Kinematics Kernels: A Reassessment of Particle Kinematics in Newtonian Mechanics*

Fereydoun F. Me'mār-Zanjāni II (فریدون معمارزنجانی)†
(Dated: January 12, 2023)

When a cannonball is shot towards the walls of a fortress, or when the ion drives on a satellite eject gas in order to keep its orbit stable, or, similarly, when we observe the ordinary day-to-day movements of cars, rivers, and other objects, and even the speed of a chemical reaction, we, in fact, are witnessing the effects that physics, or more specifically, kinematics, has on our world. The former, for instance, employs the range formula for projectiles; the satellite, on the other hand, utilizes the concept of velocity to continue being kept afloat in space. Although the latter cases are, respectively, measured in terms of acceleration, distance, and speed (or, depending on the circumstances, in terms of their vectorized counterparts: change in speed, displacement, and velocity); they all, however, do have a shared sense of motion attached to them, which in physics and engineering came to be known as "kinematics."

This paper aims to explain the fundamentals of kinematics and clarify the core topics and idioms covered within college-level physics courses, and some of the advanced high school science textbooks, straightforwardly and comprehensively; furthermore, core kinematics concepts and equations will be demonstrated alongside the algebraic proofs of their derived formulae.

PACS numbers: 45.50.-j, 83.10.Pp
Keywords: kinematics; particles; motion; newtonian mechanics; classical mechanics; advanced high school physics; college-level physics; university physics course.

CONTENTS

* Nowadays known as *Classical Mechanics*.

† Otherwise known as Fereydoun Memarzanjany (ORCID iD: 0000-0003-1393-6804) when copyright and/or legal matters are concerned. Reachable for any inquiries at
- E-Mail: F_Memarzanjany@outlook.com; and
- Website: https://thraetaona.github.io/

I. INTRODUCTION

Kinematics—initially developed in *classical mechanics* and traditionally known as a branch in the subject *theory of machines*, as a subfield of physics, focuses on the description of motions without any regard to force, as opposed to *kinetics*, or the study of forces (and masses thereof) that cause motion themselves. Thus, it could be said that **kinematics is the study of motion independent from objects' masses.**

a. Etymology and nomenclature The English term known as kinematics (pronounced /ˌkinəˈmadiks/ or ki·nuh·**ma**·tuhks) is from French *cinématique*, which in turn was derived from the Ancient Greek word *κίνημα* (kínēma, "motion"), plus the Middle English suffix *-ics*, referring to the study or knowledge of motions.

Within the equations and solutions of kinematics problems, algebra and geometry are perhaps among the most widely used areas of mathematics; so much, in fact, that inside mathematicians' communities, kinematics is often regarded as "the Geometry of Motion."

In the chapters that follow, core kinematics concepts and equations alongside their commonly associated variables will be demonstrated, and some examples in addition to their solutions will be provided as well. Several of the derived formulae will also be proven algebraically.

II. MEASUREMENTS OF POSITION

Everything in the world around us is in ever-lasting motion. Even the objects that appear inanimate *relative* to us are still in *absolute* motion within the universe due to the rotation of the earth around the sun, the rotation of the sun around the center of the milky way, and so on; moreover, even the atoms and sub-atomic particles of said objects are still vibrating in their places and are *not* motionless.

A. Pointwise Distance and Displacement

In addition to the coordinates of these objects, what else is of considerable importance is their change in position, which is expressed using the terms **distance** and **displacement**. Although these two terms are both used to measure the same unit (meters, m), they, however, are two distinct variables in physics with *different* properties.

Distance refers to the total sum of the length of the path(s) traversed by an object from its starting position up until its ending point, while displacement is as its name implies: the *shortest* path between those points.

Case in point, let us say that a person is trying to get to their friend's house that is situated one alley to the north, then another one to the east, and finally one to the south of their own home, and that all these alleys are of the same length. If there are no shortcuts in between and they only travel through said alleys, then the *distance* they have traveled is equal to the length of *three alleys*, while their *displacement* would only be the length of *one alley to the east.* Notice how we also include directions (e.g., *to the east*) when mentioning the displacement.

In order to have a clearer understanding of the differences between the former and the latter, first have a look at the following figure (Fig. 1):

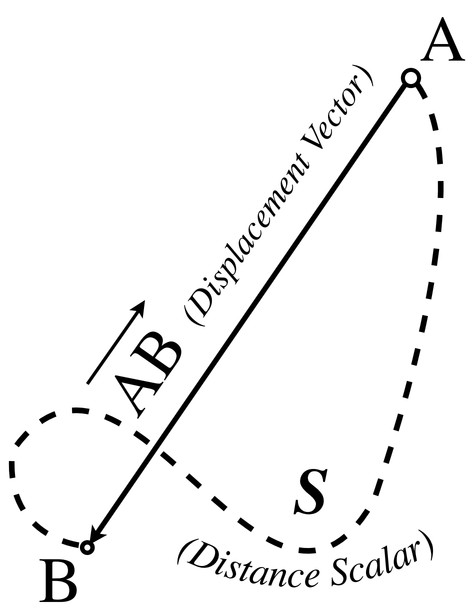


Figure 1. Distance vs. Displacement: Comparison of displacement ($\overrightarrow{AB}$) and distance ($\boldsymbol{S}$) between the two points of A and B.

As it appears in the Illustration, distance (denoted $\boldsymbol{S}$), depending on the traversed path, may not necessarily be a straight line, while displacement (denoted by a two-point vector such as $\overrightarrow{AB}$), being the shortest possible route, will always be a straight line with a direction. In other words, displacement as a directed line is a **vector quantity,** in contrast to its non-directional counterpart, distance, which is a **scalar quantity.**

Notice that since displacement is a vector quantity, it could also be *re-interpreted* by being assigned negative values. To give you an idea, negative two (-2) units to the east could also mean positive two ($+2$) units to *the direction opposite of the east,* thus positive two ($+2$) units to the west. However, this property does not hold true for scalar quantities, as they lack directions; negative values for scalar quantities are not possible.

As another example, when cars in a racetrack finish a single lap, their distance would be equal to the total length of that racetrack, whereas their displacement would be zero because they are now back at the same starting point. Even if they race for a second lap, even though then their new distance would now be equal to twice the racetrack's length, their displacement would still remain zero.

Based on the previous definitions, distance is found by simply adding the length of the different path(s) taken, and displacement, the shortest difference between two points, is defined as follows:

$$\Delta x \ (m) = x_f - x_i \ . \tag{1}$$

Equation 1. Read as *Delta-X (with the unit of meters) equals X-final minus X-initial*. If you are unfamiliar with these terms or would like to see more alternative variations of the aforementioned notation in physics, then refer to *Appendix C: Glossary and Elementary Definitions in Physics.*

Where Δx is the *change* in position, x_f is the final (ending) position, and x_i is the initial (starting) position, with all being measurements of length with the SI unit meters (m).[1]

A key takeaway about the above explanations is that the two starting–ending points can only be subtracted directly in the context of two-dimensional plane physics; for subtraction in three-dimensional spaces, the Pythagoras theorem would have to be used.[2]

[1] In some textbooks, "d" is written instead of "x"; they both carry the same meaning. From here onwards, "d" will be used in place of distance and "x" in place of displacement.

[2] 3-D and higher dimensions are outside of the teaching scope. For the Pythagoras theorem, see *Appendix B: Graphs and the Basics of Geometry.*

III. CONSTANT MOVEMENT

Prior to the experiments conducted by the prominent Italian scientist Galileo Galilei about how fast objects move, people would nominally categorize movements into "fast," "slow," or "resting"; but nowadays, with all the complex types of machinery and traffics, expressing these movements in terms of numerical values has become quite important.

A. Average Speed and Velocity

The relation between **speed** and **velocity** is similar to that of distance and displacement; speed is the scalar quantity, and velocity is its vector counterpart. In other words, the average speed is the distance (be it a straight line or curved line) covered within a timeframe, and the average velocity is the displacement traveled within a timeframe.

Here, the word *average* signifies the measurement type: an interval (i.e., more than just a single point) as a whole is being used to compute one value. If a single point were to be measured, we would instead call it the *instantaneous* speed or velocity.[3]

The formula for the *average speed* is as follows:

$$s\ (^{m}/_{s}) = \frac{distance\ in\ meters}{time\ spent\ in\ seconds} = \frac{d}{\Delta t}\ , \tag{2}$$

And the formula for the *average velocity* is comparative:

$$\overrightarrow{v}\ (^{m}/_{s}) = \frac{displacement\ in\ meters}{time\ spent\ in\ seconds} = \frac{\Delta x}{\Delta t}\ . \tag{3}$$

Considering that distance and displacement are both measured in meters (m), in addition to the fact that the SI unit of time is seconds (s), it could be said that speed and velocity have the same unit—meters per second ($^{m}/_{s}$). Furthermore, it is trivial that by solving for different variables (e.g., $\overrightarrow{v} = \frac{\Delta x}{\Delta t} \Leftrightarrow \Delta t = \frac{\Delta x}{\overrightarrow{v}}$), the unknown parameters of a problem could be determined from the known ones.[4]

To expand on what was said earlier, if the speed or velocity (both defined as meters per a single second) are known, then the amount of movement during an elapsed time could be calculated, too. For example, an object moving at an average speed of 5.5 meters per second, for a total duration of 4 seconds, would travel a total of 22 meters at the end. ($\frac{5.5\ meters}{1\ second} \times 4\ seconds = 22\ meters$.)

[3] Instantaneous rates of change, which depend on the knowledge of derivatives, are outside of the teaching scope of this paper; they will not be explained in-depth here.

[4] For more assistance regarding this, see *Appendix A: Algebraic Laws and Properties.*

IV. ACCELERATION AND VARYING MOVEMENT

A car being driven is likely to change through its gears as time goes by; it is, to put it differently, likely to **accelerate** or **decelerate** and change its speed. Acceleration is inherently a vector quantity; it is found by dividing the velocity by the time elapsed. So, by this logic, its unit has to be meters per second, per second (or meters per second squared, $^{m}/_{s^2}$), which implies that the car or object is gaining—or losing—some of its velocity and, by extension, speed each and every second.

The formula for *constant*[3] *acceleration* is as follows:

$$\overrightarrow{a}\ (^{m}/_{s^2}) = \frac{change\ of\ velocity\ in\ ^{m}/_{s}}{time\ spent\ in\ seconds} = \frac{\Delta \mathbf{v}}{\Delta t}\ . \tag{4}$$

And, as mentioned earlier in the first chapter, negative vector quantities could be re-interpreted oppositely, just like any other vector value. Thus, if the acceleration is negative, it could be said that the object is decelerating—it is accelerating in the opposite direction.

Similar to how the covered distance or displacement could be calculated based on the velocity or speed in addition to the elapsed time, the same could also be done for the acceleration in order to find the total velocity change. Even the total displacement could be evaluated from just the acceleration and time: the multiplication of acceleration by time results in the velocity change, and the multiplication of the resultant velocity by time again provides the total distance traversed; multiplying the acceleration by time squared (t^2) is a more direct way of obtaining the same outcome.

V. SUMMARIZING AND DERIVING FORMULAE

In respect to an initial position (x_i), finding the final position (x_f) when an initial velocity ($\mathbf{v_i}$) and a constant acceleration ($\mathbf{a}$) are involved is done using the following formula:

$$x_f\ (m) = x_i + \mathbf{v_i} \cdot t + \frac{1}{2} \cdot \mathbf{a} \cdot t^2. \tag{5a}$$

It is trivial that, in this way, solving for the amount of change (Δx) would simply require subtracting the initial position (x_i) from both sides of the Equation. (Remember that $\Delta x = x_f - x_i$ applies here.)

Also, if the question or problem does not supply the acceleration directly but does give a final and initial value for the velocity, the acceleration could manually be calculated based on those and then input into the above formula; these manual steps could also be embedded within the previous equation:

$$x_f\ (m) = x_i + \frac{(\mathbf{v_f} + \mathbf{v_i})}{2} \cdot t = x_i + \overline{\mathbf{v}} \cdot t\ . \tag{5b}$$

Proof of Eq. (5b):

$$
\begin{aligned}
I &= x_f \\
&= x_i + \mathbf{v_i}\cdot t + \frac{1}{2}\cdot\mathbf{a}\cdot t^2 && \text{[as per Eq. (5a)]} \\
&= x_i + t \times \left(\mathbf{v_i} + \frac{1}{2}\cdot\mathbf{a}\cdot t\right) && \text{[factorized } t\text{]} \\
&= x_i + t \times \left(\mathbf{v_i} + \frac{1}{2}\cdot\frac{(\mathbf{v_f}-\mathbf{v_i})}{t}\cdot t\right) && \text{[expanded } \mathbf{a}\text{]} \\
&= x_i + t \times \left(\mathbf{v_i} + \frac{1}{2}\cdot(\mathbf{v_f}-\mathbf{v_i})\right) && \text{[simplified } t\text{]} \\
&= x_i + t \times \left(\frac{(2\mathbf{v_i}+\mathbf{v_f}-\mathbf{v_i})}{2}\right) && \text{[multiplied, \&c.]} \\
&= x_i + t \times \left(\frac{(\mathbf{v_i}+\mathbf{v_f})}{2}\right) && \text{[subtracted } \mathbf{v_i}\text{]} \\
&= x_i + \frac{(\mathbf{v_f}+\mathbf{v_i})}{2}\cdot t && \text{[reordered]} \\
&= x_i + \overline{\mathbf{v}}\cdot t = II && \square
\end{aligned}
$$

A. The Timeless Equation for Acceleration

While sharply braking, the length of the brake lines or trail that a car's tires leave behind on the road can give law enforcement an idea of how fast that car was moving before it began to brake. Oftentimes, calculating the maximum distance that an object with its initial velocity against an opposing force (i.e., deceleration) can reach, before eventually slowing down to a halt and reaching the nill final velocity of zero, has its own practical uses. In such cases, the "timeless" equation could be used instead of manually calculating the time from other factors for the same result:

$$\overrightarrow{v_f}^2 \ ({}^m/_s) = \mathbf{v_i}^2 + 2\cdot\mathbf{a}\cdot\Delta x \ . \tag{6}$$

Proof of Eq. (6): Let $\overrightarrow{a} = \frac{\Delta\mathbf{v}}{\Delta t} \ni \overrightarrow{a} < 0 \because \mathbf{v_f} = 0\,,\ \mathbf{v_i}\in\mathbb{N} \therefore \Delta\mathbf{v} \triangleq \mathbf{v_f}-\mathbf{v_i} < 0$. So,

$$
\begin{aligned}
x_f &= x_i + (\mathbf{v_i}\cdot\Delta t) + \left(\frac{1}{2}\cdot\mathbf{a}\cdot(\Delta t)^2\right) \\
&= x_i + \Delta t\cdot\left(\mathbf{v_i} + \left(\frac{1}{2}\cdot\mathbf{a}\cdot\Delta t\right)\right) \\
&= x_i + \frac{\Delta\mathbf{v}}{\mathbf{a}}\cdot\left(\mathbf{v_i} + \left(\frac{1}{2}\cdot\mathbf{a}\cdot\frac{\Delta\mathbf{v}}{\mathbf{a}}\right)\right) \\
&= x_i + \frac{\Delta\mathbf{v}}{\mathbf{a}}\cdot\left(\mathbf{v_i} + \frac{\Delta\mathbf{v}}{2}\right)
\end{aligned}
\Bigg\}\longrightarrow
\begin{aligned}
&= x_i + \frac{\mathbf{v_f}-\mathbf{v_i}}{\mathbf{a}}\cdot\left(\mathbf{v_i} + \frac{(\mathbf{v_f}-\mathbf{v_i})}{2}\right) \\
&= x_i + \frac{\mathbf{v_f}-\mathbf{v_i}}{\mathbf{a}}\cdot\frac{2\mathbf{v_i}+\mathbf{v_f}-\mathbf{v_i}}{2} \\
&= x_i + \frac{\mathbf{v_f}-\mathbf{v_i}}{\mathbf{a}}\cdot\frac{\mathbf{v_f}+\mathbf{v_i}}{2} \\
&= x_i + \frac{\mathbf{v_f}^2-\mathbf{v_i}^2}{2\mathbf{a}}
\end{aligned}
\Bigg\}\longrightarrow
\begin{aligned}
&\Rightarrow x_f - x_i = \frac{\mathbf{v_f}^2-\mathbf{v_i}^2}{2\mathbf{a}} \\
&= \Delta x = \frac{\mathbf{v_f}^2-\mathbf{v_i}^2}{2\mathbf{a}} \\
&\Rightarrow 2\cdot\mathbf{a}\cdot\Delta x = \mathbf{v_f}^2-\mathbf{v_i}^2 \\
&\Rightarrow \mathbf{v_f}^2 = \mathbf{v_i}^2 + 2\cdot\mathbf{a}\cdot\Delta x
\end{aligned}
$$

Q.E.D.

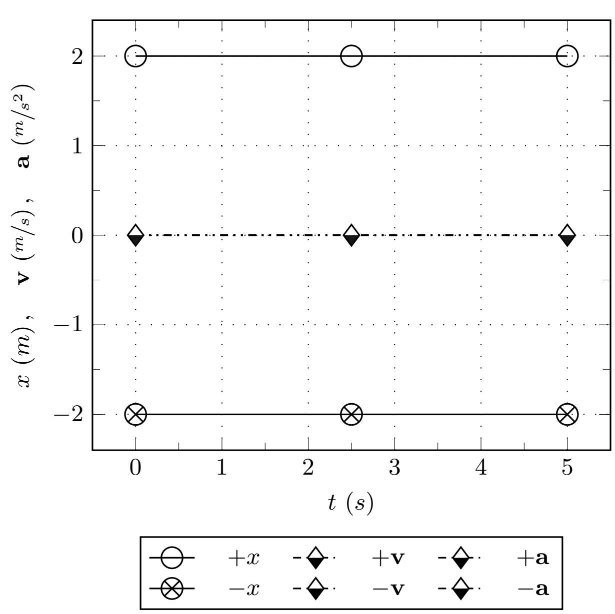
VI. GRAPHS OF PLANE KINEMATICS

Based on what was covered in the earlier chapters and Appendix B, it should now be possible to plot the previously mentioned equations of two-dimensional kinematics according to the different types of motion; in the following subchapters, stationary, constant-speed, speeding-up, and slowing-down objects will be plotted in relation to their positions, velocities, and acceleration over time.

A. Graphs of Stationary Objects

Stationary objects have the most trivial form of motion: "stationary," as the name suggests, refers to an object that lacks motion—an object that is, in other words, standing still so that its velocity (and hence its acceleration) equals zero, regardless of the object's starting position and whether or not it is in the negative side of the y-axis. Shown in the Graph (Fig. 2) are sample Position–Time, Velocity–Time, and Acceleration–Time plots for these types of movements:

Figure 2. The Plots of a Stationary Object



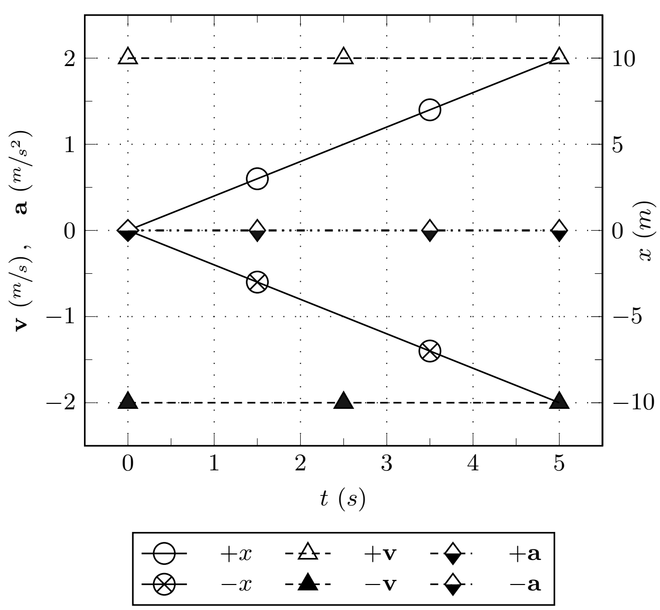
Notice how objects, as long as their x_f and x_i are equal, and regardless of the location of their initial position, are considered to be stationary; consequently, this results in both their $\Delta\mathbf{v}$ and $\Delta\mathbf{a}$ being equal to Δx—naught.

B. Graphs of Constant-Speed Objects

Objects, whose $\Delta\mathbf{v}$ equals a constant, non-changing number, gradually change their position in accordance with the direction of their velocity. Furthermore, in the case of Velocity–Time graphs, **the surface area between the velocity Graph and the x-axis will be the same as the displacement** that has occured during the elapsed timeframe with the specified velocity; for constant-speed objects, this displacement would be equal to the area of a rectangle or square (i.e., $width \times length$, with them corresponding to the time and velocity axes).

Shown below (Fig. 3) are sample plots for an object having a constant velocity:

Figure 3. The Plots of a Constant-Speed Object

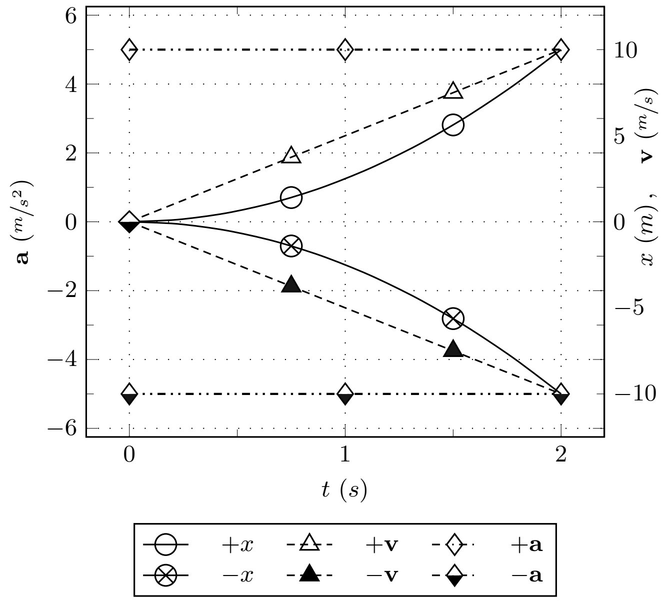


C. Graphs of Accelerating Objects

Accelerating objects shift their position in an exponential manner. As in the previous method of calculating the displacement from the surface area beneath the velocity line and the x-axis, the velocity could also be computed through finding the area covered by the x-axis and the *acceleration* Graph. Also, the displacement, or the area beneath the graph of velocity, now equals the area of a *triangle*: $\frac{(width \times length)}{2}$.

Figure 4 depicts an accelerating object's plots:

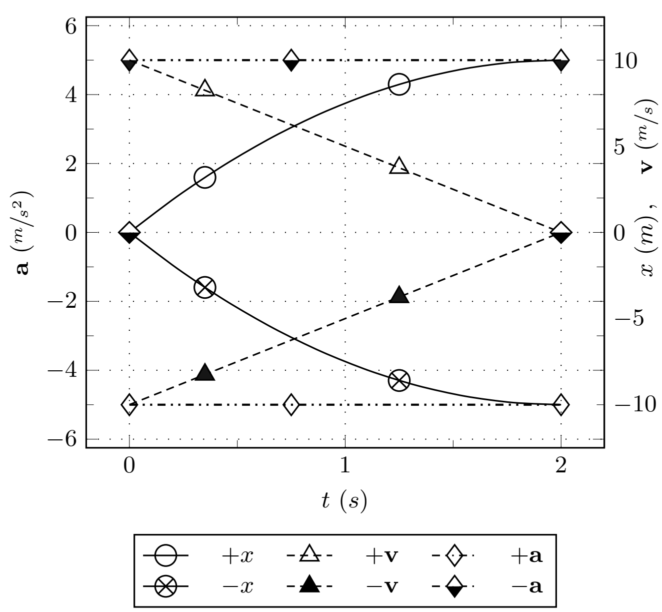
Figure 4. The Plots of an Accelerating Object



D. Graphs of Decelerating Objects

The graphs of decelerating objects are very similar to that of accelerating objects: they both have an exponential increase in position; however, compared to its accelerating counterpart, a decelerating object starts with a non-zero initial velocity ($\mathbf{v_i}$) that continuously decreases as time passes. Consequently, the instantaneous rate of change in position's graph will decrease until it reaches zero and the object ceases to move. Sample plots for a decelerating object are drawn in Figure 5:

Figure 5. The Plots of a Decelerating Object



It is worth noting that acceleration may not always be of the constant type and could be varying; in those cases, to calculate the area under the then-complex curves, one might utilize the use of anti-derivatives and integration.

Appendix A: Algebraic Laws and Properties

Written below in Table I are some of the essential algebraic properties that will be commonly used within kinematics:

Table I. The Basic Properties and Identities in Algebra

Property Name	Property	
	Add./Sub.	Mul./Div.
Commutative	$a + b = b + a$	$a \times b = b \times a$
Associative	$a + (b + c) = (a + b) + c$	$a \times (bc) = (ab) \times c$
Identity	$a \pm 0 = a$	$a \divideontimes 1 = a$
Inverse	$a + (-a) = 0$	$a \times (\frac{1}{a}) = 1$
Distributive	$a \times (b + c) = (a \times b) + (a \times c)$	

Ultimately, as a general rule, **everything that is done to one side of an equation also has to be done to the other side**; this allows for simplifications or solving for other variables. For inequalities, the same generalization applies, with the exception of negative values, in which case the lesser or greater sign would be *reversed.*

Appendix B: Graphs and the Basics of Geometry

a. 2-D graphs A two-dimensional graph has two axes: the **x-axis** and the **y-axis**. The graph may also have *labels* on said axes, in which case it should be referred to as "the graph of [*label of y*] over [*label of x*]." Additionally, points (also called coordinates) consist of one position in the x-axis and one in the y-axis and are expressed as an *ordered pair* of (x, y). Similarly, the horizontal x-axis and the vertical y-axis intersect at a specific point at the origin of the graph, which, in coordinates, is represented as $(0, 0)$. Most of the graphs in physics—especially in kinematics—only have a positive side of the x-axis.

b. Pythagoras theorem The theorem of Pythagoras states as follows: given any right triangle with the angles $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$, where $\widehat{A} = 90°$, exists the equation $a^2 = b^2 + a^2$. (Notice that a is the Hypotenuse—the side in front of which is the ninety degrees angle, $\widehat{A}$.)

c. Trigonometric functions Let α be an angle. So, $\sin(a) = \frac{\textit{opposing side}}{\textit{hypotenuse}}$, $\cos(a) = \frac{\textit{adjacent side}}{\textit{hypotenuse}}$, $\tan(a) = \frac{\textit{opposing side}}{\textit{adjacent side}}$, $\cot(a) = \frac{\textit{adjacent side}}{\textit{opposing side}} = \frac{1}{\tan(a)}$. There are also several rules and identities between these functions, but since most of the calculations in physics are commonly made using a calculator and not manually, they will not be mentioned here; however, knowing them can still assist in having a better understanding of the proof of some physics equations. Lastly, the *inverse* of these functions could be used to find angles from values, e.g., $\tan^{-1}(\frac{\textit{opposing side}}{\textit{adjacent side}}) = \arctan(\frac{\textit{opposing side}}{\textit{adjacent side}}) = \alpha$.

d. Vector addition The generic way to take the summation of two vectors is to use the following formula: $r = \sqrt{a^2 + b^2 - 2 \cdot ab \cdot \cos(\alpha)}$, where r is the resultant vector, a and b are the two additive vectors, and α is the in-between angle. Conversely, by taking the sin and cos functions of the angle α and then multiplying them by the vector r, r could be expanded into two *unit vectors:* i and j, with the former being the horizontal component and the latter the vertical one. If α is smaller than 45 degrees, then cos should be used in order to obtain the larger component, and sin for the smaller one; otherwise, it would the other way around.

Appendix C: Glossary and Elementary Definitions in Physics

a. Vector and scalar quantities Scalar quantities *only* have a magnitude, while vector quantities have a direction *in addition* to a magnitude. For example, mass (m) and heat (Q) are among the scalars, yet force (F) and displacement (x) are vectors. A simple way to understand whether a quantity is a vector or scalar is to ask, "to which direction?" e.g., "to which direction is a force of 450 newtons applied?" If it does make sense, then it is a vector. However, the same question in the context of mass, i.e., "to which direction is this object's mass of 50 kilograms?" would not make sense; thereafter, it becomes clear that mass is a scalar quantity. Lastly, vector variables are typically written with an arrow above them, such as $\overrightarrow{v}$, though they may alternatively be written in bolds, like **v**.

b. Delta (Δ) The difference or change of a given variable is denoted by Delta (Δ). For instance, Δx refers to the difference between the final and initial values of x or, in other words, $x_f - x_i$. (**Note:** Some physics textbooks might use subscripts other than $_f$ and $_i$ [e.g., $_2$ and $_1$, $_1$ and $_0$, or no subscripts and $_0$] to refer to the final and initial values.)

c. SI units The internationally agreed upon (i.e., standardized) units used in physics are referred to as SI units. In other words, even though mass could be expressed using both pounds and kilograms, only the latter (kilograms, kg) is recommended by the SI to be used in calculations. SI was originally an abbreviation for the French phrase *Le Système International d'Unités.*

[1] Organization for Educational Research and Planning (OERP), Iranian Ministry of Education, in *Experimental Sciences of the Ninth Grade of Lower-Secondary High School—134* (The Iranian Textbook Publishing and Writing Company, Tehran, Iran, 2018) Chapter 4, pp. 37–49, 4th ed.