"$n_1 \times n_2 \times \ldots \times n_k$ Dots Puzzle": An Optimal General Algorithm

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Abstract: in this paper, we show a new algorithm for the $n_1 \times n_2 \times \ldots \times n_k$ *dots puzzle* (an extension of the well-known *nine dots puzzle* of Samuel Loyd), able to solve completely the problem for the case $k=2$ and, at the same time, provide lower upper bounds for the other cases.
Keywords: combinatorics, graph theory, computational science, nine dots puzzle, covering path, minimum-link, minimum-turn, link-length, grid, point, algorithm

1 Introduction

The problem addressed in this paper is an extension of the well-known *nine dots puzzle* by Samuel Loyd (refer to [2]): the $n_1 \times n_2 \times \ldots \times n_k$ *dots puzzle*.

Given a regular k-dimensional grid of $n_1 \times n_2 \times \ldots \times n_k$ points (dimensionless), where $n_1 \geq n_2 \geq \ldots \geq n_k \geq 2$ and $k \geq 2$, the objective is to traverse all the points (at least once) using a polygonal chain composed of the minimum possible number of segments, connected sequentially to their respective endpoints.

For the two-dimensional case, an exact solution to the problem has already been found (see [1]), while, for other cases, algorithms have been proposed that provide an upper bound for the solution (see [3] and [4]). In our paper, we will present a new algorithm capable of providing both the exact solution to the two-dimensional problem and a more efficient upper bound for the other cases.

2 The case $n_1 \times n_2$

We begin the description of our algorithm with the simplest case: the two-dimensional one.

To navigate the $n_1 \times n_2$ points of the grid, we will utilize one of three different *paths* depending on the situation, solving the problem with the minimum possible number of segments.

2.1 Path 1 (when $n_1=n_2=2$ or $n_1>n_2$)

When $n_1=n_2=2$ or $n_1>n_2$, we utilize *path 1*, as shown in *Figure 1*.

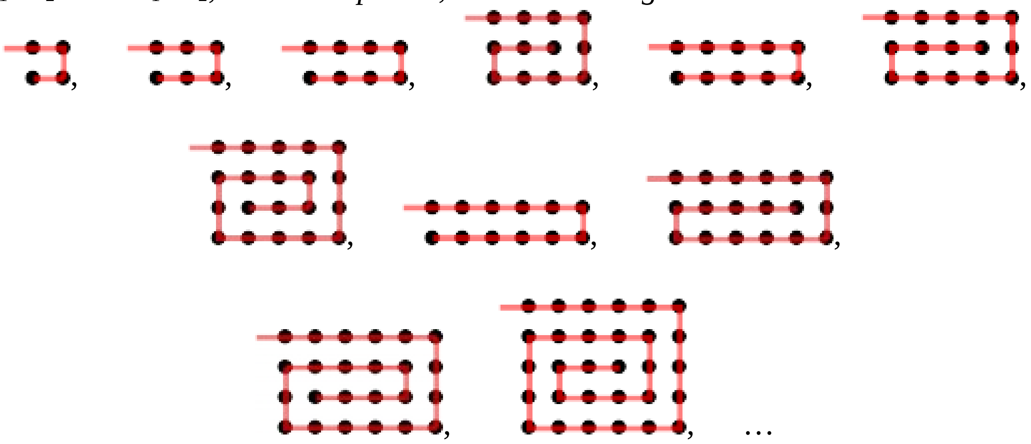


Figure 1. Some of the initial examples of *path 1*.

Path 1 begins at the upper left corner of the grid, where the number of points per row is n_1 and the number of points per column is n_2. The total number of segments used is the minimum possible, equal to $2\cdot n_2-1$ (see [1]).

Observing *Figure 1*, it can be noted that, when n_2 (the number of points in each column) is equal to 2, we use $3=2\cdot 2-1$ segments, whereas, when it is equal to 3, we use $5=2\cdot 3-1$; in general, we use 2 additional segments for each increment of n_2.

2.2 Path 2 (when $n_1=n_2>2$, with $n:=n_1=n_2$ and n is an odd number)

When $n_1=n_2>2$, with $n:=n_1=n_2$ and n being an odd number, we utilize *path 2*, shown in *Figure 2*.

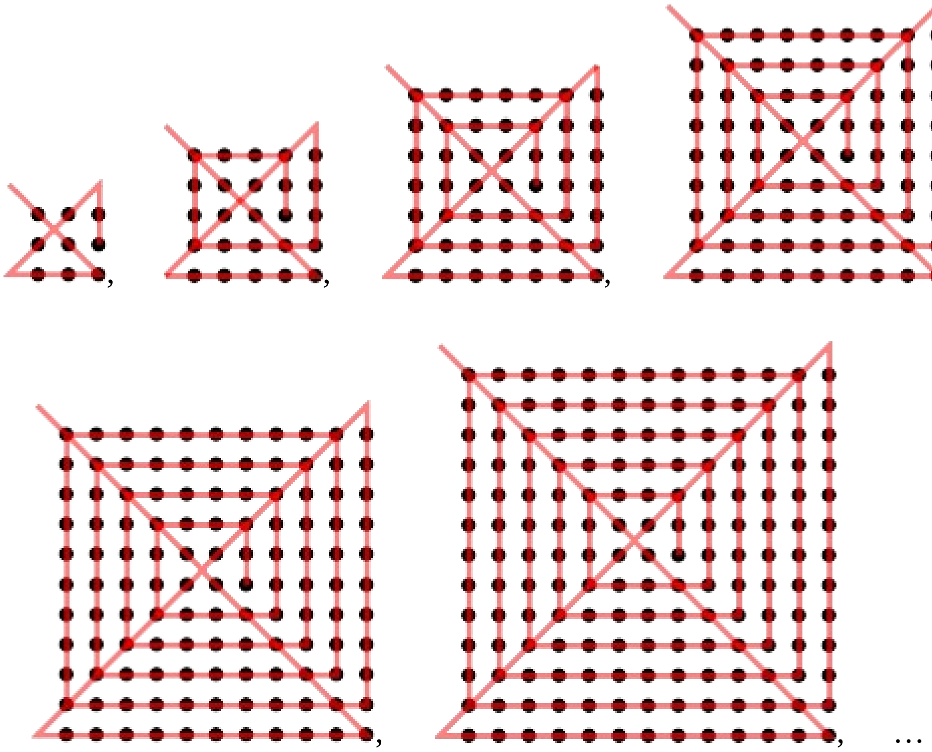


Figure 2. Some of the initial examples of *path 2*.

The starting point is again the upper left corner of the grid.

The total number of segments used is still the minimum possible: $2 \cdot n-2$ (see [1]), since, as shown in *Figure 2*, when $n=3$, we use $4=2 \cdot 3-2$ segments, and, when $n=5$, we use $8=2 \cdot 5-2$, adding 4 additional segments for each increment of n.

2.3 Path 3 (when n_1=n_2>2, with $n:=n_1=n_2$ and n is an even number)

For the final case, when $n_1=n_2>2$, with $n:=n_1=n_2$ and n is an even number, we utilize *path 3*, as illustrated in *Figure 3*.

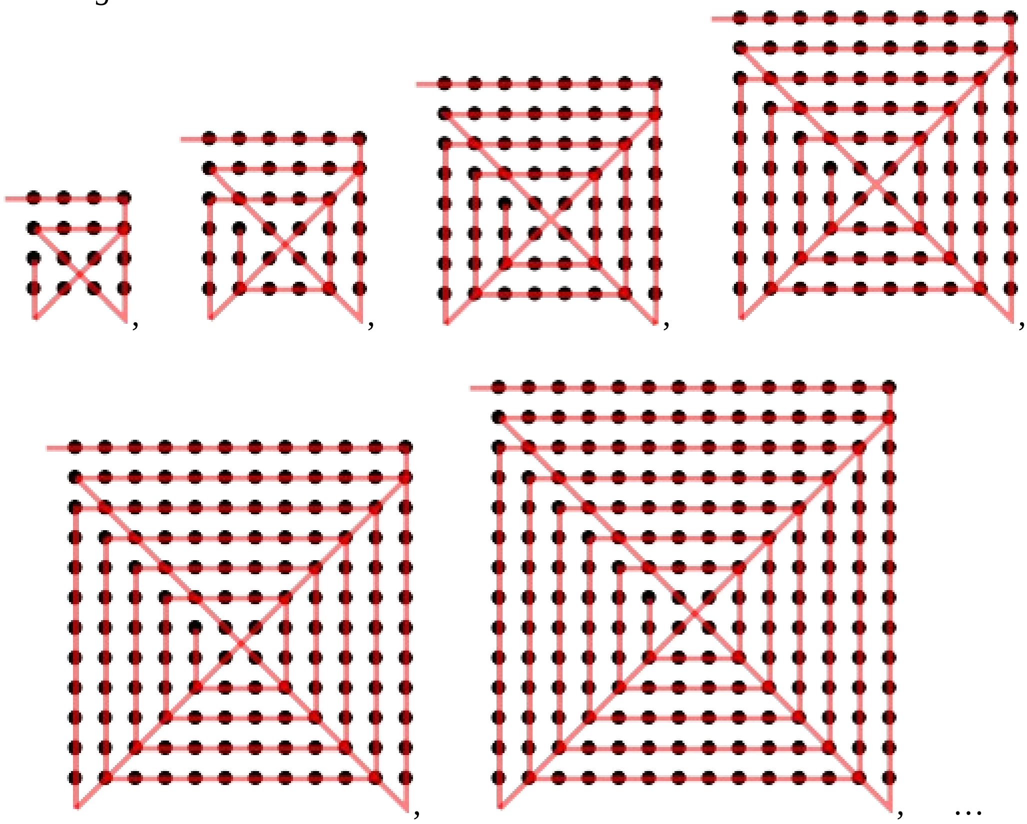


Figure 3. Some of the initial examples of *path 3*.

Path 3 starts from the upper left point of the grid. In this case, as well, the total number of segments used corresponds to the minimum possible: $2\cdot n-2$ (see [1]).

When $n=4$, we indeed use $6=2\cdot 4-2$ segments, when $n=6$, we use $10=2\cdot 6-2$, and we add 4 segments for each increase in n.

2.4 The formula for our solutions

As we have just shown, the total number of segments used $t(n_1;\ n_2)$ corresponds to the exact solution of the problem $s(n_1;\ n_2)$.

The values of our solutions are, therefore, $\forall\ n_1;\ n_2 \in \mathbb{N}-\{0;\ 1\}$:

$$t(n_1;\ n_2)=$$

$$=s(n_1;\ n_2)=\begin{cases} 2\cdot n_2-1 \\ \textbf{\textit{if}}\ n_1=n_2=2 \vee n_1>n_2 \\ \\ 2\cdot n-2 \\ \textbf{\textit{if}}\ n_1=n_2>2 \wedge n \coloneqq n_1=n_2 \end{cases} \tag{1}$$

where:

$n_1 \geq n_2$

3 The case $n_1 \times n_2 \times n_3$

In this section, we continue the description of our algorithm, extending it to the three-dimensional case and, subsequently, dedicating most of the section to finding the best optimization to save some additional segments.

3.1 Description of the algorithm

We extend the *paths* demonstrated in the two-dimensional case as illustrated in *Figure 4*, reiterating the same two-dimensional *paths* used for the case $n_1 \times n_2$ on each $n_1 \times n_2$ "plane" of the three-dimensional grid, beginning from one of the two outer "planes" and connecting each "plane" sequentially with a segment for each "plane" subsequent to the first (the green segments in *Figure 4*).

At this point, to reduce the number of used segments, we can apply a simple optimization: case by case (as in the example in *Figure 4*), we can decide not to complete the two-dimensional *paths* on the $n_1 \times n_2$ "planes", leaving a certain number of "free points" (interior) on each individual two-dimensional "plane", equal in number and aligned (like the *3* on each "plane" in the left image of *Figure 4*), so that we can subsequently visit them through a very simple "final *path*" (the blue and yellow segments of the right image in *Figure 4*), ultimately using a number of segments that could be fewer than what we would have used if visiting the "free points" with the two-dimensional *paths* immediately (as seen in the case of *Figure 5*, where the total number of segments used $t(n_1; n_2; n_3)$ is greater than that in *Figure 4*).

Regarding the "final *path*", it will alternate blue segments, through which we move to the remaining points on the two outer "planes", with yellow segments, through which we visit n_3-*1* aligned "free points" at a time, "bouncing" from one outer "plane" to another.

This *path* will thus consist of a fixed number of segments, equal to twice the number of "free points" on each individual $n_1 \times n_2$ "plane" (as shown in *Figure 4*, where we have *3* "free points" fo each $n_1 \times n_2$ "plane" and we use precisely $6=2 \cdot 3$ segments to visit them: *3* blue and *3* yellow).

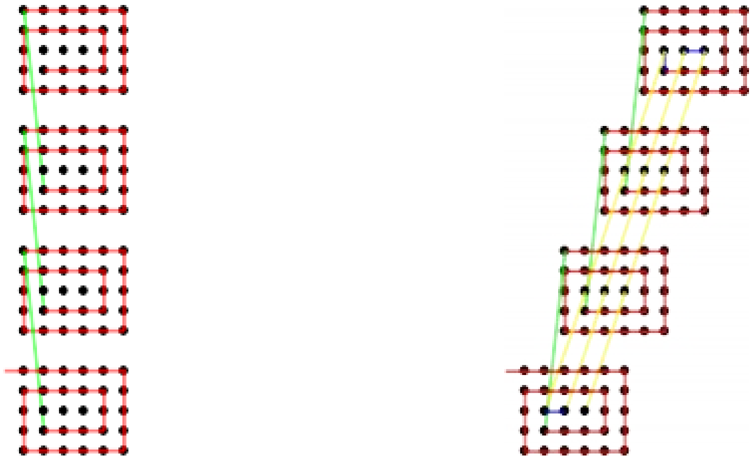


Figure 4. In the left image, we have the first part of the three-dimensional *path* for the case $n_1=6$, $n_2=5$ and $n_3=4$. They are visible the two-dimensional *paths* (*path 1*), traced on each $n_1 \times n_2$ "planes", starting from the outermost one at the bottom, and sequentially connected by green segments. Additionally, the "free points" that have not yet been visited can be noted. In the right image, we see the three-dimensional grid finally resolved through the "final *path*", alternating between blue segments, which allow us to traverse the remaining points on the two outer "planes", and yellow segments, with which we visit n_3-1 aligned "free points" at a time, "bouncing" from one outer "plane" to another. The total number of segments used to solve the grid is $t(6; 5; 4)=37$.

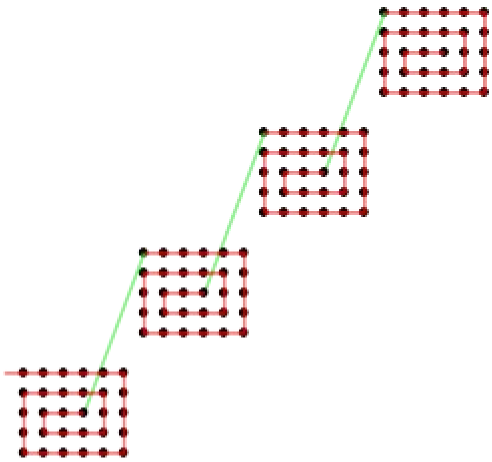


Figure 5. Again considering the case of $n_1=6$, $n_2=5$ and $n_3=4$: had we not utilized the optimization illustrated in *Figure 4*, the total number of segments used would have been *39*, indicative of two additional segments.

Having completed the description of our algorithm, one of the challenges in the upcoming subsections will be to determine the optimal number of segments for the two-dimensional *path* to use on the $n_1 \times n_2$ "planes".

This is essential in order to minimize the total number of segments required to solve the three-dimensional grid using our new algorithm.

3.2 The general formula for our solutions

As we have just demonstrated, the total number of segments used in the three-dimensional case is generally equal to one of the possible values for the number of segments in the two-dimensional *path* to be used on each individual $n_1 \times n_2$ "plane" (which we define as l, where the maximum value is, of course, the solution to the two-dimensional case, shown in *(1)*), multiplied by the number of $n_1 \times n_2$ "planes" (which is n_3), plus the number of segments (green) connecting the $n_1 \times n_2$ "planes" (thus, $n_3 - 1$), added to the number of segments used in the "final *path*", which, as mentioned earlier, is equal to twice the number of "free points" on each $n_1 \times n_2$ "plane" (which we define as p).

The general formula for our solutions is, then, $\forall\, n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\} \wedge l \in \mathbb{N}\text{-}\{0\}$:

$$\begin{aligned} t(n_1; n_2; n_3) &= \\ &= l \cdot n_3 + n_3 - 1 + 2 \cdot p = \\ &= n_3 \cdot l + n_3 + 2 \cdot p - 1 \end{aligned} \tag{2}$$

where:

$$l \leq$$

$$\leq t(n_1; n_2) = \begin{cases} 2 \cdot n_2 - 1 \\ \textbf{\textit{if}}\ n_1 = n_2 = 2 \vee n_1 > n_2 \\ \\ 2 \cdot n - 2 \\ \textbf{\textit{if}}\ n_1 = n_2 > 2 \wedge n \coloneqq n_1 = n_2 \end{cases}$$

and:

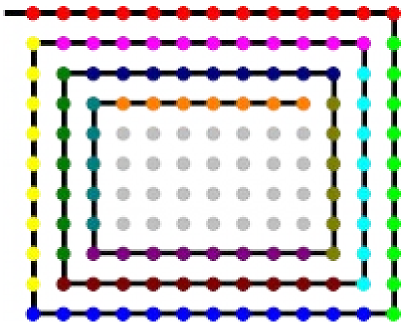
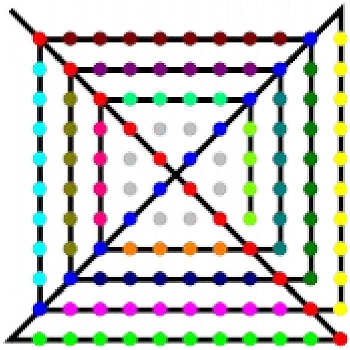
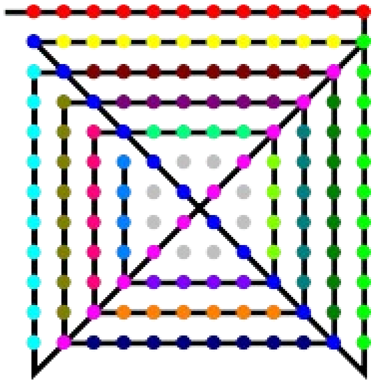
$$n_1 \geq n_2 \geq n_3$$

3.3 Calculation of the possible values of p

Since p is the number of remaining points on each $n_1 \times n_2$ "plane" after tracing l segments of a two-dimensional *path*, to find it, we calculate the number of points traversed by the first l segments of each two-dimensional *path*, simply summing the points through which each segment passes sequentially, excluding the points crossed by previous segments, this l times and for every possible value of l.

The result will be the total number of points on the $n_1 \times n_2$ "planes" minus p (thus, $n_1 \cdot n_2 - p$ or $n^2 - p$, if $n \coloneqq n_1 = n_2$), and will consequently allow us to find all possible values of p.

In *Table 1*, the values of the number of points through which each segment passes, excluding the points traversed by previous segments, are shown.

Path 1, when $n_1=n_2=2 \lor n_1>n_2$, $\forall n_1; n_2 \in \mathbb{N}-\{0; 1\}$	**Path 2, when $n_1=n_2=2\cdot m-1$, $\forall m \in \mathbb{N}-\{0; 1\}$, with $n:=n_1=n_2$**	**Path 3, when $n_1=n_2=2\cdot m$, $\forall m \in \mathbb{N}-\{0; 1\}$, with $n:=n_1=n_2$**
		
Start from the top left corner. The number of points in each row is n_1, and the number of points in each column is n_2.	Begin at the top left corner and descend diagonally to the bottom right. The number of points in each row or column is n.	Start from the top left corner. The number of points in each row or column is n.

Segment number	**Number of points on the segment, excluding the points from the previous segments**	**Segment number**	**Number of points on the segment, excluding the points from the previous segments**	**Segment number**	**Number of points on the segment, excluding the points from the previous segments**
1	n_1	1	n	1	n
2	n_2-1	2	n-1	2	n-1
3	n_1-1	3	n-1	3	n-1
4	n_2-2	4	n-1	4	n-2
5	n_1-2	5	n-3	5	n-2
6	n_2-3	6	n-3	6	n-2
7	n_1-3	7	n-3	7	n-4
8	n_2-4	8	n-3	8	n-4
9	n_1-4	9	n-5	9	n-4
10	n_2-5	10	n-5	10	n-4
11	n_1-5	11	n-5	11	n-6
12	n_2-6	12	n-5	12	n-6
13	n_1-6	13	n-7	13	n-6
14	n_2-7	14	n-7	14	n-6
15	n_1-7	15	n-7	15	n-8
16	n_2-8	16	n-7	16	n-8
17	n_1-8	17	n-9	17	n-8
18	n_2-9	18	n-9	18	n-8
…	…	…	…	…	…
$2\cdot n_2-1$	…	$2\cdot n-2$	…	$2\cdot n-2$	…

Table 1. Values of the number of points through which each segment passes, excluding the points through which previous segments pass, for each of the three *paths*.

After doing the calculations, for *path 1*, the values of $n_1 \cdot n_2 - p$ are, thus, $\forall\ n_1; n_2 \in \mathbb{N}-\{0; 1\} \wedge l \in \mathbb{N}-\{0\}$:

$$n_1 \cdot n_2 - p = \begin{cases} \frac{l+1}{2} \cdot n_1 + \frac{l-1}{2} \cdot n_2 - \frac{l^2-1}{4} \\ \textbf{\textit{if}}\ l = 2 \cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ \frac{l}{2} \cdot n_1 + \frac{l}{2} \cdot n_2 - \frac{l^2}{4} \\ \textbf{\textit{if}}\ l = 2 \cdot m,\ \forall m \in \mathbb{N} - \{0\} \end{cases}$$

therefore:

$$p = \begin{cases} n_1 \cdot n_2 - \frac{1}{2} \cdot n_1 \cdot l - \frac{1}{2} \cdot n_2 \cdot l + \frac{1}{4} \cdot l^2 - \frac{1}{2} \cdot n_1 + \frac{1}{2} \cdot n_2 - \frac{1}{4} \\ \textbf{\textit{if}}\ l = 2 \cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ n_1 \cdot n_2 - \frac{1}{2} \cdot n_1 \cdot l - \frac{1}{2} \cdot n_2 \cdot l + \frac{1}{4} \cdot l^2 \\ \textbf{\textit{if}}\ l = 2 \cdot m,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \qquad (3)$$

where:

$$l \leq 2 \cdot n_2 - 1$$

and:

$$n_1 = n_2 = 2 \vee n_1 > n_2$$

For *path 2*, we have, instead, $\forall\, n_1; n_2 \in \mathbb{N}\text{-}\{0; 1\} \wedge l \in \mathbb{N}\text{-}\{0\}$:

$$n^2-p=\begin{cases} l\cdot n-\dfrac{l^2-1}{4} \\ \textbf{\textit{if}}\ l=2\cdot m-1,\ \forall m\in\mathbb{N}-\{0\} \\ \\ l\cdot n-\dfrac{l^2}{4} \\ \textbf{\textit{if}}\ l=4\cdot m+2,\ \forall m\in\mathbb{N} \\ \\ l\cdot n-\dfrac{l^2-4}{4} \\ \textbf{\textit{if}}\ l=4\cdot m,\ \forall m\in\mathbb{N}-\{0\} \end{cases}$$

therefore:

$$p=\begin{cases} -n\cdot l+\dfrac{1}{4}\cdot l^2+n^2-\dfrac{1}{4} \\ \textbf{\textit{if}}\ l=2\cdot m-1,\ \forall m\in\mathbb{N}-\{0\} \\ \\ -n\cdot l+\dfrac{1}{4}\cdot l^2+n^2 \\ \textbf{\textit{if}}\ l=4\cdot m+2,\ \forall m\in\mathbb{N} \\ \\ -n\cdot l+\dfrac{1}{4}\cdot l^2+n^2-1 \\ \textbf{\textit{if}}\ l=4\cdot m,\ \forall m\in\mathbb{N}-\{0\} \end{cases} \tag{4}$$

where:

$$l\leq 2\cdot n-2$$

and, $\forall\, m \in \mathbb{N}\text{-}\{0; 1\}$:

$$n_1=n_2=2\cdot m-1 \wedge n\coloneqq n_1=n_2$$

For *path 3*, we finally have that, $\forall\, n_1; n_2 \in \mathbb{N}\text{-}\{0; 1\} \wedge l \in \mathbb{N}\text{-}\{0\}$:

$$n^2 - p = \begin{cases} l \cdot n - \dfrac{l^2 - 1}{4} \\ \textbf{\textit{if}}\ l = 2 \cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ l \cdot n - \dfrac{l^2}{4} \\ \textbf{\textit{if}}\ l = 2 \vee l = 4 \cdot m,\ \forall m \in \mathbb{N} - \{0\} \\ \\ l \cdot n - \dfrac{l^2 - 4}{4} \\ \textbf{\textit{if}}\ l = 4 \cdot m + 2,\ \forall m \in \mathbb{N} - \{0\} \end{cases}$$

therefore:

$$p = \begin{cases} -n \cdot l + \dfrac{1}{4} \cdot l^2 + n^2 - \dfrac{1}{4} \\ \textbf{\textit{if}}\ l = 2 \cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ -n \cdot l + \dfrac{1}{4} \cdot l^2 + n^2 \\ \textbf{\textit{if}}\ l = 2 \vee l = 4 \cdot m,\ \forall m \in \mathbb{N} - \{0\} \\ \\ -n \cdot l + \dfrac{1}{4} \cdot l^2 + n^2 - 1 \\ \textbf{\textit{if}}\ l = 4 \cdot m + 2,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \tag{5}$$

where:

$l \leq 2 \cdot n - 2$

and, $\forall\, m \in \mathbb{N}\text{-}\{0; 1\}$:

$n_1 = n_2 = 2 \cdot m \wedge n := n_1 = n_2$

3.4 Calculation of the values of l

Having found the values of *p* for each *path*, we can now substitute them into *(2)*, the general formula for our solutions.

For *path 1*, our solutions turn out to be, thus, $\forall\, n_1; n_2; n_3 \in \mathbb{N}-\{0; 1\} \wedge l \in \mathbb{N}-\{0\}$:

$$t(n_1; n_2; n_3) = \begin{cases} 2\cdot n_1\cdot n_2 - n_1\cdot l - n_2\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 - n_1 + n_2 + n_3 - \frac{3}{2} \\ \textbf{\textit{if}}\ l = 2\cdot m - 1,\ \forall\, m \in \mathbb{N} - \{0\} \\ \\ 2\cdot n_1\cdot n_2 - n_1\cdot l - n_2\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - 1 \\ \textbf{\textit{if}}\ l = 2\cdot m,\ \forall\, m \in \mathbb{N} - \{0\} \end{cases} \tag{6}$$

where:
$l \le 2\cdot n_2 - 1$

and:
$n_1 = n_2 = 2 \vee n_1 > n_2$

and:
$n_2 \ge n_3$

For *path 2*, we have, instead, $\forall\, n_1; n_2; n_3 \in \mathbb{N}-\{0; 1\} \wedge l \in \mathbb{N}-\{0\}$:

$$t(n_1; n_2; n_3) = \begin{cases} 2\cdot n^2 - 2\cdot n\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - \frac{3}{2} \\ \textbf{\textit{if}}\ l = 2\cdot m - 1,\ \forall\, m \in \mathbb{N} - \{0\} \\ \\ 2\cdot n^2 - 2\cdot n\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - 1 \\ \textbf{\textit{if}}\ l = 4\cdot m + 2,\ \forall\, m \in \mathbb{N} \\ \\ 2\cdot n^2 - 2\cdot n\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - 3 \\ \textbf{\textit{if}}\ l = 4\cdot m,\ \forall\, m \in \mathbb{N} - \{0\} \end{cases} \tag{7}$$

where:
$l \le 2\cdot n - 2$

and, $\forall\, m \in \mathbb{N}-\{0; 1\}$:
$n_1 = n_2 = 2\cdot m - 1 \wedge n \coloneqq n_1 = n_2$

and:
$n_2 \ge n_3$

For *path 3*, we finally have, ∀ $n_1; n_2; n_3 \in \mathbb{N}$-{0; 1} ∧ $l \in \mathbb{N}$-{0}:

$$t(n_1; n_2; n_3) = \begin{cases} 2\cdot n^2 - 2\cdot n\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - \frac{3}{2} \\ \textbf{\textit{if}}\ l = 2\cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ 2\cdot n^2 - 2\cdot n\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - 1 \\ \textbf{\textit{if}}\ l = 2 \vee l = 4\cdot m,\ \forall m \in \mathbb{N} - \{0\} \\ \\ 2\cdot n^2 - 2\cdot n\cdot l + n_3\cdot l + \frac{1}{2}\cdot l^2 + n_3 - 3 \\ \textbf{\textit{if}}\ l = 4\cdot m + 2,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \quad (8)$$

where:

$l \le 2\cdot n - 2$

and, ∀ $m \in \mathbb{N}$-{0; 1}:

$n_1 = n_2 = 2\cdot m \wedge n := n_1 = n_2$

and:

$n_2 \ge n_3$

Now that we have found the complete general formulas for our $t(n_1; n_2; n_3)$, we just need to determine the values of *l* that will minimize $t(n_1; n_2; n_3)$, which will yield our solution.

After doing the calculations, the values of *l* for *path 1* are, thus, ∀ $n_1; n_2; n_3 \in \mathbb{N}$-{0; 1}:

$$l = \begin{cases} = 2\cdot n_2 - 1 \\ \textbf{\textit{if}}\ -n_1 + n_2 + n_3 \le 0 \\ \\ = n_1 + n_2 - n_3 \\ \textbf{\textit{if}}\ -n_1 + n_2 + n_3 = 2\cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ = \begin{cases} n_1 + n_2 - n_3 - 1 \vee n_1 + n_2 - n_3 \vee n_1 + n_2 - n_3 + 1 \\ \textbf{\textit{if}}\ n_1 = n_2 = 2 \\ \\ n_1 + n_2 - n_3 - 1 \vee n_1 + n_2 - n_3 + 1 \\ \textbf{\textit{if}}\ n_1 > n_2 \end{cases} \\ \textbf{\textit{if}}\ -n_1 + n_2 + n_3 = 2\cdot m,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \quad (9)$$

where:

$n_1 = n_2 = 2 \vee n_1 > n_2$

and:

$n_2 \ge n_3$

The values of l for *path 2* are, instead, $\forall\, n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\}$:

$$l=\begin{cases} 2\cdot n-n_3 \\ \boldsymbol{if}\ n_3=4\cdot m+2,\ \forall\, m\in\mathbb{N} \\ \\ 2\cdot n-n_3+1 \\ \boldsymbol{if}\ n_3=4\cdot m-1,\ \forall\, m\in\mathbb{N}-\{0\} \\ \\ 2\cdot n-n_3-2 \vee 2\cdot n-n_3-1 \vee 2\cdot n-n_3 \vee 2\cdot n-n_3+1 \vee 2\cdot n-n_3+2 \\ \boldsymbol{if}\ n_3=4\cdot m,\ \forall\, m\in\mathbb{N}-\{0\} \\ \\ 2\cdot n-n_3-1 \\ \boldsymbol{if}\ n_3=4\cdot m+1,\ \forall\, m\in\mathbb{N}-\{0\} \end{cases} \tag{10}$$

where, $\forall\, m \in \mathbb{N}\text{-}\{0; 1\}$:

$n_1=n_2=2\cdot m-1 \wedge n:=n_1=n_2$

and:

$n_2\geq n_3$

For *path 3*, the values of l finally are, $\forall\, n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\}$:

$$l=\begin{cases} =2\cdot n-n_3 \\ \boldsymbol{if}\ n_3=4\cdot m+2,\ \forall\, m\in\mathbb{N} \\ \\ =2\cdot n-n_3+1 \\ \boldsymbol{if}\ n_3=4\cdot m-1,\ \forall\, m\in\mathbb{N}-\{0\} \\ \\ =\begin{cases} 2\cdot n-n_3-1 \vee 2\cdot n-n_3 \vee 2\cdot n-n_3+1 \vee 2\cdot n-n_3+2 \\ \boldsymbol{if}\ n=4 \\ \\ 2\cdot n-n_3-2 \vee 2\cdot n-n_3-1 \vee 2\cdot n-n_3 \vee 2\cdot n-n_3+1 \vee 2\cdot n-n_3+2 \\ \boldsymbol{if}\ n>4 \end{cases} \\ \boldsymbol{if}\ n_3=4\cdot m,\ \forall\, m\in\mathbb{N}-\{0\} \\ \\ =2\cdot n-n_3-1 \\ \boldsymbol{if}\ n_3=4\cdot m+1,\ \forall\, m\in\mathbb{N}-\{0\} \end{cases} \tag{11}$$

where, $\forall\, m \in \mathbb{N}\text{-}\{0; 1\}$:

$n_1=n_2=2\cdot m \wedge n:=n_1=n_2$

and:

$n_2\geq n_3$

3.5 The complete formulas for our solutions

Having found the values of *l*, we can now substitute them into the general formulas of our solutions: equation *(6)* for *path 1*, equation *(7)* for *path 2* and equation *(8)* for *path 3*.

For *path 1*, the formula for our solutions is, therefore, ∀ $n_1; n_2; n_3$ ∈ ℕ-*{0; 1}*:

$$t(n_1; n_2; n_3) = \begin{cases} 2\cdot n_2\cdot n_3 - 1 \\ \textbf{\textit{if}}\ -n_1+n_2+n_3 \le 0 \\ \\ -\frac{1}{2}\cdot n_1^2 - n_1 + n_1\cdot n_2 + n_1\cdot n_3 - \frac{1}{2}\cdot n_2^2 + n_2 + n_2\cdot n_3 - \frac{1}{2}\cdot n_3^2 + n_3 - \frac{3}{2} \\ \textbf{\textit{if}}\ -n_1+n_2+n_3 = 2\cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ -\frac{1}{2}\cdot n_1^2 - n_1 + n_1\cdot n_2 + n_1\cdot n_3 - \frac{1}{2}\cdot n_2^2 + n_2 + n_2\cdot n_3 - \frac{1}{2}\cdot n_3^2 + n_3 - 1 \\ \textbf{\textit{if}}\ -n_1+n_2+n_3 = 2\cdot m,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \tag{12}$$

where:

$n_1 = n_2 = 2 \vee n_1 > n_2$

and:

$n_2 \ge n_3$

For *path 2*, it is, instead, ∀ $n_1; n_2; n_3$ ∈ ℕ-*{0; 1}*:

$$t(n_1; n_2; n_3) = \begin{cases} 2\cdot n\cdot n_3 - \frac{1}{2}\cdot n_3^2 + n_3 - 3 \\ \textbf{\textit{if}}\ n_3 = 4\cdot m + 2,\ \forall m \in \mathbb{N} \\ \\ 2\cdot n\cdot n_3 - \frac{1}{2}\cdot n_3^2 + n_3 - \frac{5}{2} \\ \textbf{\textit{if}}\ n_3 = 4\cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ \\ 2\cdot n\cdot n_3 - \frac{1}{2}\cdot n_3^2 + n_3 - 1 \\ \textbf{\textit{if}}\ n_3 = 4\cdot m,\ \forall m \in \mathbb{N} - \{0\} \\ \\ 2\cdot n\cdot n_3 - \frac{1}{2}\cdot n_3^2 + n_3 - \frac{5}{2} \\ \textbf{\textit{if}}\ n_3 = 4\cdot m + 1,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \tag{13}$$

where, ∀ *m* ∈ ℕ-*{0; 1}*:

$n_1 = n_2 = 2\cdot m - 1 \wedge n \coloneqq n_1 = n_2$

and:

$n_2 \ge n_3$

For *path 3*, the formula for our solutions is, finally, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-*{0; 1}*:

$$t(n_1; n_2; n_3) = \begin{cases} 2 \cdot n \cdot n_3 - \frac{1}{2} \cdot {n_3}^2 + n_3 - 3 \\ \textbf{\textit{if}}\ n_3 = 4 \cdot m + 2,\ \forall m \in \mathbb{N} \\ 2 \cdot n \cdot n_3 - \frac{1}{2} \cdot {n_3}^2 + n_3 - \frac{5}{2} \\ \textbf{\textit{if}}\ n_3 = 4 \cdot m - 1,\ \forall m \in \mathbb{N} - \{0\} \\ 2 \cdot n \cdot n_3 - \frac{1}{2} \cdot {n_3}^2 + n_3 - 1 \\ \textbf{\textit{if}}\ n_3 = 4 \cdot m,\ \forall m \in \mathbb{N} - \{0\} \\ 2 \cdot n \cdot n_3 - \frac{1}{2} \cdot {n_3}^2 + n_3 - \frac{5}{2} \\ \textbf{\textit{if}}\ n_3 = 4 \cdot m + 1,\ \forall m \in \mathbb{N} - \{0\} \end{cases} \tag{14}$$

where, $\forall\ m \in \mathbb{N}$-*{0; 1}*:

$$n_1 = n_2 = 2 \cdot m \wedge n \coloneqq n_1 = n_2$$

and:

$$n_2 \geq n_3$$

4 The case $n_1 \times n_2 \times n_3 \times n_4 \times \ldots \times n_k$

In this section, we conclude the description of our algorithm by extending it to the four-dimensional case and beyond.

4.1 Description of the algorithm

To extend the three-dimensional algorithm to the case of $n_1 \times n_2 \times n_3 \times n_4 \times \ldots \times n_k$, we reiterate our three-dimensional *path* on each $n_1{\times}n_2{\times}n_3$ "space", connecting them all, sequentially, with one segment for each subsequent three-dimensional "space", until the last of the n_4 "spaces".

If necessary, we then repeat the traced four-dimensional *path* on each $n_1{\times}n_2{\times}n_3{\times}n_4$ "space", connecting all $n_5\text{-}1$ "spaces" after the first with one segment for each subsequent four-dimensional "space", until all remaining points are exhausted.

4.2 The formula for our solutions

As we have just described, the total number of segments used $t(n_1;\, n_2;\, n_3;\, n_4;\, \ldots;\, n_k)$ is equal to the number of segments in the three-dimensional *path* for each $n_1{\times}n_2{\times}n_3$ "space", that is $t(n_1;\, n_2;\, n_3)$ (which we define c), multiplied by the number of $n_1{\times}n_2{\times}n_3$ "spaces" (equal to n_4), plus the number of segments connecting the $n_1{\times}n_2{\times}n_3$ "spaces" (thus, $n_4\text{-}1$).

This product is further multiplied, if necessary, by the number of $n_1{\times}n_2{\times}n_3{\times}n_4$ "spaces" (equal to n_5), plus the segments connecting the $n_1{\times}n_2{\times}n_3{\times}n_4$ "spaces" (which is $n_5\text{-}1$), continuing in this manner until we exhaust all remaining points.

The formula for our solutions is, therefore, $\forall\, n_1;\, n_2;\, n_3;\, n_4;\, n_5;\, n_6;\, \ldots;\, n_k \in \mathbb{N}\text{-}\{0;\, 1\} \wedge k \in \mathbb{N}\text{-}\{0;\, 1;\, 2;\, 3\}$:

$$
\begin{aligned}
& t(n_1;\, n_2;\, n_3;\, n_4;\, \ldots;\, n_k) = \\
& = (\ldots(((c \cdot n_4 + n_4 - 1) \cdot n_5 + n_5 - 1) \cdot n_6 + n_6 - 1) \cdot \ldots) \cdot n_k + n_k - 1 = \\
& = (c+1) \cdot \prod_{i=4}^{k} (n_i) - 1
\end{aligned}
\tag{15}
$$

where:

$$n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq n_6 \geq \ldots \geq n_k$$

and:

$$c \coloneqq t(n_1;\, n_2;\, n_3)$$

5 Conclusion

Having completed the description of our algorithm, in this final section, we will compare it with algorithms proposed by others, demonstrating that ours offers a more efficient upper bound for the three-dimensional case and beyond.

5.1 The solutions obtained through other algorithms

Below are the formulas for the solutions $h(n_1; n_2; n_3; \ldots; n_k)$, obtained through the algorithms proposed by Ripà (see [3] and [4]) for the three-dimensional case and beyond.

For the three-dimensional case, the general formula for Ripà's solutions (see [3]) is, thus, $\forall n_1; n_2; n_3 \in \mathbb{N}-\{0; 1\}$:

$$h(n_1; n_2; n_3) = \begin{cases} = 2\cdot n_2\cdot n_3 - 1 \\ \textbf{if } n_3 < 2\cdot(n_1-n_2)+3 \\ \\ = 2\cdot n_2\cdot n_3 - 2 \\ \textbf{if } n_3 = 2\cdot(n_1-n_2)+3 \\ \\ = \begin{cases} \frac{4}{3}\cdot i_{max}^3 + (2\cdot(n_1-n_2)+7)\cdot i_{max}^2 + \left(6\cdot(n_1-n_2) - 2\cdot n_3 + \frac{35}{3}\right)\cdot i_{max} + 4\cdot(n_1-n_2) + 2\cdot n_3\cdot(n_2-1)+5 \\ \textbf{if } n_3 \le 2\cdot(i_{max}^2 + (n_1-n_2+4)\cdot i_{max} + 2\cdot(n_1-n_2)+4) \\ \\ \frac{4}{3}\cdot i_{max}^3 + (2\cdot(n_1-n_2)+9)\cdot i_{max}^2 + \left(8\cdot(n_1-n_2) - 2\cdot n_3 + \frac{59}{3}\right)\cdot i_{max} + 8\cdot(n_1-n_2) + n_3\cdot(2\cdot n_2-3)+13 \\ \textbf{if } n_3 > 2\cdot(i_{max}^2 + (n_1-n_2+4)\cdot i_{max} + 2\cdot(n_1-n_2)+4) \end{cases} \\ \textbf{if } n_3 > 2\cdot(n_1-n_2)+3 \end{cases} \quad (16)$$

where:

$$i_{max} := \left\lfloor \frac{1}{2}\cdot\left(\sqrt{n_1^2 + n_2^2 - 2\cdot n_1\cdot n_2 + 2\cdot(n_1-n_2+n_3)+1} + n_2 - n_1 - 3\right)\right\rfloor$$

and:

$$n_1 \ge n_2 \ge n_3$$

For certain special cases, the formulas for $h(n_1; n_2; n_3)$ are as follows.

For the first case (see [3]), we have that the solutions of Ripà are given by, ∀ $n_1; n_2; n_3$ ∈ ℕ-*{0; 1; 2; 3}*:

$$h(n_1; n_2; n_3) = \begin{cases} \frac{2}{3}\cdot i_{max}{}^3 + 5\cdot i_{max}{}^2 - 2\cdot\left(n - \frac{14}{3}\right)\cdot i_{max} + 2\cdot n^2 - 2\cdot n + 3 \\ \textbf{\textit{if}}\ n - i_{max}{}^2 - 5\cdot i_{max} \leq 5 \\ \frac{2}{3}\cdot i_{max}{}^3 + 6\cdot i_{max}{}^2 - \left(2\cdot n - \frac{43}{3}\right)\cdot i_{max} + 2\cdot n^2 - 3\cdot n + 8 \\ \textbf{\textit{if}}\ n - i_{max}{}^2 - 5\cdot i_{max} > 5 \end{cases} \quad (17)$$

where:

$$i_{max} := \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n + 9} - 5\right)\right\rfloor$$

and:

$$n_1 = n_2 = n_3 \wedge n := n_1 = n_2 = n_3$$

For the second case (see [3]), we have, instead, ∀ $n_1; n_2; n_3$ ∈ ℕ-*{0; 1; 2; 3}*:

$$h(n_1; n_2; n_3) = \begin{cases} \frac{2}{3}\cdot i_{max}{}^3 + 5\cdot i_{max}{}^2 + \left(\frac{28}{3} - 2\cdot n_3\right)\cdot i_{max} + 2\cdot n_2\cdot n_3 - n_3 + 3 \\ \textbf{\textit{if}}\ n_3 - i_{max}{}^2 - 5\cdot i_{max} \leq 5 \\ \frac{2}{3}\cdot i_{max}{}^3 + 6\cdot i_{max}{}^2 + \left(\frac{43}{3} - 2\cdot n_3\right)\cdot i_{max} + 2\cdot n_2\cdot n_3 - 2\cdot n_3 + 8 \\ \textbf{\textit{if}}\ n_3 - i_{max}{}^2 - 5\cdot i_{max} > 5 \end{cases} \quad (18)$$

where:

$$i_{max} := \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n_3 + 9} - 5\right)\right\rfloor$$

and:

$$n_1 - 1 = n_2 \geq n_3$$

In [4], we also have that *(17)* was improved by the same Ripà for almost all values of *n*, using a different algorithm, such as, ∀ $n_1; n_2; n_3$ ∈ ℕ-*{0; 1; 2; 3; 4}*:

$$h(n_1; n_2; n_3) = \left\lfloor \frac{3}{2}\cdot n^2 \right\rfloor + n - 1 \quad (19)$$

where:

$$n_1 = n_2 = n_3 \wedge n := n_1 = n_2 = n_3$$

Regarding the four-dimensional case and those that follow, we instead have that the formula for Ripà's solutions (see [3] and [4]) is given by, ∀ $n_1; n_2; n_3; n_4; \ldots; n_k$ ∈ ℕ-*{0; 1}* ∧ *k* ∈ ℕ-*{0; 1; 2; 3}*:

$$h(n_1; n_2; n_3; n_4; \ldots; n_k) = (q+1)\cdot\prod_{i=4}^{k}(n_i) - 1 \quad (20)$$

where:

$$n_1 \geq n_2 \geq n_3 \geq n_4 \geq \ldots \geq n_k$$

and:

$$q := h(n_1; n_2; n_3)$$

By examining *(20)*, one can observe that it aligns with our own formula for the four-dimensional case and those that follow (equation *(15)*), varying with $h(n_1; n_2; n_3)$, Ripà's solution for the three-dimensional case, rather than our $t(n_1; n_2; n_3)$.

Therefore, to compare our algorithm with those proposed by Ripà, it will not be necessary to consider all the cases present in the problem, but rather solely the three-dimensional case, which will be the focus of the next subsection.

5.2 The discrepancy between the solutions obtained through the various algorithms

In this subsection, we will analyze the discrepancy between the solutions obtained through our algorithm and those proposed by Ripà for the three-dimensional case, demonstrating that ours provides more efficient upper bounds.

By combining *(12)*, *(13)* and *(14)* with the initial cases of *(16)*, we find that the discrepancy between our algorithm and Ripà's for these particular cases is, ∀ $n_1; n_2; n_3 \in \mathbb{N}-\{0; 1\}$:

$$h(n_1; n_2; n_3)-t(n_1; n_2; n_3)=\begin{cases}=\begin{cases}0 \\ \textbf{\textit{if}}\ n_3 \le n_1-n_2 \\ =\begin{cases}\frac{1}{2}\cdot n_1^2-n_1\cdot n_2-n_1\cdot n_3+n_1+\frac{1}{2}\cdot n_2^2+n_2\cdot n_3-n_2+\frac{1}{2}\cdot n_3^2-n_3+\frac{1}{2} \\ \textbf{\textit{if}}\ -n_1+n_2+n_3=2\cdot m-1,\ \forall m\in\mathbb{N}-\{0\} \\ \frac{1}{2}\cdot n_1^2-n_1\cdot n_2-n_1\cdot n_3+n_1+\frac{1}{2}\cdot n_2^2+n_2\cdot n_3-n_2+\frac{1}{2}\cdot n_3^2-n_3 \\ \textbf{\textit{if}}\ -n_1+n_2+n_3=2\cdot m,\ \forall m\in\mathbb{N}-\{0\}\end{cases} \\ \textbf{\textit{if}}\ n_1-n_2<n_3<2\cdot(n_1-n_2)+3 \\ =\begin{cases}\frac{1}{2}\cdot n_1^2-n_1\cdot n_2-n_1\cdot n_3+n_1+\frac{1}{2}\cdot n_2^2+n_2\cdot n_3-n_2+\frac{1}{2}\cdot n_3^2-n_3-\frac{1}{2} \\ \textbf{\textit{if}}\ -n_1+n_2+n_3=2\cdot m-1,\ \forall m\in\mathbb{N}-\{0\} \\ \frac{1}{2}\cdot n_1^2-n_1\cdot n_2-n_1\cdot n_3+n_1+\frac{1}{2}\cdot n_2^2+n_2\cdot n_3-n_2+\frac{1}{2}\cdot n_3^2-n_3-1 \\ \textbf{\textit{if}}\ -n_1+n_2+n_3=2\cdot m,\ \forall m\in\mathbb{N}-\{0\}\end{cases} \\ \textbf{\textit{if}}\ n_3=2\cdot(n_1-n_2)+3\end{cases} \\ \textbf{\textit{if}}\ n_1=n_2=2 \lor n_1>n_2 \\ =\begin{cases}2 \\ \textbf{\textit{if}}\ n_3<3 \\ 2 \\ \textbf{\textit{if}}\ n_3=3\end{cases} \\ \textbf{\textit{if}}\ n_1=n_2>2 \land n := n_1=n_2\end{cases} \quad (21)$$

where:

$n_1 \ge n_2 \ge n_3$

Analyzing *(21)*, we find that the discrepancy between the algorithm proposed by Ripà and ours is always equal to or greater than zero, thus our algorithm is always at least as efficient as Ripà's algorithm in the cases studied in *(21)*.

Then, by combining *(13)* and *(14)* with *(17)* and *(19)*, we obtain, ∀ $n_1; n_2; n_3 \in \mathbb{N}$-{0; 1; 2; 3}:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} = 0 & \textbf{\textit{if}}\ n = 4 \\ = \begin{cases} 1 & \textbf{\textit{if}}\ n = 2 \cdot m - 1,\ \forall m \in \mathbb{N} - \{0; 1; 2\} \\ 2 & \textbf{\textit{if}}\ n = 4 \cdot m + 2,\ \forall m \in \mathbb{N} - \{0\} \\ 0 & \textbf{\textit{if}}\ n = 4 \cdot m,\ \forall m \in \mathbb{N} - \{0; 1\} \end{cases} & \textbf{\textit{if}}\ n > 4 \end{cases} \quad (22)$$

where:

$n_1 = n_2 = n_3 \wedge n \coloneqq n_1 = n_2 = n_3$

Finally, to analyze the discrepancy between the solutions obtained through $h(n_1; n_2; n_3)$ and $t(n_1; n_2; n_3)$ for the remaining cases, we will focus solely on verifying that the difference $h(n_1; n_2; n_3)$-$t(n_1; n_2; n_3)$, in these cases, is always equal to or greater than zero.

We won't aim to quantify it precisely, but, instead, we will rely on straightforward observations that can be easily deduced from our formulas.

First of all, it will be helpful to eliminate the various *floor* functions present in the formulas of the cases not yet studied of $h(n_1; n_2; n_3)$.

We first note that i_{max} is written in the form, ∀ $x \in \mathbb{R}$:

$$i_{max} = \lfloor x \rfloor \quad (23)$$

where:

$x \geq 0$

Next, let's consider that, ∀ $x \in \mathbb{R}$:

$$x - 1 < \lfloor x \rfloor \leq x \quad (24)$$

where:

$x \geq 0$

Thus, using equations *(23)* and *(24)*, we replace, as needed, the various $i_{max} = \lfloor x \rfloor$, in the remaining cases of *(16)*, with *x-1* or *x*, obtaining, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-*{0; 1}*:

$$h(n_1;n_2;n_3) > \begin{cases} \frac{4}{3}\cdot(x-1)^3+(2\cdot(n_1-n_2)+7)\cdot(x-1)^2+\left(6\cdot(n_1-n_2)+\frac{35}{3}\right)\cdot(x-1)-2\cdot n_3\cdot x+4\cdot(n_1-n_2)+2\cdot n_3\cdot(n_2-1)+5 \\ \textbf{\textit{if}}\ n_3 \le 2\cdot(i_{max}{}^2+(n_1-n_2+4)\cdot i_{max}+2\cdot(n_1-n_2)+4) \\ \frac{4}{3}\cdot(x-1)^3+(2\cdot(n_1-n_2)+9)\cdot(x-1)^2+\left(8\cdot(n_1-n_2)+\frac{59}{3}\right)\cdot(x-1)-2\cdot n_3\cdot x+8\cdot(n_1-n_2)+n_3\cdot(2\cdot n_2-3)+13 \\ \textbf{\textit{if}}\ n_3 > 2\cdot(i_{max}{}^2+(n_1-n_2+4)\cdot i_{max}+2\cdot(n_1-n_2)+4) \end{cases} \quad (25)$$

where:

$$x := \frac{1}{2}\cdot\left(\sqrt{n_1{}^2+n_2{}^2-2\cdot n_1\cdot n_2+2\cdot(n_1-n_2+n_3)+1}+n_2-n_1-3\right)$$

and:

$$i_{max} := \left\lfloor \frac{1}{2}\cdot\left(\sqrt{n_1{}^2+n_2{}^2-2\cdot n_1\cdot n_2+2\cdot(n_1-n_2+n_3)+1}+n_2-n_1-3\right)\right\rfloor$$

and:

$$n_3 > 2\cdot(n_1-n_2)+3$$

and:

$$n_1 \ge n_2 \ge n_3$$

By repeating the procedure just shown for *(18)*, we obtain, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-*{0; 1; 2; 3}*:

$$h(n_1;n_2;n_3) > \begin{cases} \frac{2}{3}\cdot(x-1)^3+5\cdot(x-1)^2+\frac{28}{3}\cdot(x-1)-2\cdot n_3\cdot x+2\cdot n_2\cdot n_3-n_3+3 \\ \textbf{\textit{if}}\ n_3-i_{max}{}^2-5\cdot i_{max} \le 5 \\ \frac{2}{3}\cdot(x-1)^3+6\cdot(x-1)^2+\frac{43}{3}\cdot(x-1)-2\cdot n_3\cdot x+2\cdot n_2\cdot n_3-2\cdot n_3+8 \\ \textbf{\textit{if}}\ n_3-i_{max}{}^2-5\cdot i_{max} > 5 \end{cases} \quad (26)$$

where:

$$x := \frac{1}{2}\cdot\left(\sqrt{4\cdot n_3+9}-5\right)$$

and:

$$i_{max} := \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n_3+9}-5\right)\right\rfloor$$

and:

$$n_1-1 = n_2 \ge n_3$$

At this point, since our $t(n_1; n_2; n_3)$ is divided into a significant number of cases, it will be convenient to consider that, from *(12)*, *(13)* and *(14)*, it follows that, $\forall\ n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\}$:

$$t(n_1; n_2; n_3) \le \begin{cases} -\frac{1}{2}\cdot n_1^{\ 2} - n_1 + n_1\cdot n_2 + n_1\cdot n_3 - \frac{1}{2}\cdot n_2^{\ 2} + n_2 + n_2\cdot n_3 - \frac{1}{2}\cdot n_3^{\ 2} + n_3 - 1 \\ \textbf{\textit{if}}\ n_1 > n_2 \\ \\ 2\cdot n\cdot n_3 - \frac{1}{2}\cdot n_3^{\ 2} + n_3 - 1 \\ \textbf{\textit{if}}\ n_1 = n_2 \wedge n \coloneqq n_1 = n_2 \end{cases} \tag{27}$$

where:

$$-n_1 + n_2 + n_3 \ge 1$$

and:

$$n_1 \ge n_2 \ge n_3$$

Therefore, by combining *(25)* and *(26)* with *(27)*, the differences between the solutions obtained through our algorithm and those of Ripà, for the cases not yet analyzed, are as follows.

For specific cases of *(25)*, we obtain, $\forall\ n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) > \begin{cases} \frac{1}{3}\cdot d^3 + d^2 - \frac{1}{3}\cdot d^2\cdot r + \frac{1}{6}\cdot d - \frac{1}{6}\cdot d\cdot r + r - \frac{2}{3}\cdot r\cdot n_3 + \frac{1}{2}\cdot n_3^{\ 2} - \frac{3}{2}\cdot n_3 - 1 \coloneqq a_1 \\ \textbf{\textit{if}}\ n_3 \le 2\cdot\left(i_{max}^{\ \ 2} + (n_1 - n_2 + 4)\cdot i_{max} + 2\cdot(n_1 - n_2) + 4\right) \\ \\ \frac{1}{3}\cdot d^3 + d^2 - \frac{1}{3}\cdot d^2\cdot r + \frac{7}{6}\cdot d - \frac{1}{6}\cdot d\cdot r - \frac{2}{3}\cdot r\cdot n_3 + \frac{1}{2}\cdot n_3^{\ 2} - \frac{3}{2}\cdot n_3 \coloneqq a_2 \\ \textbf{\textit{if}}\ n_3 > 2\cdot\left(i_{max}^{\ \ 2} + (n_1 - n_2 + 4)\cdot i_{max} + 2\cdot(n_1 - n_2) + 4\right) \end{cases} \tag{28}$$

where:

$$r \coloneqq \sqrt{d^2 + 2\cdot d + 2\cdot n_3 + 1}$$

and:

$$d \coloneqq n_1 - n_2$$

and:

$$i_{max} \coloneqq \left\lfloor \frac{1}{2}\cdot\left(\sqrt{n_1^{\ 2} + n_2^{\ 2} - 2\cdot n_1\cdot n_2 + 2\cdot(n_1 - n_2 + n_3) + 1} + n_2 - n_1 - 3\right)\right\rfloor$$

and:

$$n_3 > 2\cdot(n_1 - n_2) + 3$$

and:

$$n_1 > n_2 \ge n_3$$

For the last cases of *(25)*, we have, instead, ∀ $n_1; n_2; n_3 \in \mathbb{N}-\{0; 1\}$:

$$h(n_1; n_2; n_3)-t(n_1; n_2; n_3)>\begin{cases}\frac{1}{2}\cdot n_3^{\,2}-\frac{3}{2}\cdot n_3-\frac{2}{3}\cdot r\cdot n_3+r-1 \coloneqq a_3 \\ \textbf{\textit{if}}\ n_3\leq 2\cdot(i_{max}^{\,2}+4\cdot i_{max}+4) \\ \\ \frac{1}{2}\cdot n_3^{\,2}-\frac{3}{2}\cdot n_3-\frac{2}{3}\cdot r\cdot n_3 \coloneqq a_4 \\ \textbf{\textit{if}}\ n_3>2\cdot(i_{max}^{\,2}+4\cdot i_{max}+4)\end{cases} \tag{29}$$

where:

$$r \coloneqq \sqrt{2\cdot n_3+1}$$

and:

$$i_{max} \coloneqq \left\lfloor \frac{1}{2}\cdot\left(\sqrt{2\cdot n_3+1}-3\right)\right\rfloor$$

and:

$$n_3>2\cdot(n_1-n_2)+3$$

and:

$$n_1=n_2\geq n_3$$

For the cases shown in *(26)*, we finally obtain, ∀ $n_1; n_2; n_3 \in \mathbb{N}-\{0; 1; 2; 3\}$:

$$h(n_1; n_2; n_3)-t(n_1; n_2; n_3)>\begin{cases}\frac{1}{2}\cdot n_3^2+\frac{1}{6}\cdot r-\frac{2}{3}\cdot r\cdot n_3+1 \coloneqq a_5 \\ \textbf{\textit{if}}\ n_3-i_{max}^{\,2}-5\cdot i_{max}\leq 5 \\ \\ \frac{1}{2}\cdot n_3^2-\frac{5}{6}\cdot r-\frac{2}{3}\cdot r\cdot n_3+3 \coloneqq a_6 \\ \textbf{\textit{if}}\ n_3-i_{max}^{\,2}-5\cdot i_{max}>5\end{cases} \tag{30}$$

where:

$$r \coloneqq \sqrt{4\cdot n_3+9}$$

and:

$$i_{max} \coloneqq \left\lfloor \frac{1}{2}\cdot\left(\sqrt{4\cdot n_3+9}-5\right)\right\rfloor$$

and:

$$n_1-1=n_2\geq n_3$$

Since the right-hand sides of *(28)*, *(29)* and *(30)* are strictly less than the difference between the solutions obtained through our algorithm and those of Ripà, when both the two a_i of the inequality are equal to or greater than zero, it follows that the exact difference $h(n_1; n_2; n_3)-t(n_1; n_2; n_3)$ must also be.

Thus, by setting the various a_i equal to or greater than zero, we obtain the following.

After doing the calculations, from *(28)*, we have, $\forall\ n_1;\ n_2;\ n_3 \in \mathbb{N}-\{0;\ 1\}$:

$$\begin{cases} a_1 \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 11 \\ \\ a_2 \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 13 \end{cases} \rightarrow \begin{cases} h(n_1;n_2;n_3) - t(n_1;n_2;n_3) \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 13 \end{cases} \qquad (31)$$

where:

$$h(n_1;\ n_2;\ n_3) - t(n_1;\ n_2;\ n_3) > \begin{cases} a_1 \\ \textbf{\textit{if}}\ n_3 \leq 2\cdot\left(i_{max}{}^2 + (n_1 - n_2 + 4)\cdot i_{max} + 2\cdot(n_1 - n_2) + 4\right) \\ \\ a_2 \\ \textbf{\textit{if}}\ n_3 > 2\cdot\left(i_{max}{}^2 + (n_1 - n_2 + 4)\cdot i_{max} + 2\cdot(n_1 - n_2) + 4\right) \end{cases}$$

and:

$$a_1 := \frac{1}{3}\cdot d^3 + d^2 - \frac{1}{3}\cdot d^2\cdot r + \frac{1}{6}\cdot d - \frac{1}{6}\cdot d\cdot r + r - \frac{2}{3}\cdot r\cdot n_3 + \frac{1}{2}\cdot n_3{}^2 - \frac{3}{2}\cdot n_3 - 1$$

and:

$$a_2 := \frac{1}{3}\cdot d^3 + d^2 - \frac{1}{3}\cdot d^2\cdot r + \frac{7}{6}\cdot d - \frac{1}{6}\cdot d\cdot r - \frac{2}{3}\cdot r\cdot n_3 + \frac{1}{2}\cdot n_3{}^2 - \frac{3}{2}\cdot n_3$$

and:

$$i_{max} := \left\lfloor \frac{1}{2}\cdot\left(\sqrt{n_1{}^2 + n_2{}^2 - 2\cdot n_1\cdot n_2 + 2\cdot(n_1 - n_2 + n_3) + 1} + n_2 - n_1 - 3\right) \right\rfloor$$

and:

$$r := \sqrt{d^2 + 2\cdot d + 2\cdot n_3 + 1}$$

and:

$$d := n_1 - n_2$$

and:

$$n_3 > 2\cdot(n_1 - n_2) + 3$$

and:

$$n_1 > n_2 \geq n_3$$

From *(29)*, we obtain, instead, $\forall\, n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\}$:

$$\begin{cases} a_3 \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 8 \\ \\ a_4 \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 9 \end{cases} \rightarrow \begin{cases} h(n_1; n_2; n_3) - t(n_1; n_2; n_3) \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 9 \end{cases} \tag{32}$$

where:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) > \begin{cases} a_3 \\ \textbf{\textit{if}}\ n_3 \leq 2 \cdot (i_{max}^{\ 2} + 4 \cdot i_{max} + 4) \\ \\ a_4 \\ \textbf{\textit{if}}\ n_3 > 2 \cdot (i_{max}^{\ 2} + 4 \cdot i_{max} + 4) \end{cases}$$

and:

$$a_3 \coloneqq \frac{1}{2} \cdot n_3^{\ 2} - \frac{3}{2} \cdot n_3 - \frac{2}{3} \cdot r \cdot n_3 + r - 1$$

and:

$$a_4 \coloneqq \frac{1}{2} \cdot n_3^{\ 2} - \frac{3}{2} \cdot n_3 - \frac{2}{3} \cdot r \cdot n_3$$

and:

$$i_{max} \coloneqq \left\lfloor \frac{1}{2} \cdot \left(\sqrt{2 \cdot n_3 + 1} - 3\right) \right\rfloor$$

and:

$$r \coloneqq \sqrt{2 \cdot n_3 + 1}$$

and:

$$n_3 > 2 \cdot (n_1 - n_2) + 3$$

and:

$$n_1 = n_2 \geq n_3$$

From *(30)*, we finally obtain, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-*{0; 1; 2; 3}*:

$$\begin{cases} a_5 \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 9 \\ \\ a_6 \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 10 \end{cases} \rightarrow \begin{cases} h(n_1; n_2; n_3) - t(n_1; n_2; n_3) \geq 0 \\ \textbf{\textit{at least if}}\ n_3 \geq 10 \end{cases} \tag{33}$$

where:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) > \begin{cases} a_5 \\ \textbf{\textit{if}}\ n_3 - {i_{max}}^2 - 5 \cdot i_{max} \leq 5 \\ \\ a_6 \\ \textbf{\textit{if}}\ n_3 - {i_{max}}^2 - 5 \cdot i_{max} > 5 \end{cases}$$

and:

$$a_5 := \frac{1}{2} \cdot n_3^2 + \frac{1}{6} \cdot r - \frac{2}{3} \cdot r \cdot n_3 + 1$$

and:

$$a_6 := \frac{1}{2} \cdot n_3^2 - \frac{5}{6} \cdot r - \frac{2}{3} \cdot r \cdot n_3 + 3$$

and:

$$i_{max} := \left\lfloor \frac{1}{2} \cdot \left(\sqrt{4 \cdot n_3 + 9} - 5 \right) \right\rfloor$$

and:

$$r := \sqrt{4 \cdot n_3 + 9}$$

and:

$$n_1 - 1 = n_2 \geq n_3$$

By examining *(31)*, *(32)* and *(33)*, we can observe that the difference between the solutions obtained through $h(n_1; n_2; n_3)$ and $t(n_1; n_2; n_3)$ is not necessarily always greater than or equal to zero.

Therefore, in certain cases of the problem, this inequality may not hold.

In order to demonstrate that our algorithm is indeed always as efficient as or more efficient than those proposed by Ripà, it will be necessary to calculate the exact difference $h(n_1; n_2; n_3)$-$t(n_1; n_2; n_3)$ for the cases where we have yet to establish that this difference is always greater than or equal to zero.

For the cases in *(31)* where we cannot be certain that the difference is non-negative, the exact difference $h(n_1; n_2; n_3)$-$t(n_1; n_2; n_3)$ can be calculated by combining *(12)* with *(16)*.

This will be illustrated below, categorized by the value of *d*, which represents the difference between n_1 and n_2.

For the first possible value of d, we have, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-$\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} = \begin{cases} 6 & \textbf{\textit{if}}\ n_3 = 6 \\ 8 & \textbf{\textit{if}}\ n_3 = 7 \\ 12 & \textbf{\textit{if}}\ n_3 = 8 \\ 16 & \textbf{\textit{if}}\ n_3 = 9 \\ 22 & \textbf{\textit{if}}\ n_3 = 10 \\ 28 & \textbf{\textit{if}}\ n_3 = 11 \\ 36 & \textbf{\textit{if}}\ n_3 = 12 \end{cases} \\ \textbf{\textit{if}}\ d = 1 \end{cases} \quad (34)$$

where:

$d \coloneqq n_1 - n_2$

and:

$n_1 > n_2 \geq n_3$

For the second value of d, we have, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-$\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} = \begin{cases} 10 \\ \textbf{\textit{if}}\ n_3 = 8 \\ 14 \\ \textbf{\textit{if}}\ n_3 = 9 \\ 18 \\ \textbf{\textit{if}}\ n_3 = 10 \\ 24 \\ \textbf{\textit{if}}\ n_3 = 11 \\ 30 \\ \textbf{\textit{if}}\ n_3 = 12 \end{cases} \\ \textbf{\textit{if}}\ d = 2 \end{cases} \tag{35}$$

where:

$d \coloneqq n_1 - n_2$

and:

$n_1 > n_2 \geq n_3$

For the third possible value of d, we have, instead, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-$\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} = \begin{cases} 16 \\ \textbf{\textit{if}}\ n_3 = 10 \\ 20 \\ \textbf{\textit{if}}\ n_3 = 11 \\ 26 \\ \textbf{\textit{if}}\ n_3 = 12 \end{cases} \\ \textbf{\textit{if}}\ d = 3 \end{cases} \tag{36}$$

where:

$d \coloneqq n_1 - n_2$

and:

$n_1 > n_2 \geq n_3$

For the last value of d, we finally have, $\forall\ n_1; n_2; n_3 \in \mathbb{N}$-$\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} = \begin{cases} 22 \\ \textbf{\textit{if}}\ n_3 = 12 \end{cases} \\ \textbf{\textit{if}}\ d = 4 \end{cases} \tag{37}$$

where:

$d \coloneqq n_1 - n_2$

and:

$n_1 > n_2 \geq n_3$

For the cases of *(32)* for which there is no certainty that they are equal to or greater than zero, by combining *(13)* and *(14)* with *(16)*, we find that the exact difference is, instead, ∀ $n_1; n_2; n_3 \in \mathbb{N}-\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} 2 & \textbf{\textit{if}}\ n_3 = 4 \\ 5 & \textbf{\textit{if}}\ n_3 = 5 \\ 8 & \textbf{\textit{if}}\ n_3 = 6 \\ 11 & \textbf{\textit{if}}\ n_3 = 7 \\ 14 & \textbf{\textit{if}}\ n_3 = 8 \end{cases} \quad (38)$$

where:

$n_1 = n_2 \geq n_3$

For the cases of *(33)* for which there is no certainty that they are equal to or greater than zero, by combining *(12)* with *(18)*, we obtain that the exact difference $h(n_1; n_2; n_3)-t(n_1; n_2; n_3)$ is, finally, ∀ $n_1; n_2; n_3 \in \mathbb{N}-\{0; 1; 2; 3\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) = \begin{cases} 2 & \textbf{\textit{if}}\ n_3 = 4 \\ 3 & \textbf{\textit{if}}\ n_3 = 5 \\ 5 & \textbf{\textit{if}}\ n_3 = 6 \\ 7 & \textbf{\textit{if}}\ n_3 = 7 \\ 11 & \textbf{\textit{if}}\ n_3 = 8 \\ 15 & \textbf{\textit{if}}\ n_3 = 9 \end{cases} \quad (39)$$

where:

$n_1 - 1 = n_2 \geq n_3$

5.3 Conclusion

Having illustrated the discrepancy between the solutions obtained through our algorithm and the other proposed ones, we can, therefore, state that, $\forall\ n_1; n_2; n_3 \in \mathbb{N}\text{-}\{0; 1\}$:

$$h(n_1; n_2; n_3) - t(n_1; n_2; n_3) \geq 0 \tag{40}$$

and, consequently, that our algorithm provides upper bounds that are equal to or more efficient than those obtained through the algorithms proposed by others for the three-dimensional case (as well as providing the exact solution for the two-dimensional case).

Finally, by combining *(15)* with *(20)*, we can extend the result to the four-dimensional case and to subsequent cases.

In other words, $\forall\ n_1; n_2; n_3; \ldots; n_k \in \mathbb{N}\text{-}\{0; 1\} \wedge k \in \mathbb{N}\text{-}\{0; 1; 2\}$:

$$h(n_1; n_2; n_3; \ldots; n_k) - t(n_1; n_2; n_3; \ldots; n_k) \geq 0 \tag{41}$$

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