
<td>25</td>
<td>168</td>
<td>1229</td>
<td>9592</td>
<td>78498</td>
<td>664579</td>
<td>5761455</td>
<td>50847534</td>
<td>455052511</td>
</tr>
</tbody>
</table>

\( A006879 \) Number of primes with \( n \) digits. Number of primes between \( 10^{(n-1)} \) and \( 10^n \) (Sloane & Plouffe, 1995). Using our unique \( n \)-digit \( \mathbb{P} \) grouping, this statement is mathematically equivalent to Number of primes between \( 10^{(n-1)} \) and \( 10^n - 1 \) since the integer \( 10^n \) itself can never be prime.

\( A006880 \) Number of primes < \( 10^n \). Number of primes with at most \( n \) digits or Prime counting function \( \pi(\leq 10^n) \) defined as \( |\mathbb{P} < 10^n| \) (Sloane & Plouffe, 1995). Using our unique \( n \)-digit \( \mathbb{P} \) and \( n \)-digit \( \mathbb{C} \) groupings, Prime counting function \( \pi(\leq 10^n - 1) \) is defined as \( |\mathbb{P} \leq 10^n - 1| \); and Composite counting function \( C(\pi(\leq 10^n) \text{ as } |\mathbb{C} \leq 10^n - 1| \).

We note \( A006880 \) forms the partial sums of \( A006879 \). Using our unique \( n \)-digit \( \mathbb{P} \) grouping, \( A006879 \) can be alternatively defined as \( \text{The number of primes between } 10^{(n-1)} \text{ and } 10^n - 1 \) which supply precisely the original and identical \( A006879 \), as \( n \)-digit prime number values. Only by employing similar crucial step of using our unique \( n \)-digit \( \mathbb{C} \) grouping \( \text{The number of composites between } 10^{(n-1)} \text{ and } 10^n - 1 \), will we obtain the complementary-\( A006879 \) as \( n \)-digit composite number values. There are precisely \( 10^n - 1 \) minus \( 10^{(n-1)} \) plus \( 10^{(n-1)} - 10^{(n-1)} \) integer numbers between \( 10^{(n-1)} \) and \( 10^n - 1 \). The important implication is that we are now always dealing with the same \( n \)-digit integer, prime and composite numbers whereby the relationship \( n \)-digit \( \mathbb{Z} = \text{n-digit } \mathbb{P} \text{ + n-digit } \mathbb{C} \) will always hold [except for when \( n = 1 \) because 0 and 1 are neither prime nor composite]. We further note from \( A006879 \) and \( A006880 \) the number of primes that are still constituted by very large number values will rapidly decline in an acceleratingly manner with two progressively larger \( n \) values in \( 10^{(n-1)} \) and \( 10^n - 1 \). Then by pure aesthetic argument, one could [non-rigorously] speculate there will always be many allocated primes to theoretically represent all even prime gaps in the sequence of prime numbers.

For \( i = 1, 2, 3, 4, \ldots \), Set of \( \mathbb{Z} \) \{0, 1, 2, 3, 4, \ldots\} as \( \text{CIS-M-linear} = \text{Set of neither } \mathbb{P} \text{ nor } \mathbb{C} \{0, 1\} \text{ as } \text{CFS} + \text{Set of } \mathbb{P} \{2, 3, 5, 7, 11, \ldots\} \text{ as } \text{CIS-ALN-decelerating} + \text{Set of } \mathbb{C} \{4, 6, 8, 9, 10, \ldots\} \text{ as } \text{CIS-M-accelerating} \). All \( \mathbb{P} \) are odd except for the first and only even \( \mathbb{P} \) \( 2 \). There is only one solitary even \( \mathbb{P} \) \( 2 \) and one solitary odd \( \mathbb{P} \) \( 5 \) that are not \( \mathbb{C} \). Otherwise, all \( \mathbb{Z} \) with their last digit ending as even numbers 0, 2, 4, 6 or 8, or odd number 5 must always be \( \mathbb{C} \). Apart from \( \mathbb{P} \) \( 2 \) and \( \mathbb{P} \) \( 5 \), all \( \mathbb{P} \) have their last digit ending as odd numbers 1, 3, 7 or 9. But not all \( \mathbb{Z} \) with their last digit ending as odd numbers 1, 3, 7 or
9 are \( P \) – in fact, these numbers are more likely to be \( C \) than \( P \). We deduce that for \( \geq 2 \)-digit numbers, (i) \( C \) can have their last digit ending in 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 but (ii) \( P \) can only have their last digit ending in 1, 3, 7 or 9; and thus (iii) all \( Z \) with their last digit ending in 0, 2, 4, 5, 6 or 8 must be \( C \).

For \( n = 1, 2, 3, 4, \ldots \) [to an arbitrarily large number]; we apply probability theory to the generated subsets of \( n \)-digit \( P \) as \textbf{CIS-ALN-decelerating} and \( n \)-digit \( C \) as \textbf{CIS-IM-accelerating}. With probability 1, all randomly selected \( Z \) that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be \textit{almost surely} \( C \). This is equivalently stated as: With probability 0, all last digit selected \( Z \) that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be \textit{almost never} \( P \). Thus, \( P \) randomly selected \( Z \) is \( C \) with 100% certainty) = 0.6 [except for the isolated 1-digit \( Z \) 0 and 1-digit \( P \) 2 and 5 which are not \( C \)]. The terms \textit{almost surely} and \textit{almost never} can now be replaced with \textit{surely} and \textit{never} when we disregard the 1-digit \( P \) and 1-digit \( C \). Since the condition “randomly selected \( Z \) that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be \textit{surely} \( C \)” will always apply to any chosen subsets of \( \geq 2 \)-digit \( Z \), the consequently derived equivalent condition “60% of all \( Z \) being \( C \) with 100% certainty” will also always apply to these same subsets. Then, with 60% of all \( Z \) being \( C \) with 100% certainty, there will always be more \( C \) than \( P \) for any chosen corresponding subsets of \( \geq 2 \)-digit \( P \) and \( \geq 2 \)-digit \( C \).

\textbf{Constraints on Prime numbers and Prime gaps:} We define Prime gap \( = P_{i+1} - P_i \). We ignore \( P_1 = 2 \) and \( P_3 = 5 \). We convey below the paired list of (last digit for \( P_i \), last digit for \( P_{i+1} \) as full range of choices permissible for corresponding specified groupings of prime gaps.

\begin{align*}
\textbf{CIS-ALN-decelerating} \ P_i \ &\text{selected from Prime gap} = 2, 12, 22, 32 \ldots \text{[to an arbitrarily large number as CIS-ALN-decelerating]} \rightarrow (1, 3), (7, 9), (9, 1). \text{The last digit of } P_i \text{ with prime gap having last digit ending in 2 cannot end in 3 or 5 but can end in 1, 7 or 9.} \\
\textbf{CIS-ALN-decelerating} \ P_i \ &\text{selected from Prime gap} = 4, 14, 24, 34 \ldots \text{[to an arbitrarily large number as CIS-ALN-decelerating]} \rightarrow (3, 7), (7, 1), (9, 3). \text{The last digit of } P_i \text{ with prime gap having last digit ending in 4 cannot end in 1 or 5 but can end in 3, 7 or 9.} \\
\textbf{CIS-ALN-decelerating} \ P_i \ &\text{selected from Prime gap} = 6, 16, 26, 36 \ldots \text{[to an arbitrarily large number as CIS-ALN-decelerating]} \rightarrow (1, 7), (3, 9), (7, 3). \text{The last digit of } P_i \text{ with prime gap having last digit ending in 6 cannot end in 5 or 9 but can end in 1, 3 or 7.} \\
\textbf{CIS-ALN-decelerating} \ P_i \ &\text{selected from Prime gap} = 8, 18, 28, 38 \ldots \text{[to an arbitrarily large number as CIS-ALN-decelerating]} \rightarrow (1, 9), (3, 1), (9, 7). \text{The last digit of } P_i \text{ with prime gap having last digit ending in 8 cannot end in 5 or 7 but can end in 1, 3 or 9.} \\
\textbf{CIS-ALN-decelerating} \ P_i \ &\text{selected from Prime gap} = 10, 20, 30, 40 \ldots \text{[to an arbitrarily large number as CIS-ALN-decelerating]} \rightarrow (1, 1), (3, 3), (7, 7), (9, 9). \text{The last digit of } P_i \text{ with prime gap having last digit ending in 0 cannot end in 5 but can end in 1, 3, 7 or 9.}
\end{align*}

The last digit of \( P_i \) ending in 1 is associated with prime gap having last digit ending in 0, 2, 6 or 8. The last digit of \( P_i \) ending in 3 is associated with prime gap having last digit ending in 0, 4, 6 or 8. The last digit of \( P_i \) ending in 7 is associated with prime gap having last digit ending in 0, 2, 4 or 6. The last digit of \( P_i \) ending in 9 is associated with prime gap having last digit ending in 0, 2, 4 or 8.

We note \textbf{CIS-ALN-decelerating} \( P_i \) having Prime gap, \{given as multiples of 10\} with last digit ending in 0 is associated with four choices that are available arbitrarily often. Otherwise, \textbf{CIS-ALN-decelerating} \( P_i \) having Prime gap, with last digit ending in 2, 4, 6 or 8 is each associated with three choices that are available arbitrarily often. Statistically, the last digit of \( Z_r \) ending in 1, 3, 7 or 9 are more likely to be just \( O \), than [odd] \( P_i \). At ever larger range of numbers for the paired list of (last digit for \( P_i \), last digit for \( P_{i+1} \)), we can intuitively surmise that \( P \) associated with progressively larger prime gaps moving from left to right and from top to bottom should occur relatively more often than \( P \) associated with comparatively smaller prime gaps. However, both \( P \) associated with progressively larger prime gaps and \( P \) associated with comparatively smaller prime gaps should both generally occur less often at ever larger range of numbers. Thus, although prime gap having last digit ending in 0 can be associated with last digit of \( P_i \) ending in 1, 3, 7 or 9 as four choices [instead of just three choices]; these prime gaps as a unique group will still always constitute larger prime gaps that will overall intrinsically occur less often at ever larger range of numbers.

The crucial inference here is that all known last digit of \( P_i \) ending in 1, 3, 7 or 9 that will represent all existing even prime gaps must do so on the eternal basis thus confirming Polignac’s and Twin prime conjectures to be true. Our modified Polignac’s conjecture [involving all even prime gaps 2, 4, 6, 8, 10...] together with its subset Twin prime conjecture [involving only even prime gaps 2] can now be categorically stated as: \textbf{Set all even prime gaps} = \textbf{Subset odd prime numbers associated with each even prime gap} = \textbf{CIS-ALN-decelerating}. These two conjectures are traditionally stated together in a less informative manner as \textbf{Set all even prime gaps} = \textbf{Subset odd prime numbers associated with each even prime gap} = \textbf{CIS}.

\textbf{Constraints on Composite numbers and Composite gaps:} We define Composite gap \( = C_{i+1} - C_i \). For 1-digit \( P \) and 1-digit
C that are members of 1-digit \( \mathbb{Z} \), there are always more \( P \) than \( C \) except at \( \mathbb{Z} = 9 \) which is \( C \) and whereby now \( |P| = |C| = 4 \). There will always be more \( C \) as \text{CIS-IM-accelerating} than \( P \) as \text{CIS-ALN-decelerating} when \( \mathbb{Z} \geq 10 \) with \( |C| = 2 \) \( |P| \) at \( \mathbb{Z} = 14 \) and \( |C| > 2 \) \( |P| \) at \( \mathbb{Z} > 14 \).

Let \( P(\text{certain } C) \) denote \( P(\text{randomly selected } \mathbb{Z} = 100\% \text{ certainty}) \). Then, \( P(\text{certain even } C) = 0.5 \) for all subsets \( \geq 2 \)-digit \( C \) having elements with their last digit ending in 0, 2, 4, 6 or 8 [as \text{CIS-IM-linear}] and \( P(\text{certain odd } C) = 0.1 \) for subset \( \geq 2 \)-digit \( C \) having elements with their last digit ending in 5 [as \text{CIS-IM-linear}].

For a given odd \( P \), with prime gap \( = \) arbitrarily large even number \( n \); let Gap-2-E-\( C_i = 1^n \) even \( C \) with composite gap 2, \( O-P_i = \text{initial odd } P_i \), Gap-1-E-\( C = 2^n \) even \( C \) with composite gap 1, Gap-1-O-\( C = 3^n \) odd \( C \) with composite gap 1, Gap-1-E-\( C_i = 4^n \) even \( C \) with composite gap 1,..., Gap-2-E-\( C_n = n^n \) even \( C \) with composite gap 2, and \( O-P_{i+1} = \text{final odd } P_{i+1} \). We can follow the given sequence of orderly consecutive numbers conveniently called \text{P-C identifier grouping} = (Gap-2-E-\( C_1, O-P_2, \text{Gap-1-E-} C_2, \text{Gap-1-O-} C_3, \text{Gap-2-E-} C_4, O-P_{i+1}). We note the alternating pattern of Gap-1-E-\( C \) and Gap-1-O-\( C \), and that Gap-2-E-\( C_n \) is acting as the new Gap-2-E-\( C_1 \) for \( O-P_{i+1} \) that is now also representing the following new [perpetually repeating] cycle of \( O-P_i \) to \( O-P_{i+1} \) with a totally different new prime gap.

One can more closely analyze the \text{P-C identifier grouping}. Event 1: \( P(\text{uncertain even } C \text{ with last digit ending in 0, 2, 4, 6 or 8}) \text{ as \text{CIS-ALN-decelerating}} \) can [partially and orderly] represent Gap-2-E-\( C \); that always occur before every \( O-P_i \) with last digit ending in 1, 3, 7 or 9. Event 2: \( P(\text{uncertain even } C \text{ with last digit ending in 0, 2, 4 or 8}) \text{ as \text{CIS-IM-accelerating}} \) can [partially and orderly] represent Gap-1-E-\( C \) that always occur after every \( O-P_i \) with last digit ending in 1, 3, 7 or 9. Event 3: \( P(\text{uncertain odd } C \text{ with last digit ending in 1, 3, 7 or 9}) \text{ as \text{CIS-IM-accelerating}} \) can [partially and orderly] represent, and \( P(\text{certain odd } C \text{ with last digit ending in 5}) \text{ as \text{CIS-IM-linear}} \) can [totally and orderly] represent, Gap-1-O-\( C \) that always occur after Gap-1-E-\( C \). Event 4: \( P(\text{uncertain even } C \text{ with last digit ending in 0, 2, 4, 6 or 8}) \text{ as \text{CIS-IM-accelerating}} \) can [partially and orderly] represent Gap-2-E-\( C_n \) that always occur before every \( O-P_{i+1} \) with last digit ending in 1, 3, 7 or 9. Some caveats: (1) Event 3 and Event 4 will alternatingly recur on size of the prime gap \( > 2 \) for initial odd \( P \) whereby Event 3 will ultimately represents the final odd \( P \) and the penultimate/preceding Event 2 will ultimately represents even \( C \) with composite gap 2 that precede the final odd \( P \). (2) For prime gap \( = 2 \) situation, Event 3 will immediately represent the final odd \( P \) whereby Event 2 represents even \( C \) with composite gap 2 that precede the final odd \( P \).

The conjecture on all Subsets of Prime numbers derived from the Set of prime gaps manifesting cardinality \text{CIS-ALN-decelerating} can be algorithmically proven by deriving above correct and complete mathematical arguments that led to \text{P-C identifier grouping} (even \( C \) with composite gap 2, initial odd \( P_i \), even \( C \) with composite gap 1, odd \( C \) with composite gap 1..., final odd \( P_{i+1} \)) which must also obey the above \text{Constraints on Prime numbers and Prime gaps}.

We note that (i) every odd \( P \) with last digit ending in 1, 3, 7 or 9 will fully represent every known odd \( P \) and their individually associated prime gaps. We refer to all subsets \( \geq 2 \)-digit \( P \) and \( \geq 2 \)-digit \( C \). Let \( P(\text{uncertain } P) \) denote \( P(\text{randomly selected } \mathbb{Z} \text{ with } <100\% \text{ certainty}) \) having elements with their last digit ending in 1, 3, 7 or 9, and \( P(\text{certain } C) \) denote \( P(\text{randomly selected } \mathbb{Z} \text{ with } <100\% \text{ certainty}) \) having elements with their last digit also ending in 1, 3, 7 or 9. Then, \( P(\text{uncertain } P \text{ with their last digit ending in 1, 3, 7 or 9}) \) + \( P(\text{certain } C \text{ with their last digit ending in 1, 3, 7 or 9}) \) = 0.4 with the later belonging to \text{CIS-IM-accelerating} > the former belonging to \text{CIS-ALN-decelerating}.

Except for solitary odd prime gap 1 for even \( P \), all prime gaps are constituted from even numbers 2, 4, 6, 8, 10... to an arbitrarily large number as \text{CIS-ALN-decelerating} giving rise to associated odd \( P \) subsets with each subset containing an arbitrarily large number of elements as \text{CIS-ALN-decelerating}. All composite gaps are constituted from CFS containing odd number 1 and even number 2 giving rise to subset containing even and odd \( C \) all with odd composite gap 1 [as \text{CIS-IM-accelerating}] and subset containing even \( C \) all with even composite gap 2 [as \text{CIS-ALN-decelerating} that is equal in magnitude to, and occur immediately after, \( P \)]. Consider \( P_2 = 7, C_3 = 8, C_4 = 9, C_5 = 10, P_5 = 11 \). Except not applicable for twin primes with prime gap = 2 situation, even and odd \( C \) associated with odd composite gap 1 as \text{CIS-IM-accelerating} are always > even \( C \) associated with even composite gap 2 as \text{CIS-ALN-decelerating}. The later with even composite gap 2 are only constituted by uncertain even \( C \) with last digit ending in 0, 2, 4 or 8 that occur immediately after every uncertain odd \( P \) with last digit ending in 1, 3, 7 or 9 whereas the former with odd composite gap 1 can be constituted by uncertain odd \( C \) with last digit ending in 1, 3, 7 or 9 [and certain odd \( C \) with last digit ending in 5] alternating with uncertain even \( C \) with last digit ending in 0, 2, 4 or 8. Thus, (i) apart for \( P \), even \( C \) with last digit ending in 6 can never occur after a \( P \), (ii) for all odd \( P \) with prime gap \( \geq 2 \), \( P(\text{uncertain even } C \text{ with composite gap 2 with last digit ending in 0, 2, 4 or 8}) = P(\text{uncertain \( P \) with last digit ending in 1, 3, 7 or 9}) \), and (iii) for odd \( P \) with prime gap > 2, \( P(\text{uncertain odd } C \text{ with odd composite gap 1 and last digit ending in 1, 3, 7 or 9}) + P(\text{certain } C \text{ with odd composite gap 1 and last digit ending in 5}) = P(\text{uncertain even } C \text{ with odd composite gap 1 and last digit ending in 0, 2, 4, 6 or 8}) = 0.5.

In between any two \( P \) with prime gap > 2, \text{uncertain odd } C \text{ with odd composite gap 1} = \text{uncertain even } C \text{ with odd
The condition $P(\text{uncertain with odd composite gap } 1) + P(\text{uncertain with even composite gap } 2) = 0.4$ is equivalently stated as $P(\text{uncertain with odd composite gap } 1) + 2 P(\text{uncertain with even composite gap } 2) = 0.4$. Then, (i) $P(\text{uncertain with odd composite gap } 1) \geq 1 P(\text{uncertain with even composite gap } 2)$ can only occur when $P(\text{uncertain with odd composite gap } 1) \geq 2/15$ and $P(\text{uncertain with even composite gap } 2) \leq 2/15$. (ii) $P(\text{uncertain with odd composite gap } 1) \geq 2 P(\text{uncertain with even composite gap } 2)$ can only occur when $P(\text{uncertain with odd composite gap } 1) \geq 4/15$ and $P(\text{uncertain with even composite gap } 2) \leq 1/15$. (iii) $P(\text{uncertain with odd composite gap } 1) \geq 3 P(\text{uncertain with even composite gap } 2)$ can only occur when $P(\text{uncertain with odd composite gap } 1) \geq 14/45$ and $P(\text{uncertain with even composite gap } 2) \leq 2/45$, etc.

In general, for $n = 1, 2, 3, 4, 5, ...; P(\text{uncertain with odd composite gap } 1) \geq n P(\text{uncertain with even composite gap } 2)$ can only occur when $P(\text{uncertain with odd composite gap } 1) \geq 2/5 - 4/(15n)$ and $P(\text{uncertain with even composite gap } 2) \leq 2/(15n)$. The dynamic changes between $P(\text{uncertain with odd composite gap } 1)$ and $P(\text{uncertain with even composite gap } 2)$ are of greater interest whereby $P(\text{uncertain with odd composite gap } 1) = n P(\text{uncertain with even composite gap } 2)$ can only occur when $P(\text{uncertain with odd composite gap } 1) = 2/5 - 4/(15n)$ and $P(\text{uncertain with even composite gap } 2) = 2/(15n)$.

$P(\text{uncertain with even composite gap } 2) = P(\text{uncertain with even composite gap } 2)$. $P(\text{uncertain with odd composite gap } 1) = P(\text{uncertain}) - P(\text{uncertain with even composite gap } 2)$ whereby $P(\text{uncertain with even composite gap } 2)$ can never exceed $P(\text{uncertain}) = 0.6$ let alone $P(\text{uncertain with odd composite gap } 1) = 0.4$. Twin primes with prime gap = 2 are guaranteed to have a solitary even $C$ with even composite gap 2 that will always have last digit ending with 0, 2, 6 or 8 [but never 1, 3, 4, 5 or 9] which form a part of $P(\text{uncertain with last digit ending in } 0, 2, 6$ or 8) = 0.4. For primes with prime gaps > 2, there are (prime gap minus 1) $C$ in between any two $P$ with prime gaps > 2; and there are always (prime gap minus 2) $C$ with solitary $C$ having even composite gap 2 [as first even $C$], then odd composite gap 1 $C$ as second odd $C$ alternating with third even $C$, then fourth odd $C$ alternating with fifth even $C$, etc occurring in pairs. Again, primes with prime gap > 2 are guaranteed to have a solitary even $C$ with even composite gap 2 that will form part of uncertain $C$ when their last digit ends in 0, 2, 6 or 8 having $P(\text{uncertain with last digit ending in } 0, 2, 6$ or 8) = 0.4.

The following are examples of computed data on n-digit prime numbers that include their average prime gaps.

Corresponding subsets 1-digit $P$ [2, 3, 5 and 7] and $C$ [4, 6, 8 and 9] that are derived from subset 1-digit $Z$ [0, 1, 2, 3, 4, 5, 6, 7, 8 and 9] with cardinality of 10 have both equal cardinality of 4. First 1-digit $P$, occurs at $i = 1$ (odd position) and last 1-digit $P$, ends at $i = 4$ (even position). Average $P$ gap for 1-digit $P$ = 10/4 = 2.5.

Corresponding subsets 2-digit $P$ [11, 13, 17, 19, 23,...] with cardinality of 21 and $C$ [10, 12, 14, 15, 16,...] with cardinality of 69 together form subset 2-digit $Z$ [10, 11, 12, 13, 14,..., 99] with cardinality of 90. There are $60\%$ of 90 $Z = 54$ $Z$ being $C$ with 100% certainty. Consequently, there are 21 $P$ and 69 - 54 = 15 $C$ that together constitute the $P(\text{uncertain with uncertain C}) = 0.4$ whereby we note that there are more uncertain $P$ [21/36 = 58.3%] than uncertain $C$ [15/36 = 41.7%].

First 2-digit $P$, starts at $i = 5$ (odd position) and last 2-digit $P$, ends at $i = 25$ (odd position). Average $P$ gap for 2-digit $P$ = 90/21 = 4.29.

Corresponding subsets 3-digit $P$ [101, 103, 107, 109, 113,...] with cardinality of 143 and $C$ [100, 102, 104, 105, 106,...] with cardinality of 757 together form subset 3-digit $Z$ [100, 101, 103, 104,..., 999] with cardinality of 900. There are $60\%$ of 900 $Z = 540$ $Z$ being $C$ with 100% certainty. Consequently, there are 143 $P$ and 757 - 540 = 217 $C$ that together constitute the $P(\text{uncertain with uncertain C}) = 0.4$ whereby we note that there are less uncertain $P$ [143/360 = 39.7%] than uncertain $C$ [217/360 = 60.3%]. First 3-digit $P$, starts at $i = 26$ (even position) and last 3-digit $P$, ends at $i = 168$ (even position). Average $P$ gap for 3-digit $P$ = 900/143 = 6.32.

Corresponding subsets 4-digit $P$ [1009, 1013, 1019, 1021, 1031,...] with cardinality of 1061 and $C$ [1000, 1001, 1002, 1003, 1004,...] with cardinality of 7939 together form subset 4-digit $Z$ [1000, 1001, 1002, 1003, 1004,..., 9999] with cardinality of 9000. There are $60\%$ of 9000 $Z = 5400$ $Z$ being $C$ with 100% certainty. Consequently, there are 1061 $P$ and 7939 - 5400 = 2539 $C$ that together constitute the $P(\text{uncertain with uncertain C}) = 0.4$ whereby we note that there are less uncertain $P$ [1061/3600 = 29.5%] than uncertain $C$ [2539/3600 = 70.5%]. First 4-digit $P$, starts at $i = 169$ (odd position) and last 4-digit $P$, ends at $i = 1229$ (odd position). Average $P$ gap for 4-digit $P$ = 9000/1061 = 8.48.

Corresponding subsets 5-digit $P$ [100007, 100009, 100037, 100039, 100061,...] with cardinality of 8363 and $C$ [10000, 10001, 10002, 10003, 10004,...] with cardinality of 81637 together form subset 5-digit $Z$ [100000, 100001, 100002, 100003, 100004,..., 999999] with cardinality of 90000. There are $60\%$ of 90000 $Z = 54000$ $Z$ being $C$ with 100% certainty. Consequently, there are 8363 $P$ and 81637 - 54000 = 27637 $C$ that together constitute the $P(\text{uncertain with uncertain C}) = 0.4$ whereby we note that there are less uncertain $P$ [8363/36000 = 23.2%] than uncertain $C$ [27637/36000 = 76.8%]. First 5-digit $P$, starts
at \( i = 1230 \) (even position) and last 5-digit \( \mathbb{P} \), ends at \( i = 9592 \) (even position). Average \( \mathbb{P} \) gap for 5-digit \( \mathbb{P} = 90000/8363 = 10.76 \).

Corresponding subsets 6-digit \( \mathbb{P} \{100003, 100019, 100043, 100049, 100057...\} \) with cardinality of 68906 and \( \mathbb{C} \{100000, 100001, 100002, 100004, 100005...\} \) with cardinality of 831094 together form subset 6-digit \( \mathbb{Z} \{100000, 100001, 100002, 100003, 100004...\, 999999\} \) with cardinality of 900000. There are 60% of 900000 \( Z = 540000 \) being \( \mathbb{C} \) with 100% certainty. Consequently, there are \( 68906 \mathbb{P} \) and \( 831094 - 540000 = 291094 \mathbb{C} \) that together constitute the \( \mathbb{P} \) (uncertain \( \mathbb{P} \) + uncertain \( \mathbb{C} \)) = 0.4 whereby we note that there are less uncertain \( \mathbb{P} [68906/360000 = 19.1\%] \) than uncertain \( \mathbb{C} [291094/360000 = 80.9\%] \). First 6-digit \( \mathbb{P}_i \) starts at \( i = 9593 \) (odd position) and last 6-digit \( \mathbb{P}_i \) ends at \( i = 78498 \) (even position). Average \( \mathbb{P} \) gap for 6-digit \( \mathbb{P} = 900000/68906 = 13.06 \).

All the odd integers which are not prime are odd composite numbers. Examples of composite odd numbers are 9, 15, 21, 25, 27, 31, etc. All the even integers which are not prime are even composite numbers. Examples of even composite numbers are 4, 6, 8, 10, 12, 14, 16, etc. Some of the properties of co-prime numbers are as follows.

1 is coprime with every number. Any two prime numbers are coprime to each other: As every prime number has only two factors 1 and the number itself, the only common factor of two prime numbers will be 1. For example, 2 and 3 are two prime numbers. Factors of 2 are 1, 2, and factors of 3 are 1, 3. The only common factor is 1 and hence they are coprime. Any two successive numbers/ integers are always coprime: Take any consecutive numbers such as 2, 3, or 3, 4 or 5, 6, and so on; they have 1 as their highest common factor (HCF). The sum of any two coprime numbers are always coprime with their product: 2 and 3 are coprime and have 5 as their sum (2+3) and 6 as the product (2×3). Hence, 5 and 6 are coprime to each other. Two even numbers can never form a coprime pair as all the even numbers have a common factor as 2. If two numbers have their unit digits as 0 and 5, then they are not coprime to each other. For example 10 and 15 are not coprime since their HCF is 5 (or divisible by 5).

We know that coprime numbers are the numbers whose HCF is 1; i.e., two numbers whose common factor is 1 only are called coprime numbers. On the other hand, twin prime numbers are the prime numbers whose difference is always equal to 2. For example, the difference between 3 and 5 is 2, and hence 3 and 5 are twin prime numbers. The major contrasting features between twin prime and coprime numbers are as follows. The difference between two twin prime numbers is always equal to 2, whereas the difference between two coprime numbers can be any number. Twin prime numbers are always prime numbers, whereas the coprime numbers can also be a composite number. It is a mathematical impossibility that the \( P_{i+1} - P_i \) be constituted by a random integer with last digit ending in 3 and 7. Thus, twin prime (Gap 2) can only arise with last digit combination (1, 3), (7, 9), (9, 1). In general:

Gap 2 can end with last digit combinations as (1, 3), (7, 9), (9, 1).
Gap 4 can end with last digit combinations as (3, 7), (7, 1), (9, 3).
Gap 6 can end with last digit combinations as (1, 7), (3, 9), (7, 3).
Gap 8 can end with last digit combinations as (1, 9), (3, 1), (9, 7).
Gap 10 can end with last digit combinations as (1, 1), (3, 3), (7, 7), (9, 9).
Gap 12 can end with last digit combinations as (1, 3), (7, 9), (9, 1).

...repeating cycles...

We outline the following interesting concepts from formal language theory (FLT). A string \( a \) is a subsequence of another string \( b \), if \( a \) can be obtained from \( b \) by deleting zero or more of the characters in \( b \). Example, 517 is a substring of 251667. The empty string is a subsequence of every string. Two strings \( a \) and \( b \) are comparable if either \( a \) is a substring of \( b \), or \( b \) is a substring of \( a \). A string \( a \) in a set of strings \( S \) is minimal if whenever \( b \) (an element of \( S \)) is a substring of \( a \), we have \( b = a \). Then, every set of pairwise incomparable strings is finite (Lothaire, 1983). Consequently, from any set of strings we can find its minimal elements.

Prime numbers are defined as All Natural numbers apart from 1 that are evenly divisible by itself and by 1. Every prime number, when written in base ten, has one of the following [finite] 26 primes as a substring: 2, 3, 5, 7, 11, 19, 41, 61, 89, 409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 66649, 946669, 60000049, 66000049, 66600049. Algorithmically defined using above concepts from FLT (Shallit, 1999 – 2000), we named these 26 primes the Minimal set of prime-strings in base 10 or Minimal primes.

Composite numbers are defined as All Natural numbers apart from 1 that are evenly divisible by numbers other than itself and 1. Every composite number, when written in base ten, has one of the following [finite] 32 composites as a substring: 4, 6, 8, 9, 10, 12, 15, 20, 21, 22, 25, 27, 30, 32, 33, 35, 50, 51, 52, 55, 57, 70, 72, 75, 77, 111, 117, 171, 371, 711, 713, 731. Algorithmically defined using above concepts from FLT, we named these 32 composites the Minimal set of composite-strings in base 10 or Minimal composites.
2.6 Anatomy of Nontrivial Zeros-Gram Points Varying Loop

Let Origin intercept point = nontrivial zero (or NTZ) = Gram[x=0,y=0] point (or G[x=0,y=0]P aka the ’usual’ / ’traditional’ Gram point); and y-axis intercept point = Gram[x=0] point (or G[x=0]P). We follow the peculiar choice of the index n used for Gram points and NTZ [depicted in order of their initial appearances for $\sigma = \frac{1}{2}$ and positive t values]: $n = -3$ for 1st $-ve$ G[y=0]P, $n = -1$ for 1st $-ve$ G[x=0]P, $n = -2$ for 2nd $+ve$ G[y=0]P, $n = -1$ for 3rd $+ve$ G[y=0]P, $n = 1$ for 1st NTZ, $n = 0$ for 2nd $+ve$ G[x=0]P, $n = 0$ for 4th $+ve$ G[y=0]P, $n = 1$ for 3rd $-ve$ G[x=0]P, $n = 2$ for 2nd NTZ, $n = 1$ for 5th $+ve$ G[y=0]P, $n = 3$ for 3rd NTZ, $n = 2$ for 4th $+ve$ G[x=0]P, $n = 2$ for 6th $+ve$ G[y=0]P, $n = 3$ for 5th $-ve$ G[x=0]P, and so on. Thus, we observe the following different varieties of Nontrivial Zeros-Gram Points Varying Loops commencing from 1st NTZ: (A) NTZ, $+ve$ G[x=0]P, $+ve$ G[y=0]P, $-ve$ G[x=0]P, NTZ; (B) NTZ, $+ve$ G[y=0]P, NTZ; (C) NTZ, $+ve$ G[x=0]P, $+ve$ G[y=0]P, $-ve$ G[x=0]P, NTZ; (D) NTZ, $+ve$ G[y=0]P, NTZ; (E) NTZ, $+ve$ G[x=0]P, $+ve$ G[y=0]P, $-ve$ G[x=0]P, NTZ; (F) NTZ, $+ve$ G[y=0]P, $-ve$ G[x=0]P, NTZ; (G) NTZ, $+ve$ G[y=0]P, (H) NTZ; (I) NTZ, $+ve$ G[x=0]P, $+ve$ G[y=0]P, $-ve$ G[x=0]P, NTZ; and so on.

We provide four figures that geometrically depict OUTPUT for $\sigma = \frac{1}{2}$ as Gram points, Close-up view of virtual Origin points when $\sigma = \frac{1}{2}$, and Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{3}{2}$ and $\sigma = \frac{5}{2}$.

In Figure 7 for the $\sigma = \frac{1}{2}$ [or $\sigma < \frac{1}{2}$ situation], there are relatively more virtual Gram[x=0] points existing as y-axis intercept points. On the contrary $\sigma > \frac{1}{2}$ situation e.g. when $\sigma = \frac{3}{2}$, there will instead be virtual Origin intercept points (as additional positive virtual Gram[y=0] points on x-axis) at the “varying” [infinitely many] virtual Origin points with relatively less virtual Gram[x=0] points existing as y-axis intercept points. Then the proof for Riemann hypothesis to be true can be stated as fulfilling two conditions: The position of Origin point when $\sigma = \frac{1}{2}$ is uniquely a solitary point, and the positions of virtual Origin points for any $\sigma$ values when $\sigma \neq \frac{1}{2}$ are non-uniquely infinitely many points but these cannot include the
Figure 6. Close-up view of virtual Origin points when $\sigma = \frac{1}{2}$. OUTPUT for $\sigma = \frac{1}{2}$ [$\sigma < \frac{1}{2}$ situation] as virtual Gram points. Polar graph of $\zeta\left(\frac{1}{2} + it\right)$ plotted along non-critical line for real values of $t$ running between 0 and 100, horizontal axis: $\text{Re}\{\zeta\left(\frac{1}{2} + it\right)\}$, and vertical axis: $\text{Im}\{\zeta\left(\frac{1}{2} + it\right)\}$. Total absence of all Origin intercept points at the "static" Origin point. Total presence of all virtual Origin intercept points (as additional negative virtual Gram[$y=0$] points on x-axis) at the "varying" [infinitely many] virtual Origin points.

Figure 7. Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$. Horizontal axis: $\text{Re}\{\zeta(\sigma + it)\}$, and vertical axis: $\text{Im}\{\zeta(\sigma + it)\}$. Total absence of all Origin intercept points at the [static] Origin point. Total presence of all virtual Origin intercept points (as additional negative virtual Gram[$y=0$] points on the x-axis) at the [infinitely many varying] virtual Origin points; viz, these negative virtual Gram[$y=0$] points on the x-axis cannot exist at the solitary Origin point since the two trajectories form two co-lines [as two parallel curved lines that will never cross over].
position of the Origin point.

The Incompletely Predictable Nontrivial zeros-Gram points Varying Loops, indicating NTZ gaps as geometrically depicted in Figure 2, are dynamically defined by the line tracing joining $n^{th}$ NTZ to $(n+1)^{th}$ NTZ with the [solitary] Origin point acting as the unique $\sigma = \frac{1}{2}$-Attractor. The four boundaries in a usual NTZ-VL on the short range scale will typically consist of the two sequential patterns $n^{th}$ NTZ, then a [alternatingly] positive and negative $G[x=0]P$ (or $vice versa$), then a positive $G[y=0]P$, and finally $(n+1)^{th}$ NTZ. The area enclosed by each NTZ-VL can be obtained by integrating the relevant equation for each Varying Loop in interval from $0\pi$ to $2\pi$.

2.7 Anatomy of Prime-Composite Varying Loop

We provide the figure that geometrically depict Incompletely Predictable Prime-Composite Varying Loops. This allows a visual representation of two combined algorithms in action; viz, Sieve-of-Eratosthenes algorithm that generate all prime numbers and Complement-Sieve-of-Eratosthenes algorithm that generate all composite numbers. In addition, the tabulated and graphed Prime-Composite finite scale mathematical landscape, and its derivation are provided.

Let $N$ = natural numbers, $P$ = prime numbers, and $C$ = composite numbers. Based on our innovative Dimension $(2x - N)$ system with $N = 2x - \Sigma PC_x$-Gap and $x$ = all integers commencing from 1; Dimension $(2x - N)$ when expanded is numerically just equal to $\Sigma PC_x$-Gap since Dimension $(2x - N) = 2x - 2x + \Sigma PC_x$-Gap = $\Sigma PC_x$-Gap. Definition for this system is explained using position $x = 31$ and 32. For $i$ and $x \in N$ [as per the data in Table 3]; $\Sigma PC_x$-Gap = $\Sigma PC_{x-1}$-Gap + Gap value at $P_{x-1}$ or Gap value at $C_{x-1}$ whereby (i) $P$, or $C$ at position $x$ is determined by whether relevant $x$ value belongs to a $P$ or $C$, and (ii) both $\Sigma PC_1$-Gap and $\Sigma PC_2$-Gap = 0. Example, for position $x = 31: 31 is P (P_{11}). Desired Gap value at $P_{10}$ = 2. Thus $\Sigma PC_{31}$-Gap (55) = $\Sigma PC_{30}$-Gap (53) + Gap value at $P_{10}$ (2). Example, for position $x = 32: 32 is C (C_{20}). Desired Gap value at $C_{19}$ = 2. Thus $\Sigma PC_{32}$-Gap (57) = $\Sigma PC_{31}$-Gap (55) + Gap value at $C_{20}$ (2).

Plus-Minus Gap 2 Composite Number Alternating Law refers to rhythmic patterns of alternating presence and absence for relevant Gap 2 Composite Numbers. It has built-in intrinsic mechanism to automatically generate all the prime numbers from prime gaps $\geq 4$ in a mathematically consistent ad infinitum manner. Plus Gap 2 Composite Number Continuous Law refers to (non-)rhythmic patterns with continual presence for relevant Gap 2 Composite Numbers. It has built-in intrinsic mechanism to automatically generate all the prime numbers from prime gap = 2 appearances in a mathematically consistent ad infinitum manner. These two deduced Laws that must crucially involve both prime and composite numbers being dependently and algorithmically tabulated together with subsequent analysis on their [consequently combined] corresponding gaps will qualitatively confirm Polignac’s and Twin prime conjectures to be true.

3. Equations derived from Riemann zeta function and its related functions

This section follows the abbreviations listed previously.
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<th>Gaps</th>
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<th>Dim</th>
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Legend: C = composite, P = prime. Dim = Dimension, Y = 2x - 7 (for visual clarity). N/A = Not Applicable.

Table 3. Prime-Composite finite scale mathematical (tabulated) landscape. Data for x = 2 to 64.

Figure 9. Prime-Composite finite scale mathematical (graphed) landscape. Data for x = 2 to 64.
Derived \( f(n) = 0 \) and \( F(n) = 0 \) equations – representative examples given below comply with exact DA homogeneity at \( \sigma = \frac{1}{2} \) critical line and inexact DA homogeneity at \( \sigma \neq \frac{1}{2} \) non-critical lines. NTZ are synonymous with Gram[\( x=0, y=0 \)] points which is one type of Gram points. Whenever applicable, all modified equations below are expressed using trigonometric identities. Together with Gram[\( y=0 \)] points and Gram[\( x=0 \)] points as remaining two types of Gram points, these three types of Gram points are fully located in their complex equations (akin to Complex Containers) as IP entities whereby their overall location [but not actual positions] are intrinsically incorporated in these complex equations. Eqs. (1), (3), (5), (6), (7) and (8) that comply with exact DA homogeneity at \( \sigma = \frac{1}{2} \) all have fractional exponents \( \frac{1}{2} \). Eqs. (2) and (4) that comply with inexact DA homogeneity at \( \sigma = \frac{3}{4} \) have fractional exponents \( \frac{3}{4} \) in the former and \( \frac{3}{5} \) in the later that are mixed with fractional exponents \( \frac{1}{2} \).

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos \left( \frac{t \ln(2n)}{4} \right) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos \left( \frac{t \ln(2n-1)}{4} \right) = 0 \quad (1)
\]

With exact DA homogeneity, Eq. (1) is \( f(n) \) sim-\( \eta(s) \) at \( \sigma = \frac{1}{2} \) that will incorporate all NTZ [as Zeroes]. There is total absence of (non-existent) virtual NTZ [as virtual Zeroes].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos \left( \frac{t \ln(2n)}{4} \right) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos \left( \frac{t \ln(2n-1)}{4} \right) = 0 \quad (2)
\]

With inexact DA homogeneity, Eq. (2) is \( f(n) \) sim-\( \eta(s) \) at \( \sigma = \frac{3}{4} \) that will incorporate all (non-existent) virtual NTZ [as virtual Zeroes]. There is total absence of NTZ [as Zeroes].

\[
\frac{1}{2^t} \left[ t^2 + 1 \right] \cdot \left( \left(2n\right)^{\frac{1}{4}} \cos \left( \frac{t \ln(2n)}{4} \right) - \left(2n-1\right)^{\frac{1}{4}} \cos \left( \frac{t \ln(2n-1)}{4} \right) + C \right]_{\infty}^{1} = 0 \quad (3)
\]

With exact DA homogeneity, Eq. (3) is \( F(n) \) DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all NTZ [as Pseudo-zeroes to Zeroes conversion]. There is total absence of (non-existent) virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion].

\[
\frac{1}{2^t} \left[ t^2 + \frac{9}{25} \right] \cdot \left( \left(2n\right)^{\frac{1}{4}} \cos \left( \frac{t \ln(2n)}{4} \right) - \left(2n-1\right)^{\frac{1}{4}} \cos \left( \frac{t \ln(2n-1)}{4} \right) + C \right]_{\infty}^{1} = 0 \quad (4)
\]

With inexact DA homogeneity, Eq. (4) is \( F(n) \) DSPL at \( \sigma = \frac{3}{4} \) that will incorporate all (non-existent) virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion]. There is total absence of NTZ [as Pseudo-zeroes to Zeroes conversion].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \sin\left( \frac{t \ln(2n)}{4} \right) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \sin\left( \frac{t \ln(2n-1)}{4} \right) = 0 \quad (5)
\]

Eq. (5) can also be equivalently written as

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos \left( \frac{t \ln(2n)}{2} \right) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos \left( \frac{t \ln(2n-1)}{2} \right) = 0.
\]

With exact DA homogeneity, Eq. (5) is \( f(n) \) Gram[\( y=0 \)] points-sim-\( \eta(s) \) at \( \sigma = \frac{1}{2} \) that will incorporate all Gram[\( y=0 \)] points [as Zeroes]. There is total absence of virtual Gram[\( y=0 \)] points [as virtual Zeroes].

\[
-\frac{1}{2(t^2 + \frac{1}{4})} \left[ \left(2n\right)^{\frac{1}{4}} \cos \left( \frac{t \ln(2n)}{4} \right) - \cos \left( \frac{t \ln(2n-1)}{4} \right) + C \right]_{\infty}^{1} = 0 \quad (6)
\]

Eq. (6) can also be equivalently written as

\[
\frac{1}{2(t^2 + \frac{1}{4})} \cdot \left( \left(2n\right)^{\frac{1}{4}} \cos \left( \frac{t \ln(2n)}{4} \right) - \cos \left( \frac{t \ln(2n-1)}{4} \right) + C \right]_{\infty}^{1} = 0.
\]

With exact DA homogeneity, Eq. (6) is \( F(n) \) Gram[\( y=0 \)] points-DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all Gram[\( y=0 \)] points [as Pseudo-zeroes to Zeroes conversion]. There is total absence of virtual Gram[\( y=0 \)] points [as virtual Pseudo-zeroes to virtual Zeroes conversion].
\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1)) = 0
\]  (7)

With exact DA homogeneity, Eq. (7) is \( f(n) \) Gram\[x=0\] points-sim-\( \eta \)\( (s) \) at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\[x=0\] points [as Zeros]. There is total absence of virtual Gram\[x=0\] points [as virtual Zeros].

\[
\frac{1}{2(t^2 + \frac{1}{4})^2} \left[ (2n)^{\frac{1}{2}} (\cos(t \ln(2n) - \frac{3}{4} \pi) - \cos(t \ln(2n-1) - \frac{3}{4} \pi)) + C \right]_1^\infty = 0
\]  (8)

With exact DA homogeneity, Eq. (8) is \( F(n) \) Gram\[x=0\] points-DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\[x=0\] points [as virtual pseudo-zeroes to virtual Zeros conversion]. There is total absence of virtual Gram\[x=0\] points [as virtual Pseudo-zeroes to virtual Zeros conversion].

We outline sim-\( \eta \)\( (s) \) as Eq. (2) and DSPL as Eq. (4) that comply with inexact DA homogeneity at \( \sigma = \frac{1}{2} \) non-critical line (depicted by Figure 3) whereby \( \sigma = \frac{1}{2} \) instead of \( \sigma = \frac{1}{4} \) is substituted into these two equations. Using [selective] trigonometric identity for linear combination of sine and cosine function whenever applicable to relevant validly treated as a proportionality factor. We analyze \( f(n) = 0 \) and \( F(n) = 0 \) equations at \( \sigma = \frac{1}{2} \) critical line for NTZ situation where \( R = 2^\frac{1}{2}(2n)^{\frac{1}{2}} \) or \( 2^\frac{1}{2}(2n-1)^{\frac{1}{2}} \) in \( f(n) \)'s Eq. (1) and \( R = \frac{1}{2^\frac{1}{2}(2n)^{\frac{1}{2}}} \) or \( \frac{1}{2^\frac{1}{2}(2n-1)^{\frac{1}{2}}} \) in \( F(n) \)'s Eq. (3).

**Remark 7.** Whereas for NTZ \( F(n) \) Eq. (3) that exactly represent precise Areas of Varying Loops and \( f(n) \) Eq. (1) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude \( R \) from Eq. (3) which is dependent on parameter \( t \) and Eq. (1) which is independent of parameter \( t \) represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

We analyze \( f(n) = 0 \) equations [relevant to approximate Areas of Varying Loops] at \( \sigma = \frac{1}{2} \) critical line for Gram\[y=0\] points as Eq. (5) and Gram\[x=0\] points as Eq. (7) whereby we validly designate \( R = (2n)^{\frac{1}{2}} \) or \( (2n-1)^{\frac{1}{2}} \) as the assigned scaled amplitude and [unwritten] \( \alpha = 0 \) as the assigned phase shift.

Relevant to precise Areas of Varying Loops at \( \sigma = \frac{1}{2} \) critical line for Gram\[y=0\] points \( F(n) \) Eq. (6) with \( R = \frac{1}{2^\frac{1}{2}(2n)^{\frac{1}{2}}} \) or \( \frac{1}{2^\frac{1}{2}(2n-1)^{\frac{1}{2}}} \) and Gram\[x=0\] points \( F(n) \) Eq. (8) with \( R = \frac{1}{2^\frac{1}{2}(2n)^{\frac{1}{2}}} \) or \( \frac{1}{2^\frac{1}{2}(2n-1)^{\frac{1}{2}}} \), we observe the former \( R \) to be the negative of the later \( R \). However, this observation is context-sensitive because when Eq. (6) is written in its equivalent format above, the former \( R \) is identical to the later \( R \). Both \( R \) are now just given by \( \frac{1}{2^\frac{1}{2}(2n)^{\frac{1}{2}}} \) or \( \frac{1}{2^\frac{1}{2}(2n-1)^{\frac{1}{2}}} \).
Remark 8. Whereas for Gram[y=0] points F(n) Eq. (6) that exactly represent precise Areas of Varying Loops and f(n) Eq. (5) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R in Eq. (6) which is dependent on parameter t and Eq. (5) which is independent of parameter t represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Remark 9. Whereas for Gram[x=0] points F(n) Eq. (8) that exactly represent precise Areas of Varying Loops and f(n) Eq. (7) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R in Eq. (8) which is dependent on parameter t and Eq. (7) which is independent of parameter t represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Finally, we analyze f(n) = 0 and F(n) = 0 equations at \( \sigma = \frac{1}{2} \pi \) critical line for NTZ situation where phase shift \( \alpha = \frac{1}{4} \pi \) in NTZ f(n) Eq. (1) and \(-\frac{1}{4} \pi \) in NTZ F(n) Eq. (3); and F(n) = 0 equations at \( \sigma = \frac{1}{2} \pi \) critical line for Gram[y=0] points and Gram[x=0] points situations where phase shift \( \alpha = -\frac{1}{4} \pi \) (or \( \frac{3}{4} \pi \) when written in its equivalent format above) in Gram[y=0] points F(n) Eq. (6) and \(-\frac{3}{4} \pi \) in Gram[x=0] points F(n) Eq. (8). Always being \( \frac{1}{2} \pi \) out-of-phase with each other, trigonometric functions sine and cosine are cofunctions with \( \sin n = \cos (\frac{\pi}{2} - n) \) or \( \cos n = \sin (\frac{\pi}{2} - n) \) or \( \sin (n + \frac{\pi}{2}) \), \( \frac{d}{dn} \sin n = \cos n \cdot \frac{d}{dn} \cos n = -\sin n \). \( \int \sin n \cdot dn = -\cos n + C \) and \( \int \cos n \cdot dn = \sin n + C \). Last two integrals explain relation between f(n)’s Zeros and F(n)’s Pseudo-zeros when they involve simple sine and/or cosine terms viz, f(n)’s IP Gram[y] points or f(n)’s IP virtual Gram[y] points and Gram[x] points or f(n)’s IP virtual Gram[x] points situations where phase shift \( \alpha = \frac{1}{4} \pi \[or \pi \]\) out-of-phase with each other. Peculiar to IP NTZ as critical line for NTZ situation where phase shift \( \alpha = \frac{1}{2} \pi \) radian. Involving trigonometric functions as complex sine and/or cosine terms: f(n)’s IP NTZ or [non-existent] f(n)’s IP virtual NTZ (in t values) = F(n)’s IP Pseudo-NTZ or [non-existent] F(n)’s IP virtual Pseudo-NTZ (in t values) \(-\frac{1}{4} \pi \); f(n)’s IP Gram[y=0] points or f(n)’s IP virtual Gram[y=0] points (in t values) = F(n)’s IP Pseudo-Gram[y=0] points or F(n)’s IP virtual Pseudo-Gram[y=0] points (in t values) \(-\frac{3}{4} \pi \); and f(n)’s IP Gram[x=0] points or f(n)’s IP virtual Gram[x=0] points (in t values) \(-\frac{3}{4} \pi \).

Underlying f(n) as equation and F(n) as law (equation) that generate their CIs of IP Zeros, IP virtual Zeros, IP Pseudo-zeros and IP virtual Pseudo-zeros are precisely related as \( \frac{1}{2} \pi \) (for NTZ case) or \( \frac{3}{4} \pi \) (for Gram[y=0] points and Gram[x=0] points cases) out-of-phase with each other. Peculiar to IP NTZ as Origin intercept points, we crucially note only they will uniquely behave in accordance with complex sine and/or cosine terms present in their equations that generate corresponding IP Zeros and IP Pseudo-zeros which are \( \frac{1}{2} \pi \) [but not \( \frac{3}{4} \pi \)] out-of-phase with each other.

We show that corresponding paired IP two types of Gram points [as Zeros] situation, paired IP two types of virtual Gram points [as virtual Zeros] situation, paired IP two types of Pseudo-Gram points [as Pseudo-zeros] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeros] situation are always \( \frac{1}{2} \pi \) out-of-phase with each other in every one of these situations.

The x-axis and y-axis are orthogonal to each other with angle between them \( \frac{1}{2} \pi \) radian. Involving trigonometric functions as complex sine and/or cosine terms: f(n)’s IP Gram[y=0] points or f(n)’s IP virtual Gram[y=0] points (in t
values) = f(n)’s IP Gram[x=0] points or f(n)’s IP virtual Gram[x=0] points (in t values) + \frac{1}{2}\pi; and F(n)’s IP Pseudo-Gram[y=0] points or F(n)’s IP virtual Pseudo-Gram[y=0] points (in t values) = F(n)’s IP Pseudo-Gram[x=0] points or F(n)’s IP virtual Pseudo-Gram[x=0] points (in t values) + \frac{1}{2}\pi.

These observations imply underlying f(n) as equation and F(n) as law (equation) that generate corresponding paired IP two types of Gram points [as Zeroes] situation, paired IP two types of virtual Gram points [as virtual Zeroes] situation, paired IP two types of Pseudo-Gram points [as pseudo-zeroes] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] situation are always \frac{1}{2}\pi out-of-phase with each other in every one of these mentioned situations.

The \( \sigma = \frac{1}{2} \) NTZ computed from Eq. (1) – \( \sigma \neq \frac{1}{2} \) (non-existent) virtual NTZ computed from Eq. (2) Pairing. For \( i = 1, 2, 3, ..., \infty \); let mutually exclusive \( i^{th} \) NTZ = NTZ, and \( i^{th} \) virtual NTZ = vNTZ, and \( i^{th} \) NTZ gaps = NTZ-Gap, and \( i^{th} \) virtual NTZ gaps = vNTZ-Gap. Eq. (1) and Eq. (2) are dependently identical except for associated \( \sigma \) values. They are used to precisely, tediously and dependently calculate all NTZs and vNTZs, with their \( i^{th} \) positions being IP.

**I. NTZ or Gram \([x=0,y=0]\) points** as geometrical Origin intercept points are mathematically defined by \( \sum \Re \Im |\eta(s)| = \Re |\eta(s)| + \Im |\eta(s)| = 0 \). General equation for f(n)’s sim-\( \eta(s) \) as Zeroes is given by

\[
\sum_{n=1}^{\infty} (2n)^{-\sigma} (\sin(t \ln(2n)) - \cos(t \ln(2n))) = 0
\]

General equation for F(n)’s DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

\[
\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot \left| (2n)^{1-\sigma} ((t + \sigma - 1) \sin(t \ln(2n)) + (t - \sigma + 1) \cos(t \ln(2n))) - (2n - 1)^{1-\sigma} ((t + \sigma - 1) \sin(t \ln(2n)) + (t - \sigma + 1) \cos(t \ln(2n))) + C \right| = 0
\]

**II. Gram[y=0] points** as geometrical x-axis intercept points are mathematically defined by \( \sum \Re \Im |\eta(s)| = \Re |\eta(s)| + 0 \), or simply \( \Im |\eta(s)| = 0 \). General equation for f(n)’s Gram[y=0] points-sim-\( \eta(s) \) as Zeroes is given by

\[
\sum_{n=1}^{\infty} (2n)^{-\sigma} \sin(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \sin(t \ln(2n - 1)) = 0
\]

General equation for F(n)’s Gram[y=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

\[
-\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot \left| (2n)^{1-\sigma} ((\sigma - 1) \sin(t \ln(2n)) + t \cos(t \ln(2n))) - (2n - 1)^{1-\sigma} ((\sigma - 1) \sin(t \ln(2n - 1)) + t \cos(t \ln(2n - 1))) + C \right| = 0
\]

**III. Gram[x=0] points** as geometrical y-axis intercept points are mathematically defined by \( \sum \Re \Im |\eta(s)| = 0 + \Im |\eta(s)| \), or simply \( \Re |\eta(s)| = 0 \). General equation for f(n)’s Gram[x=0] points-sim-\( \eta(s) \) as Zeroes is given by

\[
\sum_{n=1}^{\infty} (2n)^{-\sigma} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \cos(t \ln(2n - 1)) = 0
\]

General equation for F(n)’s Gram[x=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

\[
-\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot \left| (2n)^{1-\sigma} (t \sin(t \ln(2n)) - (\sigma - 1) \cos(t \ln(2n))) - (2n - 1)^{1-\sigma} (t \sin(t \ln(2n - 1)) - (\sigma - 1) \cos(t \ln(2n - 1))) + C \right| = 0
\]
4. Riemann zeta function and its related functions

This section follows the abbreviations listed previously.

\[ \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \]

The common convention is to write \( s = \sigma + it \) with \( t = \sqrt{-1} \), and with \( \sigma \) and \( t \) real. Valid for \( \sigma > 0 \), we write \( \zeta(s) \) as \( \text{Re}[\zeta(s)]+i\text{Im}[\zeta(s)] \) and note that \( \zeta(\sigma + it) \) when \( 0 < t < +\infty \) is the complex conjugate of \( \zeta(\sigma - it) \) when \( -\infty < t < 0 \).

Also known as alternating zeta function, \( \eta(s) \) must act as proxy for \( \zeta(s) \) in critical strip (viz. \( 0 < \sigma < 1 \)) containing critical line (viz. \( \sigma = \frac{1}{2} \)) because \( \zeta(s) \) only converges when \( \sigma > 1 \). This implies \( \zeta(s) \) is undefined to left of this \( \sigma > 1 \) region [in the critical strip] which then requires \( \eta(s) \) representation instead. They are related to each other as \( \zeta(s) = \gamma \cdot \eta(s) \) with proportionality factor \( \gamma = \frac{1}{1 - 2^{1-s}} \) and \( \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \).

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \]

\[ = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \]

\[ = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \]

\[ = \frac{1}{(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s})(1 - 7^{-s})(1 - 11^{-s}) \cdots} \]

Eq. (15) is defined for only \( 1 < \sigma < \infty \) region where \( \zeta(s) \) is absolutely convergent with no zeros located here. In Eq. (15), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] also represents \( \zeta(s) \) if \( s \) all prime and, by default, composite numbers are (intrinsically) encoded in \( \zeta(s) \).

\[ \zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \cdot \Gamma(1-s) \cdot \zeta(1-s) \]

With \( \sigma = \frac{1}{2} \) as symmetry line of reflection, Eq. (16) is Riemann’s functional equation valid for \( -\infty < \sigma < \infty \). It can be used to find all trivial zeros on horizontal line at \( t = 0 \) occurring when \( \sigma = -2, -4, -6, -8, -10, \ldots \), \( \infty \) whereby \( \zeta(s) = 0 \) because factor \( \sin(\frac{\pi s}{2}) \) vanishes. \( \Gamma \) is gamma function, an extension of factorial function [a product function denoted by \( ! \)] notation whereby \( n! = n(n-1)(n-2) \ldots (n-(n-1)) \) with its argument shifted down by 1, to real and complex numbers. That is, if \( n \) is a positive integer, \( \Gamma(n) = (n-1)! \).

\[ \zeta(s) = \frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \]

\[ = \frac{1}{(1 - 2^{1-s})} \left( \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \right) \]

Eq. (17) is defined for all \( \sigma > 0 \) values except for simple pole at \( \sigma = 1 \). As alluded to above, \( \zeta(s) \) without \( \frac{1}{(1 - 2^{1-s})} \) viz. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \) is \( \eta(s) \). It is a holomorphic function of \( s \) defined by analytic continuation and is mathematically defined at \( \sigma = 1 \) whereby analogous trivial zeros with presence for \( \eta(s) \) [but not for \( \zeta(s) \)] on vertical straight line \( \sigma = 1 \) are found at \( s = 1 \pm i \cdot \frac{2\pi k}{\ln(2)} \) where \( k = 1, 2, 3, 4, \ldots, \infty \).

Euler formula can be stated as \( e^{it} = \cos n + i \cdot \sin n \). Euler identity (where \( n = \pi \)) is \( e^{i\pi} = \cos \pi + i \cdot \sin \pi = -1 + 0 \) [or stated as \( e^{i\pi} + 1 = 0 \)]. The \( n^t \) of \( \zeta(s) \) is expanded to \( n^t = n^{(\sigma + it)} = n^\sigma e^{it\ln(n)} \) since \( n^t = e^{t\ln(n)} \). Apply Euler formula to
result in \( n' = n'^{\frac{1}{2}} \cos(t \ln(n)) + \cdot \sin(t \ln(n)) \). This is written in trigonometric form [designated by short-hand notation \( n'(Euler) \)] whereby \( n' \) is modulus and \( t \ln(n) \) is polar angle (argument).

We apply \( n'(Euler) \) to Eq. (17) to obtain \( f(n) \) general sim-\( \eta(s) \) for determining \( \sigma = \frac{1}{2} \) NTZ versus (non-existent) \( \sigma \neq \frac{1}{2} \) virtual NTZ (Ting, 2020, section 4, p. 24 - 28). At \( \sigma = \frac{1}{2} \), this is given as Eq. (9) and with the trigonometric identity application as Eq. (1). Integrate \( f(n) \) general sim-\( \eta(s) \) to obtain \( F(n) \) general DSPL for determining \( \sigma = \frac{1}{2} \) Pseudo-zeros versus (non-existent) \( \sigma \neq \frac{1}{2} \) virtual Pseudo-zeros. Pseudo-zeros and (non-existent) virtual Pseudo-zeros can be converted to Zeros (NTZ) and (non-existent) virtual Zeros (virtual NTZ). At \( \sigma = \frac{1}{2} \), this is given as Eq. (10) and with the trigonometric identity application as Eq. (3).

We provide \( f(n) \) general Gram[y=0] points-sim-\( \eta(s) \) for determining \( \sigma = \frac{1}{2} \) Gram[y=0] points versus \( \sigma \neq \frac{1}{2} \) virtual Gram[y=0] points (Ting, 2020, section 5, p. 28 - 30). At \( \sigma = \frac{1}{2} \), this is given as Eq. (11) but we are unable to apply trigonometric identity. Integrate \( f(n) \) general Gram[y=0] points-sim-\( \eta(s) \) to obtain \( F(n) \) general Gram[y=0] points-DSPL for determining \( \sigma = \frac{1}{2} \) Pseudo-zeros versus \( \sigma \neq \frac{1}{2} \) virtual Pseudo-zeros. Pseudo-zeros and virtual Pseudo-zeros can be converted to Zeros (Gram[y=0] points) and virtual Zeros (virtual Gram[y=0] points). At \( \sigma = \frac{1}{2} \), this is given as Eq. (12) and with the trigonometric identity application as Eq. (6).

We provide \( f(n) \) general Gram[x=0] points-sim-\( \eta(s) \) for determining \( \sigma = \frac{1}{2} \) Gram[x=0] points versus \( \sigma \neq \frac{1}{2} \) virtual Gram[x=0] points (Ting, 2020, section 5, p. 28 - 30). At \( \sigma = \frac{1}{2} \), this is given as Eq. (13) but we are unable to apply trigonometric identity. Integrate \( f(n) \) general Gram[x=0] points-sim-\( \eta(s) \) to obtain \( F(n) \) general Gram[x=0] points-DSPL for determining \( \sigma = \frac{1}{2} \) Pseudo-zeros versus \( \sigma \neq \frac{1}{2} \) virtual Pseudo-zeros. Pseudo-zeros and virtual Pseudo-zeros can be converted to Zeros (Gram[x=0] points) and virtual Zeros (virtual Gram[x=0] points). At \( \sigma = \frac{1}{2} \), this is given as Eq. (14) and with the trigonometric identity application as Eq. (8).

5. Conclusions

Treated as Incompletely Predictable problems, we have provided a comparatively elementary algorithm-type proof for Polignac’s and Twin prime conjectures. In particular, this statement can be conceptually stated as Plus-Minus Gap 2 Composite Number Alternating Law and Plus Gap 2 Composite Number Continuous Law that are applicable on the finite (small) scale, are also applicable on the infinite (large) scale. There is zero probability that any particular prime gaps from eternal repeated groupings of small and/or large prime gaps that faithfully generate all the countably arbitrarily large number of odd primes will abruptly terminate or disappear.

Treated as Incompletely Predictable problems, we have provided a comparatively elementary equation-type proof on Riemann hypothesis while explaining the existence of mutually exclusive three types of Gram points and two types of virtual Gram points. We also conduct appropriate analysis on complex properties present in Dirichlet Sigma-Power Law, Gram[y=0] points-Dirichlet Sigma-Power Law and Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant Pseudo-Gram points; and in virtual Gram[y=0] points-Dirichlet Sigma-Power Law and virtual Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant virtual Pseudo-Gram points. There is zero probability that any of the countably infinitely large number of nontrivial zeros can be located away from [geometric] Origin point, which is equivalent to [mathematical] critical line.

The geometrical-mathematical unified approach used in our proofs is analogically similar to the algebra-geometry unified approach of geometric Langlands program formalized by Professor Peter Scholze and Professor Laurent Fargues (Fargues & Scholze, 2021). Our achievements represent solving colloquially, from outside the dazzling big container overall complex (meta-)properties on Incompletely Predictable problems in relation to prime numbers and nontrivial zeros. Extra dedicated research is required to determine their many more interesting [unsolved] complex properties colloquially, from inside the dazzling big container at the finer level; e.g., on our proposed Gram’s Prime k-tuple law for \( k \geq 6 \) values (in Remark 4) and Rosser’s Prime k-tuple rule for \( k \geq 2 \) values (in Remark 5).

Declaration

Professor John Ting (Diploma of Ageing Studies and Services, FARGP, ANZCA Primary Fellowship, FRACGP, MBBS). Mr. Jim Entwood, arXiv.org Operations Manager, has helped to solve some technical issues from submitting the arXiv version of this paper. Fields of interest from Doctor of Philosophy (PhD) viewpoint: Ageing, Dementia, Sleep, Learning, Memory and Number Theory. An autistic savant is someone with autism who also has a single extraordinary area of knowledge or ability. Savant skills are typically linked to massive memory retention ability such as required for advance Concrete Mathematics on rapid calculations, memorizing whole phone book, etc. The author only possess average level of working, short-term and long-term memory, and Concrete Mathematics ability. While conducting active research for this paper which requires advance Abstract Mathematics, he practices behavioral augmentation on his personal Stage 3 Deep Sleep which contributes to insightful thinking, creativity and memory, and Stage 4 REM Sleep which is essential to cognitive functions like memory, learning and creativity.
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References


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References


Appendix

A. Gram’s Law and Rosser Rule for Gram points

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), ‘traditional’/‘usual’ Gram points or (mathematical) Gram[y]0 points or (geometrical) x-axis intercept points are other conjugate pairs values in Riemann zeta function ζ(s) on σ = 1/2 critical line. Then s = 1/2 + it gives rise to ζ(1/2 + it) on critical line; and Gram points when defined in terms of ζ(s) is given by ∑ReIm(ζ(s)) = ReIm(ζ(s)) + 0, or simply Im(ζ(s)) = 0. Alternatively defined using expression denoting ζ(s) on critical line ζ(1/2 + it) = Z(t)e−θ(t) whereby Hardy’s function, Z, is real for real t, and θ is Riemann–Siegel theta function given in terms of gamma function as θ(t) = arg(Γ(1/4 + t/2)) − log π t for real values of t; we note that ζ(s) is real when sin(θ(t)) = 0. This implies that θ(t) is an integer multiple of π which allows for location of Gram points to be calculated easily by inverting the formula for θ. Gram points are historically [crudely] numbered as gn for n = 0, 1, 2, 3,…., whereby gn is the unique solution of θ(t) = nr. Here, n = 0 is the [first] g0 value of 17.845995405… which is larger than the smallest [first] positive nontrivial zeros (NTZ) value of 14.13472515…. Thus, n = -3 correspond to g_{-3} = 0, n = -2 correspond to g_{-2} = 3.4362182261…, and n = -1 correspond to g_{-1} = 9.6669080561…

Paired [infinite-length] integer sequences with prestigious connections:
A100967+0, which is A100967 (Noe, 2004), is precisely defined as ”Least k such that binomial(2k+1, k-n-1) ≥ binomial(2k, k) viz. (2k+1)!k!(k-n-1)!/(k-n-1)!/(k+n+1)!”. The terms commencing from Position 0, 1, 2, 3,… of A100967+0 are listed below: 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, 3535,…..

A100967+1 is precisely defined as ”Add 1 to each and every terms from A100967+0”. The terms commencing from Position 0, 1, 2, 3,… of A100967+1 are listed below: 4, 10, 19, 30, 45, 62, 82, 105, 131, 160, 192, 226, 264, 304, 348, 394, 443, 495, 550, 607, 668, 731, 798, 867, 939, 1014, 1092, 1173, 1256, 1343, 1432, 1525, 1620, 1718, 1819, 1923, 2030, 2139, 2252, 2367, 2486, 2607, 2731, 2858, 2988, 3120, 3256, 3395, 3536,…..

A228186 (Ting, 2013) is defined as ”Greatest natural number k > n such that calculated peak values for ratio R = \[\frac{(k + n - 1)!(n - k)!}{n!(n - 1)!}\] belong to maximal rational numbers < 1. It is also defined as ”Smallest natural number k > n such that (k+n+1)!(k-n-2)! < 2k!(k-1)!”. The terms commencing from Position 0, 1, 2, 3,… of A228186 are listed below: 4, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 226, 263, 304, 347, 393, 442, 494, 549, 607, 667, 731, 797, 866, 938, 1013, 1091, 1172, 1256, 1342, 1432, 1524, 1619, 1717, 1818, 1922, 2029, 2139, 2251, 2367, 2485, 2606, 2730, 2857, 2987, 3120, 3255, 3394, 3535,…..

Unexpected connection [and unrelated to NTZ and Gram points]: A228186 can be considered an innovative [infinite-length] ”Hybrid integer sequence” identical to ”non-Hybrid integer sequence” A100967+0 except for the interspersed [finite] 21 ‘exceptional’ terms located at Position 0, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their corresponding 21 values exactly specified by [infinite-length] ”non-Hybrid integer sequence” A100967+1.

A114856-”bad”-Gram-points, which is A114856 (Weisstein, 2006), is precisely defined as ”Indices n of Gram points gn for which (-1)^nZ(gn) < 0 with Z(t) being Riemann-Siegel Z-function and full given range of values n = 0, 1, 2, 3,….”. The terms of A114856-”bad”-Gram-points are listed below: 126, 134, 195, 211, 232, 254, 288, 367, 377, 379, 397, 400, 461, 507, 518, 529, 567, 578, 595, 618, 626, 637, 654, 668, 692, 694, 703, 715, 728, 766, 777, 793, 795, 807, 819, 848, 857, 869, 887, 964, 992, 995, 1016, 1028, 1034, 1043, 1046, 1071, 1086,…. A114856-”good”-Gram-points, given by ”total”-Gram points minus A114856-”bad”-Gram-points, is precisely defined as ”Indices n of Gram points gn for which (-1)^nZ(gn) > 0 with Z(t) being Riemann-Siegel Z-function and full given range of values n = 0, 1, 2, 3,….”. The derived terms of A114856-”good”-Gram-points: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,….

A216700 (Greathouse IV, 2012) is precisely defined as ”Violations of Rosser Rule: numbers n such that the Gram block [g_n, g_{n+1}] contains fewer than k points t such that Z(t) = 0 with Z(t) being Riemann-Siegel Z-function and full given range of values n = 0, 1, 2, 3,….”. The terms of A216700 are listed below: 13999525, 30783329, 30930927, 37592215, 40870156, 43628107, 46082042, 46875667, 49624541, 50799238, 55221454, 56948780, 60515663, 6131766, 69784844, 75052114, 79545241, 79652248, 83088043, 83689523, 85348958, 86513820, 87947597,…..

Expected connection: All NTZ (as conjectured by Riemann hypothesis) and Gram points (by definition) are located on
the same critical line of Riemann zeta function. Counting NTZ can be validly reduced to counting all Gram points where Gram’s Law is satisfied and adding count of NTZ inside each Gram block. With this process, we need not locate NTZ but just have to accurately compute \( Z(t) \) to show that it changes sign.

Gram’s Law is the observation that there is [usually] exactly one NTZ (Gram\([x=0,y=0]\) points or Origin intercept points) between any two “good” Gram points. Examples of closely related statements equivalent to Gram’s law are: \((-1)^n Z(g_n)\) is [usually] positive or \( Z(t) \) [usually] has opposite sign at consecutive Gram points. Thus, a \( t \)-valued Gram point is called a “good” Gram point if \( \zeta(s) \) is positive at \( \frac{1}{2} + it \) with \((-1)^n Z(g_n) > 0\) and a “bad” Gram point if \( \zeta(s) \) is negative at \( \frac{1}{2} + it \) with \((-1)^n Z(g_n) < 0\). The indices of “bad" Gram points where \( Z \) has the ‘wrong’ sign are given by A114856 in OEIS. A Gram block \([g_n, g_{n+k}]\) is a half-open interval bounded by two “good” Gram points \( g_n \) and \( g_{n+k} \) such that all Gram points \( g_{n+1}, \ldots, g_{n+k-1} \) between them are "bad" Gram points. A refinement of Gram’s Law is known as Rosser Rule (Rosser, Yohe & Schoenfeld, 1969) which stated that Gram blocks [usually] have the expected number of NTZ in them (identical to number of Gram intervals), even though some of the individual Gram intervals in the block may not have exactly one NTZ in them. Example, the interval bounded by \( g_{125} \) and \( g_{127} \) is a Gram block containing a unique ”bad" Gram point \( g_{126} \) and the expected number 2 of NTZ although neither of its two Gram intervals contains a unique NTZ.

Gram’s Law and Rosser Rule both imply that in some sense NTZ do not stray too far from their expected positions, and that they hold most of the time but are violated infinitely often (in an Incompletely Predictable manner) (Trudgian, 2011 & Trudgian, 2014). Professor Timothy Trudgian in 2011 explicitly showed that both Gram’s Law and Rosser Rule fail in a positive proportion of cases. In particular, it is expected that in about 73% \( \approx \frac{3}{4} \) one NTZ is enclosed by two successive Gram points [and thus Gram’s Law fails for about 27% \( \approx \frac{1}{4} \) of all Gram intervals to contain exactly one NTZ], but in about 14% no NTZ and in about 13% two NTZ are in such a Gram interval on the long run.

**B. Freebasic programme to elucidate all possible patterns of Prime k-tuplets for \( k = 2 \) to 50**

Actual patterns for first 50 Prime k-tuples including data on \( p_1 \) congruent to \( p \) (modulo \( q \)) can be obtained from the website **Patterns of prime k-tuplets & the Hardy-Littlewood constants** with URL https://pzktupel.de/ktpatt_hl.php that is maintained by Norman Luhn (Email address: pzktupel@pzktupel.de). Luhn’s programme to provide all possible patterns of Subtype I Admissible Prime k-tuplets for \( k = 2 \) to 50 is reproduced below with permission.

```vb
#INCLUDE "windows.bi"
#INCLUDE "vbcompat.bi"
DIM AS UINTEGER G,i,j,k,mini,l,m,n,o,p
DIM AS STRING S1,S2

START:
INPUT "k= "; k
mini=500
G=2*3*5*7*11*13*17*19*23

REDIM F(G\7) AS UBYTE
REDIM Z(100000000) AS UINTEGER
REDIM S(2000000) AS STRING
REDIM R(2000000) AS UINTEGER

i=3
WHILE i<24
FOR j=i+i TO G STEP i*2
F(j SHR 3)=BITSET(F(j SHR 3),j MOD 8)
NEXT j
i+=2
WEND

j=0
```
REM Collect all free Cells with no factor <29 and count in Array Z()

FOR i = 3 TO G\8
  IF BIT(F(I),1)=0 THEN j++=1:Z(j)=i*8+1
  IF BIT(F(I),3)=0 THEN j++=1:Z(j)=i*8+3
  IF BIT(F(I),5)=0 THEN j++=1:Z(j)=i*8+5
  IF BIT(F(I),7)=0 THEN j++=1:Z(j)=i*8+7
NEXT i

REM start with 29, calculate the diameter Z(i+k-condition)-Z(i) < mini ? Yes, set mini=Z(i+k)-Z(i)

FOR i = 29 TO j STEP 2
  IF Z(i+k)-Z(i)¡mini THEN mini=Z(i+k)-Z(i)
NEXT i

PRINT "Diameter found for k: ";mini

REM Musterermittlung

FOR i = 1 TO j
  IF Z(i+k)-Z(i)<>mini THEN GOTO NXTI
  S1=""
  FOR l = i TO i+k-1
    S1=S1+STR(Z(l)-Z(i))+" 
  NEXT l
  m+=1:S(m)=S1:R(m)=Z(i) MOD 30
  IF m>9999 THEN m=9999
  NXTI:NEXT i

PRINT "Write pattern in file pattern.txt"

OPEN "pattern.txt" FOR OUTPUT as #1
PRINT "k=":k
FOR n = 1 TO m
  FOR o = 1 TO n-1
    IF S(o)=S(n) THEN GOTO NXTN
  NEXT o
  PRINT "Member of: N=";R(n);"+30k, Pattern d=";S(n)
NXTN:NEXT n
CLOSE #1
PRINT "done, ENTER!"

SLEEP

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