THE TWIN PRIME CONJECTURE IS TRUE

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Abstract. Every prime number \( p \geq 5 \) has the form \( 6x - 1 \) or \( 6x + 1 \). We call \( x \) the generator of \( p \). Twin primes are distinguished by a common generator for each pair. Therefore it makes sense to search for the Twin Primes on the level of their generators. This paper presents a new approach to prove the Twin Prime Conjecture by a method to extract all Twin Primes on the level of the Twin Prime Generators. We define the \( \omega_{pn} \)–numbers \( x \) as numbers for that holds that \( 6x - 1 \) and \( 6x + 1 \) are coprime to the primes \( 5, 7, \ldots, p_n \). By dint of the average size \( \delta(p_n) \) of the \( \omega_{pn} \)–gaps we can prove the Twin Prime Conjecture.

1. Introduction

The question on the infinity of the twin primes keeps busy many mathematicians for a long time. 1919 V. Brun [3] had proved that the series of the inverted twin primes converges while he had tried to prove the Twin Prime Conjecture. Several authors worked on bounds for the length of prime gaps (see f.i. [4, 5, 6]). In 2008 B. Green and T. Tao [7] succeeded in proving that there are arbitrarily long arithmetic progressions containing only prime numbers. 2014 Y. Zhang [8] obtained a great attention with his proof that there are infinitely many consecutive primes with a gap of \( 70,000,000 \) at most. With the project "PolyMath8", in particular forced by T. Tao [9], this bound could be lessen down to \( 246 \) respectively to \( 12 \) assuming the validity of the Elliott–Halberstam Conjecture [10].

We present in this paper another approach as in the most works on this topic. We transfer the looking for twin primes to the level of their generators because each twin prime has a common generator.

2. Twin Prime Generators

It is well known that every prime number \( p \geq 5 \) has the form \( 6x - 1 \) or \( 6x + 1 \). We will call \( x \) the generator of \( p \). Twin primes are distinguished by a common generator for each pair.
Definition 2.1. Let

\( \mathbb{N} \) be the set of the positive integers,
\( \mathbb{P} \) the set of the prime numbers, \( \mathbb{P}^* \) primes \( \geq 5 \),
\( \mathbb{P}_- = \{ p \in \mathbb{P}^* \mid p \equiv -1(\text{mod } 6) \} \), \( \mathbb{P}_+ = \{ p \in \mathbb{P}^* \mid p \equiv +1(\text{mod } 6) \} \)

and

\[ \kappa(n) := \left\lfloor \frac{n + 1}{6} \right\rfloor \text{ for } n \in \mathbb{N} \quad (2.1) \]

the generator function of the pair \((6\kappa(n) - 1, 6\kappa(n) + 1)\). If a pair \((6x - 1, 6x + 1)\) is a twin prime then we call \(x\) as a twin prime generator and

\[ \mathbb{G} := \{ x \in \mathbb{N} \mid 6x - 1 \in \mathbb{P}_-, 6x + 1 \in \mathbb{P}_+ \} \]

is the set of all twin prime generators. Hence the pair \((5, 7)\) is the least twin prime in our consideration.

In order to transfer the searching for twin primes to the level of their generators we need a criterion for checking a natural number to be a twin prime generator.

Theorem 2.2. A number \(x\) is a twin prime generator, a member of \(\mathbb{G}\), if and only if there is no \(p \in \mathbb{P}^*\) with \(p < 6x - 1\) that one of the following congruences fulfills:

\[ x \equiv -\kappa(p)(\text{mod } p) \quad (2.2) \]
\[ x \equiv +\kappa(p)(\text{mod } p) \quad (2.3) \]

Proof. At first we assume that there is a prime \(p \in \mathbb{P}^*\) with \(p < 6x - 1\) such that (2.2) or (2.3) is valid. There are two cases.

1. \(p \in \mathbb{P}_-\), what means \(p = 6\kappa(p) - 1\):
   - If (2.2) is true then there is an \(n \in \mathbb{N}\) with
     \[ x = -\kappa(p) + n \cdot (6\kappa(p) - 1) \]
     \[ 6x = -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) \]
     \[ 6x + 1 = -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) + 1 \]
     \[ = (6n - 1)(6\kappa(p) - 1) \]
     \[ = (6n - 1) \cdot p \]
     \[ \implies 6x + 1 \equiv 0(\text{mod } p) \implies x \notin \mathbb{G} \]
   - For (2.3) the proof will be done with \(6x - 1\):
     \[ 6x - 1 = 6\kappa(p) + 6n \cdot (6\kappa(p) - 1) - 1 \]
     \[ = (6n + 1)(6\kappa(p) - 1) \]
     \[ = (6n + 1) \cdot p \]
     \[ \implies 6x - 1 \equiv 0(\text{mod } p) \implies x \notin \mathbb{G} \]

2. \(p \in \mathbb{P}_+\), what means \(p = 6\kappa(p) + 1\):
   - We go the same way with (2.2) and \(6x - 1\) as well as (2.3) and \(6x + 1\):
     \[ 6x - 1 = (6n - 1)(6\kappa(p) + 1) \implies 6x - 1 \equiv 0(\text{mod } p) \]
     \[ 6x + 1 = (6n + 1)(6\kappa(p) + 1) \implies 6x + 1 \equiv 0(\text{mod } p) \]
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With these it’s shown that \( x \notin \mathbb{G} \) if the congruences (2.2) or (2.3) are valid. They cannot be true both because they exclude each other.

If on the other hand \( x \notin \mathbb{G} \), then \( 6x - 1 \) or \( 6x + 1 \) is no prime. Let be \( 6x - 1 \equiv 0 \pmod{p} \) for any \( p \in \mathbb{P}_- \). Then we have

\[
\begin{align*}
6x - 1 & \equiv (6\kappa(p) - 1) \pmod{p} \\
6x & \equiv 6\kappa(p) \pmod{p} \\
x & \equiv \kappa(p) \pmod{p} \implies (2.3).
\end{align*}
\]

For any \( p \in \mathbb{P}_+ \) we have

\[
\begin{align*}
6x - 1 & \equiv -p \pmod{p} \\
& \equiv -(6\kappa(p) + 1) \pmod{p} \\
6x & \equiv -6\kappa(p) \pmod{p} \\
x & \equiv -\kappa(p) \pmod{p} \implies (2.2).
\end{align*}
\]

The other both cases we can handle in the same way. Therefore either (2.2) or (2.3) is valid if \( x \notin \mathbb{G} \). \( \square \)

If we consider that the least proper divisor of a number \( 6x + 1 \) is less or equal to \( \sqrt{6x + 1} \) then \( p \) in the congruences (2.2) and (2.3) can be limited by

\[ \hat{p}(x) = \max(p \in \mathbb{P}^* \mid p \leq \sqrt{6x + 1}). \]

Remark 2.3. Henceforth we will use the letter \( p \) for a general prime number and \( p_n \) if we describe an element of a sequence of primes.

With \( p_n \) as the \( n \)-the prime number\(^1\) and \( \pi(z) \) as the number of primes \( \leq z \) we have with

\[
\begin{align*}
x & \equiv -\kappa(p_n) \pmod{p_n} \text{ or } x \equiv +\kappa(p_n) \pmod{p_n} \quad (2.4)
\end{align*}
\]

for \( 3 \leq n \leq \pi(\hat{p}(x)) \) a provable system of criteria to exclude all numbers \( x \geq 4 \) being no twin prime generators. Since the modules are primes the criteria are independent among each other.

Hence we can square the congruences (2.4) and get

\[
x^2 \equiv \kappa(p_n)^2 \pmod{p_n} \text{ for } 3 \leq n \leq \pi(\hat{p}(x)). \quad (2.5)
\]

This results in a system of indicator functions \( \psi(x, p_n) \) for that holds for \( 3 \leq n \leq \pi(\hat{p}(x)) \)

\[
x^2 - \kappa(p_n)^2 \equiv \psi(x, p_n) \pmod{p_n} \text{ respectively} \\
\psi(x, p_n) = (x^2 - \kappa(p_n)^2) \pmod{p_n}.
\] \( (2.6) \]

For the sake of completeness we define \( \psi(x, p_n) = 0 \) for \( x \leq \kappa(p_n) \).

Obviously holds that if \( \psi(x, p) = 0 \) for any \( p \leq \hat{p}(x) \) then \( x \) cannot be a twin prime generator.\(^2\) Due to modulo the indicator function \( \psi(x, p) \) is periodical in \( x \)

\(^1\)It is \( p_1 = 2 \)

\(^2\)We consider only cases \( x > \kappa(p_n) \) in the sequel.
with a period length of $p$. With the indicator functions for all modules $p_3, \ldots, p_n$ we build the aggregate indicator functions

$$
\Psi(x, p_n) = \prod_{i=3}^{n} \frac{\psi(x, p_i)}{p_i}
$$

and

$$
\hat{\Psi}(x) = \Psi(x, \hat{p}(x)).
$$

Because the co-domain of $\psi(x, p)$ consists of integers between 0 and $p - 1$, the aggregate functions $\Psi(x, p)$ and $\hat{\Psi}(x)$ have rational values between 0 and $\frac{1}{p}$.

**Definition 2.4.** Let be

$$
\xi_n := \min(x \in \mathbb{N} \mid \hat{p}(x) = p_n).
$$

It is the first integer $x$ for that the modul $p_n$ could be a prime factor of $6x \pm 1$. Therefore we denote it as the origin of the modul $p_n$.

Up from $\xi_n$ in every $\psi$-period there are just $p_n - 2$ positions with $\psi(x, p_n) > 0$ and two positions with $\psi(x, p_n) = 0$, once if (2.2) and on the other hand if (2.3) holds. Obviously the distance between these both positions is $2\kappa(p_n)$ respectively $4\kappa(p_n) \pm 1$ since the period length is $6\kappa(p_n) \pm 1$.

It is $p_n \leq \hat{p}(x) \leq \sqrt{6x + 1}$ and therefore $p_n^2 \leq 6x + 1$. Then $\frac{p_n^2 - 1}{6}$ is the least number that meets this relation. Comparing with (2.8) we get

$$
\xi_n = \frac{p_n^2 - 1}{6}.
$$

**Lemma 2.5.** At the origin $\xi_n$ cannot be a twin prime generator since we have $\psi(\xi_n, p_n) = 0$.

**Proof.** We substitute $p_n$ by $6\kappa(p_n) \pm 1$. With this and (2.9) holds

$$
\xi_n = \frac{(6\kappa(p_n) \pm 1)^2 - 1}{6}
$$

$$
= \frac{6\kappa(p_n) (6\kappa(p_n) \pm 1 \pm \kappa(p_n)}{6}
$$

$$
= \frac{\kappa(p_n) \cdot p_n \pm \kappa(p_n)}{6}
$$

$$
= \pm \kappa(p_n)(\text{mod } p_n) \implies \psi(\xi_n, p_n) = 0.
$$

In the proof we have seen that for every prime $p$ holds that $p^2 - 1$ is an integer divisible by 6. For every $x \geq \xi_n$ the local position relative to the period start $^3$ can be determined by the position function $\tau(x, p_n)$:

$$
x + \kappa(p_n) \equiv \tau(x, p_n)(\text{mod } p_n) \quad \text{respectively}
$$

$$
\tau(x, p_n) = (x + \kappa(p_n)) \text{ Mod } p_n.
$$

$^3$For $p_n \in \mathbb{P}$ the period start is $\xi_n$ and else it is $\xi_n - 2\kappa(p_n)$. 
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Hence the co-domain of the position function \( \tau(x, p) \) are the integers \( 0, 1, \ldots, p - 1 \) and it has in \( x \) a period length of \( p \). Between the indicator function \( \psi(x, p) \) and the position function \( \tau(x, p) \) there is the following relationship:

\[
\psi(x, p) = \tau(x, p) \cdot (x - \kappa(p)) \mod p
\]

Observe that \( \psi(x, p) = 0 \) if and only if \( \tau(x, p) = 0 \) or \( \tau(x, p) = 2\kappa(p) \). From (2.10) we see that the values of the position function \( \tau(x, p) \) consists exactly of the \( p \) values \( 0, 1, 2, \ldots, p - 1 \). Because the values 0 and \( 2\kappa(p) \) indicate that \( x \) cannot be a twin prime generator (see Theorem 2.2) we will call them as \( \tau \)-bad values and the others as \( \tau \)-good values. Hence there are two \( \tau \)-bad values and \( p - 2 \) \( \tau \)-good values for each modul \( p \).

3. The \( \omega_{p_n} \)-numbers

For every natural number \( x \) in the interval

\[
\mathcal{A}_n := [\xi_n, \xi_{n+1} - 1]
\]

\( \hat{p}(x) \) persists constant on the value \( p_n \). The length of this interval \(^4\) will be denoted as \( d_n \). It is depending on the distance between successive primes. Since they can only be even, we have with \( a = 2, 4, 6, \ldots \)

\[
d_n = \frac{(p_n + a)^2 - 1}{6} - \frac{p_n^2 - 1}{6}
= \frac{2ap_n + a^2}{6}
= \frac{a}{3}(p_n + \frac{a}{2})
\geq \frac{2}{3}(p_n + 1).
\]

(3.2)

On the other hand we obtain because of \( p_{n+1} < 2p_n \) (see [2], p. 188)

\[
d_n = \frac{p_{n+1}^2 - 1}{6} - \frac{p_n^2 - 1}{6}
= \frac{p_{n+1}^2 - p_n^2}{6}
= \frac{(p_{n+1} + p_n)(p_{n+1} - p_n)}{6}
\leq \frac{3p_n \cdot p_n}{6} = \frac{p_n^2}{2}.
\]

And since \( d_n \) is an integer and \( p_n^2 \) is odd it holds

\[
d_n \leq \frac{p_n^2 - 1}{2} = 3\xi_n \quad \text{and hence}
\]

\[
\xi_{n+1} \leq 4\xi_n.
\]

(3.4)

The congruences in (2.10)

\[
x + \kappa(p_i) \equiv \tau(x, p_i)(\mod p_i), \quad 3 \leq i \leq n
\]

(3.5)

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\(^4\)Really is \( \mathcal{A}_n := [\xi_n, \xi_{n+1} - 1] \cap \mathbb{N} \). Henceforth all intervals will be defined as sections of the number line.
fulfill the requirements of the Chinese Remainder Theorem (see [1], p. 89). Therefore it is modulo $5 \cdot 7 \cdot \ldots \cdot p_n$ uniquely resolvable. With

$$p_n \#_5 := \prod_{i=3}^{n} p_i = 5 \cdot 7 \cdot \ldots \cdot p_n$$  \hspace{1cm} (3.6)$$
it’s \pmod{p_n \#_5}$ uniquely resolvable. Therefore the aggregate indicator function $\Psi(x, p_n)$ has the period length $p_n \#_5$ and it holds:

$$\Psi(x + a \cdot p_n \#_5, p_n) = \Psi(x, p_n) \mid a \in \mathbb{N}.$$  

**Definition 3.1.** A positive integer $x$ will be called as an $\omega_{p_n}$–number if both $6x - 1$ and $6x + 1$ are coprime to $p_n \#_5$. Then is $\Psi(x, p_n) > 0$.

**Corollary 3.2.** Because of the periodicity of the aggregate indicator function $\Psi(x, p_n)$ in $x$ there are infinitely many $\omega_{p_n}$–numbers.

**Definition 3.3.** Let be $P_n := [\xi_n, \xi_n + p_n \#_5 - 1]$ the interval of one period of the aggregate indicator function $\Psi(x, p_n)$. We’ll denote it henceforth as period section. Evidently is $A_n \subset P_n$ for all $n \geq 3$.

**Lemma 3.4.** The period section $P_n$ contains

$$\phi(p_n) := \prod_{k=3}^{n} (p_k - 2)$$  \hspace{1cm} (3.7)$$
$\omega_{p_n}$–numbers.

**Proof.** Due to Definition 3.3 the period section $P_n$ contains $p_n \#_5$ successive natural numbers $\xi_n, \xi_n + 1, \ldots, \xi_n + p_n \#_5 - 1$. Each sequence of $p_k$ successive members of them for $3 \leq k \leq n$ contains due to (2.11) two members $x_{\mp}$ with the two $\tau$–bad values

$$\tau(x_{-}, p_k) = 0 \text{ and } \tau(x_{+}, p_k) = 2\kappa(p_k)$$

and $p_k - 2$ members $y$ with the $p_k - 2$ $\tau$–good values. Each $\omega_{p_n}$–number $y \in \mathcal{P}_n$ can be represented by exactly one $m$–tuple with $m = n - 2$

$$\tau_3, \tau_4, \ldots, \tau_n),$$

where all values $\tau_k = \tau(y, p_k)$ are $\tau$–good values and since by virtue of (2.11) for only $\tau$–good values holds

$$\Psi(y, p_n) = \prod_{k=3}^{n} \tau_k (\tau_k - 2\kappa(p_k) \text{ Mod } p_k) > 0.$$  

Because the primes $5, 7, \ldots, p_n$ are independent modules, all the $m$–tuples are different and their number is

$$\prod_{k=3}^{n} (p_k - 2).$$

Since each such $m$–tuple represents exactly one different $\omega_{p_n}$–number in $\mathcal{P}_n$ it holds

$$\phi(p_n) = \prod_{k=3}^{n} (p_k - 2).$$

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5It is $p_n \#_5 = \xi_n \#$, with the primorial $p_n \#$.

6Then is $\gcd(36x^2 - 1, p_n \#_5) = 1$. 
Corollary 3.5. Let $x$ be an $\omega_{p_n}$–number as a member of $A_n$. Then $x$ is a twin prime generators because by virtue of Definition 3.1 $6x - 1$ as well as $6x + 1$ are prime to $5, 7, \ldots, p_n$ and it holds $\Psi(x, p_n) = \hat{\Psi}(x) > 0$.

Vice versa an $\omega_{p_m}$–number as member of $A_n$ by $m < n$ is not necessarily a twin prime generator.

The ratio between (3.7) and the period length by virtue of (3.6) results in

$$\eta(p_n) := \frac{\phi(p_n)}{p_n \#_5} = \prod_{i=3}^{n} \frac{p_i - 2}{p_i},$$

as the average density of the $\omega_{p_n}$–numbers in $P_n$. Obviously $\eta(p)$ is a strictly monotonic decreasing function.

Lemma 3.6. The average density function $\eta(p_n)$ is double-sided bounded by

$$\frac{3}{p_n} < \eta(p_n) < \frac{3}{\log p_n}.$$

Proof.

A) At first we prove the left inequality.

Let be $Q_n = \{ p \in P^* \mid p \leq p_n \}$ and $U_n$ the set of all odd numbers from 5 to $p_n$. Because all primes $> 2$ are odd numbers it holds $Q_n \subseteq U_n$. All factors of $\eta(p_n)$ are less than 1. Thus we get with $m = \frac{p_n+1}{2}$

$$\eta(p_n) = \prod_{k=3}^{n} \frac{p_k - 2}{p_k} > \prod_{k=3}^{m} \frac{u_k - 2}{u_k} = \prod_{k=3}^{m} \frac{2k - 2}{2k - 1} = \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdots \frac{p_n - 4}{p_n - 2} \cdot \frac{p_n - 2}{p_n} = \frac{3}{p_n}.$$

B) It’s well known that (see [1], p. 40 above)

$$\log x < \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{k=1}^{\pi(x)} \left(\frac{p_k - 1}{p_k}\right)^{-1}.$$

Therefore is

$$\frac{1}{\log p_n} > \prod_{k=1}^{n} \frac{p_k - 1}{p_k} = \frac{1}{2} \cdot 2 \cdot \prod_{k=3}^{n} \frac{p_k - 1}{p_k} = \frac{1}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 1}{p_k} = \frac{1}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 2}{p_k} = \frac{1}{3} \cdot \eta(p_n) = \frac{1}{3} \cdot \frac{\eta(p_n)}{3}.$$

Hence we get for the right inequality $\eta(p_n) < \frac{3}{\log p_n}$.

With it the proof is completed. □
Corollary 3.7. Since both bounds in Lemma 3.6 go to zero holds
\[ \lim_{n \to \infty} \eta(p_n) = 0. \]
This means that the \( \omega_{p_n} \)-numbers have an asymptotic zero-density. Therefore the twin prime generators as subset of the \( \omega_{p_n} \)-numbers have an asymptotic zero-density too. This result is in accordance with the fact that also the set of the primes has an asymptotic zero-density.

3.1. The Symmetry of the \( \omega_{p_n} \)-Numbers. In the period section \( \mathcal{P}_n \) the element
\[ x^{(0)}_n := p_n \#_5 \]
has a particular importance. Because \( p_n \#_5 \) is divisible by all primes between 5 and \( p_n \) it holds
\[ p_n \#_5 \equiv 0 (\text{mod } p_m) \mid 3 \leq m \leq n, p_m \in \mathbb{P}^* \]
and hence
\[ x^{(0)}_n \not\equiv \pm \kappa(p_m)(\text{mod } p_m) \mid 3 \leq m \leq n, p_m \in \mathbb{P}^*. \]
Hence \( x^{(0)}_n \) is an \( \omega_{p_n} \)-number and it holds
\[ \Psi(x^{(0)}_n, p_n) > 0. \tag{3.9} \]
Because of \( \xi_n < p_n \#_5 < \xi_n + p_n \#_5 \) the number \( x^{(0)}_n \) is in the inner of \( \mathcal{P}_n \) but near to the end.

Theorem 3.8. The \( \omega_{p_n} \)-numbers are symmetrically distributed around the axis
\[ x^{(0)}_n \]
for any positive integer \( a < x^{(0)}_n \).

Proof. Since by virtue of (2.7) the aggregate indicator function \( \Psi(x, p_n) \) consists of the product of the indicator functions \( \psi(x, 5), \ldots, \psi(x, p_n) \), the \( \omega_{p_n} \)-numbers are symmetrically arranged around the axis \( x^{(0)}_n \) if and only if holds for \( m = 3, \ldots, n \)
\[ \psi(x^{(0)}_n - a, p_m) = \psi(x^{(0)}_n + a, p_m) \tag{3.10} \]
with any number \( a < x^{(0)}_n \). From (2.6) we have
\[ \psi(x, p_m) = (x^2 - \kappa(p_m)^2) \text{ Mod } p_m. \]
With \( x^{(0)}_n \pm a \) for \( x \) we get
\[ \psi(x^{(0)}_n \pm a, p_m) = \left((x^{(0)}_n \pm a)^2 - \kappa(p_m)^2\right) \text{ Mod } p_m \]
\[ = \left(x^{(0)}_n x^{(0)}_n \pm 2a + a^2 - \kappa(p_m)^2\right) \text{ Mod } p_m \]
and since \( x^{(0)}_n \equiv 0 (\text{mod } p_m) \)
\[ = (a^2 - \kappa(p_m)^2) \text{ Mod } p_m. \tag{3.11} \]
Since \( \psi(x^{(0)}_n \pm a, p_m) \) result in a common value it follows (3.10). \qed

\footnote{Unless otherwise specified the use of the variable \( p_m \) means below \( p_m \in \mathbb{P}^* \mid 3 \leq m \leq n. \)}
Additionally from (3.11) we see that
\[ \psi(x_n^{(0)} \pm a, p_m) = 0 \text{ for } a = \kappa(p_m). \]
This means that at these positions there cannot be \( \omega_{p_n} \)-numbers. Around the axis \( x_n^{(0)} \) there are \( \omega_{p_n} \)-gaps with the length \( \kappa(p_n) \) but an \( \omega_{p_n} \)-gap cannot reach over \( x_n^{(0)} \).

If we limit our consideration to the period section \( \mathcal{P}_n \) then there is a section of symmetry with a length of \( 2\xi_n \) at the end of \( \mathcal{P}_n \) around \( x_n^{(0)} = p_n \# 5 \). But in the remaining of the period section there is symmetry too. Let \( x_n^{(1)} := \frac{p_n \# 5}{2} \) be a rational number as the middle between the integers
\[ x_n^{(1-)} := \frac{p_n \# 5 - 1}{2} \quad \text{and} \quad x_n^{(1+)} := \frac{p_n \# 5 + 1}{2}. \]

**Theorem 3.9.** The \( \omega_{p_n} \)-numbers are symmetrically distributed around the axis \( x_n^{(1)} \)
\[ \psi(x_n^{(1-)} - a, p_n) = \psi(x_n^{(1+)} + a, p_n) \]
for any positive integer \( a < x_n^{(1)} \).

**Proof.** Because of the periodicity of the aggregate indicator function we have
\[ \psi(\xi_n + a, p_n) = \psi(p_n \# 5 + \xi_n + a, p_n) \]
and because of Theorem 3.8
\[ = \psi(p_n \# 5 - \xi_n - a, p_n). \quad (3.12) \]
Therefore there is symmetry around
\[ \frac{\xi_n + a + p_n \# 5 - \xi_n - a}{2} = \frac{p_n \# 5}{2} = x_n^{(1)}. \]
We set \( x_n^{(1+)} \) instead of \( \xi_n \) in (3.12) and get
\[ \psi \left( x_n^{(1+)} + a, p_m \right) = \psi \left( p_n \# 5 - (x_n^{(1+)} + a), p_m \right) \]
\[ = \psi \left( \frac{p_n \# 5 - p_n \# 5 + 1}{2} - a, p_m \right) \]
\[ = \psi \left( \frac{p_n \# 5 - 1}{2} - a, p_m \right) \]
\[ = \psi \left( x_n^{(1-)} - a, p_m \right) \]
\[ \square \]

Figure 1 shows schematically (and not in scale) the sections of symmetry in \( \mathcal{P}_n \).
The section around \( x_n^{(1)} \) has a length of \( p_n \# 5 - 2\xi_n \). It increases more quickly than the length of the section around \( x_n^{(0)} \) with \( 2\xi_n \) because of \( p_n \# 5 \) increases more quickly than \( \xi_n \).
3.2. The Overlapping of the period sections. The intervals \( A_n, n \geq 3 \) defined by (3.1) cover the positive integers \( \geq 4 \) gapless and densely. It is
\[
\mathbb{N} = \{1, 2, 3\} \cup \bigcup_{n=3}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=3}^{\infty} A_n = \emptyset.
\]
They are the beginnings of the period sections \( P_n \) of the \( \omega_{p_n} \)-numbers. Hereafter let’s say \textbf{A-sections} to the intervals \( A_n \). Every \( \omega_{p_n} \)-number that lies in an A-section is a twin prime generator (see Corollary 3.5). In contrast to the A-sections the period sections \( P_n \) overlap each other very closely. So the period section \( P_9 \) reaches over 1739 A-sections up to the beginning of the period section \( P_{1748} \) and the next \( P_{10} \) over 7863 A-sections up to the beginning of \( P_{7873} \).

**Lemma 3.10.** Each origin \( \xi_n \) cannot be located at the beginning \( \xi_m + a \cdot p_m \#_5 \) of any period of the aggregate indicator function \( \Psi(x, p_m) \) for \( n > m \) and \( a \in \mathbb{N} \). Therefore it holds for \( n > m \)
\[
\xi_n \not\equiv \xi_m \pmod{p_m \#_5}.
\]
**Proof.** The equation
\[
\frac{p_{m}^{2} - 1}{6} + a \cdot p_m \#_5 = \frac{p_{n}^{2} - 1}{6} \quad \text{and hence} \quad p_m(p_m + a \cdot p_{m-1} \#) = p_{n}^{2}
\]
is for no prime \( p_n > p_m \) solvable since holds \( \gcd(p_n, p_m) = 1 \).

Vice versa every period section \( P_{n+1} \) starts always inside of the previous period section \( P_n \) nearby to its origin because (see (3.3) too)
\[
\xi_{n+1} = \xi_n + d_n \quad \text{and} \quad d_n < \frac{p_{n}^{2}}{2} \ll \frac{p_n \#_5}{2}.
\]

**Lemma 3.11.** Even it holds
\[
\xi_n \not\equiv \xi_m \pmod{p_m} \quad \text{for all} \quad n > m.
\]
**Proof.** We assume contrarily \( \xi_n = \xi_m + a \cdot p_m \) for any \( a \in \mathbb{N} \). Analogously to the proof of Lemma 3.10 we multiply by 6 and get
\[
p_{n}^{2} = p_m(p_m + 6a).
\]
Also this equation is since \( \gcd(p_{m}, p_n) = 1 \) for no \( p_m, p_n \in \mathbb{P} \) solvable. \( \square \)
4. The $\omega_{p_n}$-Gaps

**Definition 4.1.** The inverse of $\eta(p_n)$ means the **average distance** between two immediately successive $\omega_{p_n}$-numbers in the period section $P_n$

$$\overline{\delta}(p_n) := \frac{1}{\eta(p_n)},$$  \hspace{1cm} (4.1)

the average size of the so called $\omega_{p_n}$-gaps.

**Corollary 4.2.** From Lemma 3.6 follows also

$$\lim_{n \to \infty} \overline{\delta}(p_n) = \infty.$$

**Theorem 4.3.** For $p_n > 200$ the length of the interval $A_n$ is greater than the square of the average size of the $\omega_{p_n}$-gaps

$$d_n > \overline{\delta}(p_n)^2 \text{ for } p_n > 200.$$

**Proof.** At first we prove that $u(p_n) := p_n \eta(p_n)^2$

is an increasing function by trend. We consider their properties for two cases:

**A)** $p_{n+1} \geq p_n + 4$:

$$u(p_{n+1}) - u(p_n) = \eta(p_n)^2 \left( p_{n+1} \frac{(p_{n+1} - 2)^2}{p_{n+1}^2} - p_n \right)$$

$$= \eta(p_n)^2 \left( \frac{p_{n+1}(p_{n+1} - 4) + 4}{p_{n+1}} - p_n \right)$$

$$= \eta(p_n)^2 \left( p_{n+1} - 4 - p_n + \frac{4}{p_{n+1}} \right)$$

$$\geq \frac{4\eta(p_n)^2}{p_{n+1}} > 0.$$

Hence it holds in this case $u(p_{n+1}) > u(p_n)$.

**B)** $p_{n+1} = p_n + 2$:

$$u(p_{n+1}) - u(p_n) = \eta(p_n)^2 \left( p_{n+1} \frac{(p_{n+1} - 2)^2}{p_{n+1}^2} - p_n \right)$$

$$= p_n \eta(p_n)^2 \left( \frac{p_n}{p_{n+1}} - 1 \right)$$

$$= p_n \eta(p_n)^2 \cdot \left( \frac{p_n - p_{n+1}}{p_{n+1}} \right)$$

$$= -\frac{2p_n}{p_n + 2} \eta(p_n)^2 < 0.$$  

Hence holds $u(p_{n+1}) < u(p_n)$. Now we set $u(p_{n+1}) = u(p_n) - v(p_n)$ with the "loss function"

$$v(p_n) := \frac{2p_n}{p_n + 2} \eta(p_n)^2.$$
At first we'll look for the behavior of \( u(p_{n+2}) \) depending on the prime distance
\[
a := p_{n+2} - p_{n+1} = p_{n+2} - p_{n} - 2 \quad \text{for } a = 4, 6, 10, 12, 16, \ldots
\]
With it we get
\[
u(p_{n+2}) = p_{n+2} \cdot \eta(p_{n+2})^2
= (p_n + a + 2) \cdot \eta(p_n + a + 2)^2
= (p_n + a + 2) \cdot \frac{(p_n + a)^2}{(p_n + a + 2)^2} \cdot \frac{p_n^2}{(p_n + 2)^2} \cdot \eta(p_n)^2
= u(p_n) \cdot \frac{(p_n + a)^2 \cdot p_n}{(p_n + a + 2)(p_n + 2)^2}
\]
We consider the difference between nominator and denominator of the fraction
\[
(p_n + a)^2 \cdot p_n - (p_n + a + 2)(p_n + 2)^2
= (p_n^2 + 2ap_n + a^2)p_n - (p_n + a + 2)(p_n^2 + 4p_n + 4)
= p_n^3 + 2ap_n^2 + a^2p_n - p_n^3 - 4p_n^2 - 4p_n - ap_n^2 - 4ap_n - 4a - 2p_n^2 - 8p_n - 8
= (a - 6)p_n^2 + (a^2 - 4a - 12)p_n - 4a - 8
\]
and get for
\[
a = 4 \rightarrow -2p_n^2 - 12p - 24 < 0
a = 6 \rightarrow -32 < 0
a = 10 \rightarrow 4p_n^2 + 48p - 48 > 0 \quad | p \geq 2.
\]
This means that \( u(p_{n+2}) < u(p_n) \) for \( a = 4, 6 \) and \( u(p_{n+2}) > u(p_n) \) for \( a \geq 10 \).

With an analogous procedure we can demonstrate even for the case \( p_{n+3} = p_{n+2} + 2 \) that also holds
\[
u(p_{n+3}) > u(p_n)
\]
if \( a \geq 10 \). In the case \( p_{n+3} > p_{n+2} + 2 \) it holds \( u(p_{n+3}) > u(p_n) \) because of \( A \). It seems to be important to emphasize that all these results hold for the case \( p_{n+1} - p_n = 2 \).

The loss function \( v(p_n) \) is strictly monotonic decreasing, because for two twin primes \( p_n, p_n + 2 \) und \( p_n + 2 + a, p_n + 4 + a \) with the distance \( a \) holds
\[
v(p_n + 2 + a) = \frac{2(p_n + 2 + a)}{p_n + 4 + a} \eta(p_n + 2 + a)^2
\]
and since all factors of \( \eta(p) < 1 \)
\[
\leq \frac{2}{p_n + 4 + a} \cdot \frac{(p_n + a)^2}{p_n + 2 + a} \eta(p_n + 2)^2
= \frac{2}{p_n + 4 + a} \cdot \frac{(p_n + a)^2}{p_n + 2 + a} \cdot \frac{p_n^2}{(p_n + 2)^2} \eta(p_n)^2
= v(p_n) \cdot \frac{p_n}{p_n + 2} \cdot \frac{p_n + a}{p_n + 2 + a} \cdot \frac{p_n + a}{p_n + 4 + a} < v(p_n).
\]
From twin prime to twin prime the loss function \( v(p_n) \) monotonically decreases for each twin distance \( a \).

As upshot we see that in the majority of cases \( u(p) \) is an increasing function while the loss function \( v(p) \) from twin prime to twin prime decreases. The function \( u(p) \) tends to result in an increasing function.

The greatest twin prime \(< 200 \) is \((197, 199)\). For the next prime number \( 211 \) holds \( u(211) > 1.5159 = \frac{3}{2} + 0.0159 \). Since the next prime after \( 211 \) follows only at \( 223 \) therefore for no prime \( p > 211 \) is \( u(p) < u(211) \). On the other hand since \( v(p_n) \) is monotonically decreasing, is \( v(197) < 0.0148 < 0.0159 \) and all further \( v(p_{n+k}) \) are even less. Therefore we get for \( p_n > 200 \) with (3.2) and (4.1)

\[
u(p_n) = p_n\eta(p_n)^2 > \frac{3}{2} \rightarrow \delta(p_n)^2 < \frac{2}{3} p_n < \frac{2}{3} (p_n + 1) \leq d_n.
\]

This completes the proof. \( \square \)

**Corollary 4.4.** Since \( u(p) \) is an increasing function by trend there is always a prime \( p \) and a number \( c_p > 1 \) such that holds

\[c_p \cdot \delta(p_n)^2 < d_n \text{ for } p_n > p.\]

For instance we get for

\[
\begin{align*}
p_n > 1,277 &\rightarrow 2\delta(p_n)^2 < d_n, \\
p_n > 25,561 &\rightarrow 10\delta(p_n)^2 < d_n, \\
p_n > 77,291 &\rightarrow 20\delta(p_n)^2 < d_n \text{ or for} \\
p_n > 830,293 &\rightarrow 100\delta(p_n)^2 < d_n.
\end{align*}
\]

**Corollary 4.5.** Since by virtue of Theorems 3.8 and 3.9 the \( \omega_{p_n} \)-numbers are symmetrically distributed around the axes \( x_n(0) = p_n \#_5 \) as well as \( x_n(1) = \frac{p_n \#_5}{2} \) therefore the \( \omega_{p_n} \)-gaps are symmetrically distributed with respect to their sizes.

Since \( 4 \not\equiv 0 \pmod{p_n \#_5} \) we can multiply (2.6) by 4 and have

\[4\psi(x, p_m) \Mod p_m = ((2x)^2 - (2\kappa(p_m))^2) \Mod p_m \]

and get for \( x = \frac{p_n \#_5 \pm 1}{2} \)

\[4\psi\left(\frac{p_n \#_5 \pm 1}{2}, p_m\right) \Mod p_m = \left((p_n \#_5 \pm 1)^2 - 4\kappa(p_m)^2\right) \Mod p_m = \left(1 - 4\kappa(p_m)^2\right) \Mod p_m.\]

Because this never can be zero therefore at the positions \( \frac{p_n \#_5 \pm 1}{2} \) are \( \omega_{p_n} \)-numbers. Hence no \( \omega_{p_n} \)-gap can reach over the symmetry axis \( x_n(1) \). Each \( \omega_{p_n} \)-gap has in \( P_n \) always a symmetry partner, each \( \omega_{p_n} \)-gap occurs twice in \( P_n \).
With the transition $p_n \to p_{n+1}$ the symmetry of the $\omega_{p_n}$-numbers in their period sections $\mathcal{P}_n$ repeats $p_{n+1}$-times oneself in $\mathcal{P}_{n+1}$, disturbed by the
\[
\begin{align*}
p_{n+1} \cdot \phi(p_n) - \phi(p_{n+1}) &= p_{n+1} \cdot \phi(p_n) - \phi(p_n) \cdot (p_{n+1} - 2) \\
&= 2\phi(p_n)
\end{align*}
\]

(4.2)

excludings by the indicator function $\psi(x, p_{n+1})$. How the $2\phi(p_n)$ positions are distributed in $\mathcal{P}_{n+1}$ is uncertainly. Nevertheless in $\mathcal{P}_{n+1}$ we have symmetry again (see Corollary 4.5), but only on the whole.

5. Generic Extensions

Definition 5.1. Let $x \in \mathcal{P}_n$. We will denote with
\[
\gamma^{(m)}_n(x) := x + m \cdot p_n # 5
\]
the $m^{th}$ generic extension of $x$ of the order $p_n$.

In order to study the distribution of the $\omega_{p_n}$-numbers in their period section $\mathcal{P}_n$ we will it partition in the following subsections
\[
\mathcal{P}_n = \bigcup_{m=0}^{p_n-1} \mathcal{P}_n^{(m)}
\]
with
\[
\mathcal{P}_n^{(m)} := \{ x \in \mathbb{N} \mid \gamma^{(m)}_{n-1}(\xi_n) \leq x < \gamma^{(m+1)}_{n-1}(\xi_n) \}, \quad 0 \leq m \leq p_n - 1.
\]

Now we shift these subsections to the left by $\xi_n - 1$ and obtain
\[
\mathcal{G}_n = \mathcal{P}_n - (\xi_n - 1) = \bigcup_{m=0}^{p_n-1} \mathcal{G}_n^{(m)} = \{ x \in \mathbb{N} \mid 1 \leq x \leq p_n # 5 \}
\]
with
\[
\mathcal{G}_n^{(m)} := \mathcal{P}_n^{(m)} - (\xi_n - 1) = \{ x \in \mathbb{N} \mid \gamma^{(m)}_{n-1}(1) \leq x < \gamma^{(m+1)}_{n-1}(1) \}
\]
for $0 \leq m \leq p_n - 1$.

Lemma 5.2. The sections $\mathcal{P}_n$ and $\mathcal{G}_n$ as well as the subsections $\mathcal{P}_n^{(m)}$ and $\mathcal{G}_n^{(m)}$ for $0 \leq m \leq p_n - 1$ are fully equivalent with respect of the positions of the $\omega_{p_n}$-numbers inside of them for all $n \geq 3$.

Proof. Since the periodicity of the aggregate indicator function $\Psi(x, p_n)$ and the setting $\psi(x, p_n) = 0 \mid x \leq \kappa(p_n)$ holds for all $x \mid 1 \leq x \leq \xi_n - 1$
\[
\Psi(x, p_n) = 0 \text{ if and only if } \Psi(x + p_n # 5, p_n) = 0
\]
for each $n \geq 3$. Hence the sets
\[
A_n = \{ x \in \mathbb{N} \mid 1 \leq x \leq \xi_n - 1 \}
\]
and
\[
B_n = \{ x \in \mathbb{N} \mid p_n # 5 + 1 \leq x \leq p_n # 5 + \xi_n - 1 \}
\]
are equivalent with respect to the positions of the $\omega_{p_n}$-numbers and have the same number of $\omega_{p_n}$-numbers inside.

On the other hand is
\[
\mathcal{G}_n \cap \mathcal{P}_n = \{ x \in \mathbb{N} \mid \xi_n \leq x \leq p_n # 5 \}
\]
and
$$\mathcal{G}_n = A_n \cup (\mathcal{G}_n \cap \mathcal{P}_n)$$
as well as
$$\mathcal{P}_n = (\mathcal{G}_n \cap \mathcal{P}_n) \cup B_n.$$Hence the sets \(\mathcal{P}_n\) and \(\mathcal{G}_n\) have the same number of \(\omega_{p_n}\)-numbers inside and they are fully equivalent with respect to the positions of the \(\omega_{p_n}\)-numbers.

Since it holds \(\mathcal{G}_n^{(0)} = \mathcal{G}_{n-1}\), the relations above are valid for all \(\mathcal{G}_n^{(0)}\) and \(\mathcal{P}_n^{(0)}\) and because of the periodicity also for all \(\mathcal{G}_n^{(m)}\) and \(\mathcal{P}_n^{(m)}\) with \(1 \leq m \leq p_n - 1\). This completes the proof. \(\square\)

Obviously the \(\omega_{p_{n-1}}\)-numbers are in all subsections \(\mathcal{P}_n^{(0)}, \ldots, \mathcal{P}_n^{(p_n - 1)}\) symmetrically distributed. Due to Lemma 5.2 this is valid for \(\mathcal{G}_n^{(0)}, \ldots, \mathcal{G}_n^{(p_n - 1)}\) too.

**Definition 5.3.** Analogously to (3.7) we denote with
\[
\varphi^{(m)}(p_n) := |\{x \in \mathcal{P}_n^{(m)} | \Psi(x, p_n) > 0\}|
\]the number of \(\omega_{p_n}\)-numbers in the subsection \(\mathcal{P}_n^{(m)}\). Due to Lemma 5.2 holds also
\[
\varphi^{(m)}(p_n) = |\{x \in \mathcal{G}_n^{(m)} | \Psi(x, p_n) > 0\}|
\]

**Lemma 5.4.** Let \(x\) be a fixed member of \(\mathcal{G}_n\), \(p\) a prime number, \(m\) an integer varying between 0 and \(p - 1\) and \(\gamma_n^{(m)}(x)\) the \(m\)th generic extension of order \(p_n\). Then for all \(p > p_n\) the value of
\[
\tau(\gamma_n^{(m)}(x), p) = \tau(x + m \cdot p_n \#_5, p)
\]is uniquely determined by \(m\).

**Proof.** Contrarily we assume that two different values \(m_1, m_2\) of \(m\) result in the same value of the position function
\[
\tau(x + m_1 \cdot p_n \#_5, p) = \tau(x + m_2 \cdot p_n \#_5, p).
\]By virtue of (2.4) we have
\[
\tau(x + m \cdot p_n \#_5, p) = (x + m \cdot p_n \#_5 + \kappa(p)) \mod p
\]where
\[
\tau(x, p) + m \cdot p_n \#_5 \mod p.
\]Since \(x\) is a fixed member of \(\mathcal{G}_n\) it must hold
\[
(m_1 \cdot p_n \#_5) \mod p = (m_2 \cdot p_n \#_5) \mod p.
\]Because of \(m_1, m_2 < p\) and \(\gcd(p_n \#_5, p) = 1\) this equation is only solvable with \(m_1 = m_2\). \(\square\)

**Lemma 5.5.** Let \(x\) as a member of \(\mathcal{G}_n\) be an \(\omega_{p_n}\)-number. Then \(p_{n+1} - 2\) generic extensions \(\gamma_n^{(m)}(x)\) are \(\omega_{p_{n+1}}\)-numbers and two aren’t.

**Proof.** Because of the periodicity of \(\Psi(x, p_n)\) it’s evident that \(\gamma_n^{(m)}(x)\) are \(\omega_{p_n}\)-numbers for \(0 \leq m \leq p_{n+1} - 1\), it holds
\[
\Psi(\gamma_n^{(m)}(x), p_n) > 0 \text{ since } \Psi(x + m \cdot p_n \#_5, p_n) = \Psi(x, p_n) > 0.
\]We consider for a fixed \(x_0\) the position function with respect to \(p_{n+1}\)
\[
\tau(\gamma_n^{(m)}(x_0), p_{n+1}) = \tau(x_0 + m \cdot p_n \#_5, p_{n+1}).
\]
It has \( p_{n+1} \) different values and by virtue of Lemma 5.4 exactly one value 0 for \( m = m_- \) and one value \( 2\kappa(p_{n+1}) \) for \( m = m_+ \)

\[
\tau(\gamma_n^{m_+}(x_o), p_{n+1}) \neq 0 \quad \text{and} \quad \tau(\gamma_n^{m_-}(x_o), p_{n+1}) = 2\kappa(p_{n+1})
\]

By virtue of (2.11) we get for the indicator function

\[
\psi(\gamma_n^{m_+}(x_o), p_{n+1}) = 0
\]

and since

\[
\Psi(\gamma_n^{m_+}(x_o), p_{n+1}) = \Psi(\gamma_n^{m_-}(x_o), p_n) \cdot \psi(\gamma_n^{m_+}(x_o), p_{n+1})
\]

we get finally for 2 values \( m_\pm \)

\[
\Psi(\gamma_n^{m_\pm}(x_o), p_{n+1}) = 0
\]

while for \( p_{n+1} - 2 \) values of \( m \) holds \( \psi(\gamma_n^m(x), p_{n+1}) > 0 \), which results in

\[
\Psi(\gamma_n^m(x), p_{n+1}) > 0
\]

Hence there are \( p_{n+1} - 2 \) generic extensions of the order \( p_n \) that are \( \omega_{p_{n+1}} \)-numbers.

\[\square\]

**Theorem 5.6.** The number \( \phi^m(p_n) \) of \( \omega_{p_n} \)-numbers in any subsection \( \mathcal{P}_n^{(m)} \) with \( 0 \leq m \leq p_n - 1 \) meets the following criterion

\[
\phi^m(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}).
\]

**Proof.** We consider a fixed \( \omega_{p_{n-2}} \)-number \( x_o \in \mathcal{G}_{n-2} \) with \( \Psi(x_o, p_{n-2}) > 0 \) and their generic extensions of order \( p_{n-2} \)

\[
\gamma_n^{m}(x_o) \quad \text{for} \quad 0 \leq m \leq p_{n-1} - 1.
\]

By virtue of Lemma 5.5 we know that there are \( p_{n-1} - 2 \) generic extensions with \( \Psi(\xi_{n-2}(x_o, m), p_{n-1}) > 0 \), which means they are \( \omega_{p_{n-1}} \)-numbers. As generic extensions they are members of \( \mathcal{G}_{n-1} \). Hence they are also members of \( \mathcal{G}_{n-1}^{(0)} \). In what follows we will denote these \( \omega_{p_{n-1}} \)-numbers as \( x_o \)-candidates. Let \( y \) be any \( x_o \)-candidate. Then holds \( \Psi(y, p_{n-1}) > 0 \) and we can use the position function \( \tau(y, p_n) \) to check whether the \( x_o \)-candidates are also \( \omega_{p_n} \)-numbers. By virtue of Lemma 5.4 all the \( p_{n-1} - 2 \) \( x_o \)-candidates have different \( \tau \)-values. But since the position function \( \tau(y, p_n) \) has \( p_n > p_{n-1} - 2 \) values, the \( \tau \)-bad values

\[
\tau(y, p_n) = 0 \quad \text{or} \quad \tau(y, p_n) = 2\kappa(p_n)
\]

do not necessarily occur among the \( x_o \)-candidates, but they can occur single or both. Hence there are at least \( (p_{n-1} - 2) - 2 = p_{n-1} - 4 \) \( x_o \)-candidates that are also \( \omega_{p_n} \)-numbers in \( \mathcal{G}_{n-2}^{(0)} \). This holds for one (fixed) \( \omega_{p_{n-2}} \)-number \( x_o \). But since there are \( \phi(p_{n-2}) \) different \( \omega_{p_{n-2}} \)-numbers in \( \mathcal{G}_{n-2} \) we have

\[
\phi^{(0)}(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}) \quad \text{(5.3)}
\]

\( \omega_{p_n} \)-numbers in the subsection \( \mathcal{G}_{n-1} = \mathcal{G}_{n-1}^{(0)} \).

The above described situation related to the uniqueness of the \( \tau \)-values holds also for the generic extensions of order \( p_{n-1} \) of all \( p_{n-1} - 2 \) \( x_o \)-candidates \( y \)

\[
\gamma_{n-1}^{(r)}(y) \quad 1 \leq r \leq p_n - 1
\]
because if \( y_1, y_2 \) are two different \( x_o \)-candidates then holds by virtue of Lemma 5.4

\[
\tau(y_1, p_n) \neq \tau(y_2, p_n) \quad \text{and hence} \\
\tau(\gamma_{n-1}^{(r)}(y_1), p_n) = (y_1 + r \cdot p_{n-1} \#_5 + \kappa(p_n)) \mod p_n \\
= (\tau(y_1, p_n) + r \cdot p_{n-1} \#_5) \mod p_n \\
\neq (\tau(y_2, p_n) + r \cdot p_{n-1} \#_5) \mod p_n \\
= \tau(\gamma_{n-1}^{(r)}(y_2), p_n)
\]

for \( 1 \leq r \leq p_n - 1 \) and we have in all subsections \( G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(p_n-1)} \) with respect to the \( \tau \)-values the same situation like in \( G_n^{(0)} \). Therefore and since the right side of (5.3) is independent from the number of a subsection it holds in all subsections

\[
\phi^{(r)}(p_n) \geq (p_n - 4) \cdot \phi(p_{n-2}), \quad 0 \leq r \leq p_n - 1.
\]

Due to Lemma 5.2 this is also valid for the subsections \( P_n^{(1)}, P_n^{(2)}, \ldots, P_n^{(p_n-1)} \) and the period section \( P_n \). This completes the proof. \( \square \)

**Theorem 5.7.** The \( \omega_{p_n} \)-numbers in \( P_n \) are over the subsections \( P_n^{(1)}, \ldots, P_n^{(p_n-1)} \) asymptotically uniform distributed

\[
\phi^{(m)}(p_n) \sim \frac{\phi(p_n)}{p_n} \quad \text{for} \quad 0 \leq m \leq p_n - 1.
\]

**Proof.** We consider the following ratio and get by virtue of Theorem 5.6

\[
\frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} \geq \frac{p_n(p_{n-1} - 4) \cdot \phi(p_{n-2})}{\phi(p_n)} \\
= \frac{p_n(p_{n-1} - 4)}{(p_{n-1} - 2)(p_n - 2)} \\
\text{and with} \quad p_{n-1} = p_n - d \\
= \frac{p_n(p_n - (d + 4))}{(p_n - (d + 2))(p_n - 2)} \\
= \frac{1 - \frac{d + 4}{p_n}}{\left(1 - \frac{d + 2}{p_n}\right) \left(1 - \frac{2}{p_n}\right)} \rightarrow 1.
\]

On the other hand holds \( G_n^{(0)} = G_{n-1} \) and from

\[
x \in G_n^{(0)} \mid \Psi(x, p_{n-1}) > 0 \quad \text{follows} \quad \Psi(\gamma_{n-1}^{(m)}(x), p_{n-1}) > 0 \quad \text{for} \quad 1 \leq m \leq p_n - 1.
\]

Hence we have

\[
\phi(p_{n-1}) = |\{x \in G_n^{(0)} \mid \Psi(x, p_{n-1}) > 0\}| \\
\text{and for} \quad 1 \leq m \leq p_n - 1 \\
= |\{x \in G_n^{(m)} \mid \Psi(x, p_{n-1}) > 0\}|
\]

and therefore

\[
\phi(p_{n-1}) = |\{x \in G_n^{(m)} \mid \Psi(x, p_{n-1}) > 0\}| \\
\geq |\{x \in G_n^{(m)} \mid \Psi(x, p_{n}) > 0\}| = \phi^{(m)}(p_n).
\]
Due to Lemma 5.2 this holds also for the subsets $P^{(0)}, P^{(1)}, \ldots, P^{(p_n-1)}$. Finally with
\[
\frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} \leq \frac{p_n \cdot \phi(p_{n-1})}{\phi(p_n)} = \frac{p_n}{p_n - 2} \to 1
\]
holds
\[
\frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} \sim 1.
\]

**Corollary 5.8.** The in the previous section shown symmetry of the $\omega_{p_m}$-numbers in their period section $P_m$ and the above demonstrated asymptotically uniform distribution of the $\omega_{p_m}$-numbers over the subsections $P_m^{(0)}, \ldots, P_m^{(p_m-1)}$ avert the formation of extreme configurations of the $\omega_{p_m}$-gaps in the period section $P_n$ for all $n > m$.

6. Proof of the Twin Prime Conjecture

**Proof.** The proof will be done by contradiction. We assume contrarily that there is only a finite number of twin primes and therefore only a finite number of twin prime generators. Let $y_o$ be the greatest one. It lies in the A-section $A_{n_o}$ with $n_o = \pi(\hat{p}(y_o))$, the beginning of the period section $P_{n_o}$. W.l.o.g. we can assume that $n_o > 200$. In the successive A-sections $A_t$ with $t > n_o$ consequently there cannot be any twin prime generators and hence by virtue of Corollary 3.5 no $\omega_{p_t}$-numbers. But then we have $\omega_{p_t}$-gaps with sizes $> d_t$ in all (infinitely many) period sections $P_t$ for $t > n_o$.

Because

- the squared average size of the $\omega_{p_t}$-gaps $\bar{\delta}(p_t)^2$ is less than $\frac{d_t}{c_p}$ with a number $c_p$ by virtue of Corollary 4.4,
- all period sections $P_t$ are very closely overlapped and due to the symmetrical distribution of the $\omega_{p_t}$-numbers around $\frac{p_t + p_t}{2}$ and $p_t \# 5$ and $p_t \# 5$ and
- since the asymptotic uniform distribution of the $\omega_{p_t}$-numbers over the subsections $P_t^{(0)}, \ldots, P_t^{(p_t-1)}$ extreme configurations in $P_t$ cannot occur therefore it is not possible to have for all $t > n_o$ only period sections $P_t$ with $\omega_{p_t}$-gaps at their beginnings that all are greater than $d_t$.

Therefore the proof assumption cannot be valid and hence the Twin Prime Conjecture must be true.

\[\Box\]

**References**

[3] Brun, V.: La série $\frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \ldots$ où les dénominateurs sont "nombres premiers jumeaux" est convergente où finie. Bull. Sci. Math. (2) 43 (1919), 100-104 and 124-128.

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