ON THE TWIN PRIME CONJECTURE

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ABSTRACT. Every prime number $p \geq 5$ has the form $6x-1$ or $6x+1$. We call x the **generator** of p. Twin primes are distinguished by a **common generator** for each pair. Therefore it makes sense to search for the Twin Primes on the level of their generators. This paper presents a new approach to prove the **Twin Prime Conjecture** by a method to extract all Twin Primes on the level of the Twin Prime Generators. We define the ω_{p_n}–numbers x as numbers for that holds that $6x-1$ and $6x+1$ are coprime to the primes $5, 7, \ldots, p_n$. By dint of the average size $\bar{\delta}(p_n)$ of the ω_{p_n}–gaps we can prove the **Twin Prime Conjecture**..

1. INTRODUCTION

The question on the infinity of the twin primes keeps busy many mathematicians for a long time. 1919 V. Brun [3] had proved that the series of the inverted twin primes converges while he had tried to prove the Twin Prime Conjecture. Several authors worked on bounds for the length of prime gaps (see f.i. [4, 5, 6]). In 2008 B. Green and T. Tao [7] succeeded in proving that there are arbitrarily long arithmetic progressions containing only prime numbers. 2014 Y. Zhang [8] obtained a great attention with his proof that there are infinitely many consecutive primes with a gap of $70,000,000$ at most. With the project "PolyMath8", in particular forced by T. Tao [9], this bound could be lessened down to 246 respectively to 12 assuming the validity of the Elliott–Halberstam Conjecture [10].

In [11] an attempt to prove the Twin Prime Conjecture only by elementary instruments is shown. Another approach as in the most works on this topic is presented there. The looking for twin primes is transfered to the level of their generators because each twin prime has a common generator. However the proof based on a vague assumption over the distribution of the ω_p–numbers. With this new paper this gap will be closed. For a better understanding some definitions, statements and results from [11] are repeated in the next two sections.

2. TWIN PRIME GENERATORS

It is well known that every prime number $p \geq 5$ has the form $6x - 1$ or $6x + 1$. We will call x the **generator** of p. Twin primes are distinguished by a common generator for each pair.

Date: March 3, 2023.

2010 *Mathematics Subject Classification.* 11A41.

Key words and phrases. Number theory, prime numbers, twin prime conjecture.

The author thanks **Dr. John Sellers**, Colorado, USA for the creative and pleasant cooperation on this topic and the brilliant ideas that are developed in this cooperation. They are the basis for the statements in the section 5.

Definition 2.1. Let

$\mathbb{N}$ be the set of the positve integers,

$\mathbb{P}$ the set of the prime numbers, $\mathbb{P}^*$ primes ≥ 5,

$\mathbb{P}_- = \{p \in \mathbb{P}^* \mid p \equiv -1(\operatorname{mod} 6)\}, \ \mathbb{P}_+ = \{p \in \mathbb{P}^* \mid p \equiv +1(\operatorname{mod} 6)\}$

and

$$\kappa(n) := \left\lfloor \frac{n+1}{6} \right\rfloor \text{ for } n \in \mathbb{N} \tag{2.1}$$

the **generator** function of the pair $(6\kappa(n)-1, 6\kappa(n)+1)$. If a pair $(6x-1, 6x+1)$ is a twin prime then we call x as a **twin prime generator** and

$$\mathbb{G} := \{x \in \mathbb{N} \mid 6x-1 \in \mathbb{P}_-, 6x+1 \in \mathbb{P}_+\}$$

is the set of all twin prime generators. Hence the pair $(5, 7)$ is the least twin prime in our consideration.

In order to transfer the searching for twin primes to the level of their generators we need a criterion for checking a natural number to be a twin prime generator.

Theorem 2.2. *A number x is a **twin prime generator**, a member of $\mathbb{G}$, if and only if there is **no** $p \in \mathbb{P}^*$ with $p < 6x-1$ that one of the following congruences fulfills:*

$$x \equiv -\kappa(p)(\operatorname{mod} p) \tag{2.2}$$

$$x \equiv +\kappa(p)(\operatorname{mod} p) \tag{2.3}$$

Proof. At first we assume that there is a prime $p \in \mathbb{P}^*$ with $p < 6x-1$ such that (2.2) or (2.3) is valid. There are two cases.

(1) $p \in \mathbb{P}_-$, what means $p = 6\kappa(p) - 1$:

If (2.2) is true then there is an $n \in \mathbb{N}$ with

$$\begin{aligned} x &= -\kappa(p) + n \cdot (6\kappa(p) - 1) \\ 6x &= -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) \\ 6x + 1 &= -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) + 1 \\ &= (6n - 1)(6\kappa(p) - 1) \\ &= (6n - 1) \cdot p \\ \implies 6x + 1 &\equiv 0(\operatorname{mod} p) \implies x \notin \mathbb{G} \end{aligned}$$

For (2.3) the proof will be done with $6x - 1$:

$$\begin{aligned} 6x - 1 &= 6\kappa(p) + 6n \cdot (6\kappa(p) - 1) - 1 \\ &= (6n + 1)(6\kappa(p) - 1) \\ &= (6n + 1) \cdot p \\ \implies 6x - 1 &\equiv 0(\operatorname{mod} p) \implies x \notin \mathbb{G} \end{aligned}$$

(2) $p \in \mathbb{P}_+$, what means $p = 6\kappa(p) + 1$:

We go the same way with (2.2) and $6x - 1$ as well as (2.3) and $6x + 1$:

$$6x - 1 = (6n - 1)(6\kappa(p) + 1) \implies 6x - 1 \equiv 0(\operatorname{mod} p)$$

$$6x + 1 = (6n + 1)(6\kappa(p) + 1) \implies 6x + 1 \equiv 0(\operatorname{mod} p)$$

With these it's shown that $x \notin \mathbb{G}$ if the congruences (2.2) or (2.3) are valid. They cannot be true both because they exclude each other.

If on the other hand $x \notin \mathbb{G}$, then $6x-1$ or $6x+1$ is no prime. Let be $6x-1 \equiv 0 (\bmod p)$ for any $p \in \mathbb{P}_-$. Then we have

$$\begin{aligned} 6x-1 &\equiv 0 (\bmod p) \equiv p (\bmod p) \\ &\equiv (6\kappa(p)-1)(\bmod p) \\ 6x &\equiv 6\kappa(p)(\bmod p) \\ x &\equiv \kappa(p)(\bmod p) \implies (2.3). \end{aligned}$$

For any $p \in \mathbb{P}_+$ we have

$$\begin{aligned} 6x-1 &\equiv -p (\bmod p) \\ &\equiv -(6\kappa(p)+1)(\bmod p) \\ 6x &\equiv -6\kappa(p)(\bmod p) \\ x &\equiv -\kappa(p)(\bmod p) \implies (2.2). \end{aligned}$$

The other both cases we can handle in the same way. Therefore either (2.2) or (2.3) is valid if $x \notin \mathbb{G}$. □

If we consider that the least proper divisor of a number $6x+1$ is less or equal to $\sqrt{6x+1}$ then p in the congruences (2.2) and (2.3) can be limited by

$$\hat{p}(x) = \max(p \in \mathbb{P}^* \mid p \leq \sqrt{6x+1}).$$

Remark 2.3. Henceforth we will use the letter p for a general prime number and p_n if we describe an element of a sequence of primes.

With p_n as the n-the prime number[1] and $\pi(z)$ as the number of primes $\leq z$ we have with

$$x \equiv -\kappa(p_n)(\bmod p_n) \text{ or } x \equiv +\kappa(p_n)(\bmod p_n) \tag{2.4}$$

for $3 \leq n \leq \pi(\hat{p}(x))$ a provable system of criteria to exclude all numbers $x \geq 4$ being no twin prime generators. Since the modules are primes the criteria are independent among each other.

Hence we can square the congruences (2.4) and get

$$x^2 \equiv \kappa(p_n)^2 (\bmod p_n) \text{ for } 3 \leq n \leq \pi(\hat{p}(x)). \tag{2.5}$$

This results in a system of **indicator functions** $\psi(x, p_n)$ for that holds for $3 \leq n \leq \pi(\hat{p}(x))$

$$\begin{aligned} x^2 - \kappa(p_n)^2 &\equiv \psi(x, p_n)(\bmod p_n) \text{ respectively} \\ \psi(x, p_n) &= \left(x^2 - \kappa(p_n)^2\right) \operatorname{Mod} p_n. \end{aligned} \tag{2.6}$$

For the sake of completeness we define $\psi(x, p_n) = 0$ for $x \leq \kappa(p_n)$.

Obviously holds that if $\psi(x, p) = 0$ for any $p \leq \hat{p}(x)$ then x cannot be a twin prime generator. [2] Due to *modulo* the indicator function $\psi(x, p)$ is periodical in x

[1] It is $p_1 = 2$

[2] We consider only cases $x > \kappa(p_n)$ in the sequel.

with a period length of p. With the indicator functions for all modules $p_3, \dots, p_n$ we build the *aggregate* indicator functions

$$\Psi(x, p_n) = \prod_{i=3}^{n} \frac{\psi(x, p_i)}{p_i}$$
$$\text{and} \tag{2.7}$$
$$\widehat{\Psi}(x) = \Psi(x, \hat{p}(x)).$$

Because the co-domain of $\psi(x, p)$ consists of integers between 0 and $p - 1$, the aggregate functions $\Psi(x, p)$ and $\widehat{\Psi}(x)$ have rational values between 0 and < 1.

Definition 2.4. Let be

$$\xi_n := \min(x \in \mathbb{N} \mid \hat{p}(x) = p_n). \tag{2.8}$$

It is the first integer x for that the modul p_n could be a prime factor of $6x \pm 1$. Therefore we denote it as the **origin** of the modul p_n.

Up from ξ_n in every ψ–period there are just $p_n - 2$ positions with $\psi(x, p_n) > 0$ and two positions with $\psi(x, p_n) = 0$, once if (2.2) and on the other hand if (2.3) holds. Obviously the distance between these both positions is $2\kappa(p_n)$ respectively $4\kappa(p_n) \pm 1$ since the period length is $6\kappa(p_n) \pm 1$.

It is $p_n \leq \hat{p}(x) \leq \sqrt{6x+1}$ and therefore $p_n^2 \leq 6x + 1$. Then $\frac{p_n^2 - 1}{6}$ is the least number that meets this relation. Comparing with (2.8) we get

$$\xi_n = \frac{p_n^2 - 1}{6}. \tag{2.9}$$

It is easy to prove that for every integer $n \geq 3$ holds that ξ_n is an integer divisible by 4.

Lemma 2.5. *At the origin ξ_n cannot be a twin prime generator since we have $\psi(\xi_n, p_n) = 0$.*

Proof. We substitute p_n by $6\kappa(p_n) \pm 1$. With this and (2.9) holds

$$\begin{aligned}
\xi_n &= \frac{(6\kappa(p_n) \pm 1)^2 - 1}{6} \\
&= \frac{6\kappa(p_n)\,(6\kappa(p_n) \pm 2)}{6} \\
&= \kappa(p_n)\,(6\kappa(p_n) \pm 1) \pm \kappa(p_n) \\
&= \kappa(p_n) \cdot p_n \pm \kappa(p_n) \\
&\equiv \pm\kappa(p_n) (\operatorname{mod} p_n) \implies \psi(\xi_n, p_n) = 0.
\end{aligned}$$

□

In the proof we have seen that for every prime p holds that $p^2 - 1$ is an integer divisible by 6. For every $x \geq \xi_n$ the local position relative to the period start [3] can be determined by the **position function** $\tau(x, p_n)$:

$$\begin{aligned}
x + \kappa(p_n) &\equiv \tau(x, p_n) (\operatorname{mod} p_n) \text{ respectively} \\
\tau(x, p_n) &= (x + \kappa(p_n)) \operatorname{Mod} p_n.
\end{aligned} \tag{2.10}$$

[3] For $p_n \in \mathbb{P}_-$ the period start is ξ_n and else it is $\xi_n - 2\kappa(p_n)$.

Hence the co-domain of the position function $\tau(x,p)$ are the integers $0,1,\ldots,p-1$ and it has in x a period length of p. Between the indicator function $\psi(x,p)$ and the position function $\tau(x,p)$ there is the following relationship:

$$\begin{aligned}\psi(x,p) &= \tau(x,p)\cdot(x-\kappa(p))\,\mathrm{Mod}\,p\\ &= \tau(x,p)\cdot(\tau(x,p)-2\kappa(p))\,\mathrm{Mod}\,p.\end{aligned} \tag{2.11}$$

Obviously holds $\psi(x,p)=0$ if and only if $\tau(x,p)=0$ or $\tau(x,p)=2\kappa(p)$. From (2.10) we see that the values of the position function $\tau(x,p)$ consists exactly of the p values $0,1,2,\ldots,p-1$. Because the values 0 and $2\kappa(p)$ indicate that x cannot be a twin prime generator (see Theorem 2.2) we will call them as τ*–bad values* and the others as τ*–good values*. Hence there are two τ–bad values and $p-2$ τ–good values for each modul p.

3. The ω_{p_n}–numbers

For every natural number x in the interval

$$\mathcal{A}_n := \{x\in\mathbb{N}\mid \xi_n\le x<\xi_{n+1}\} \tag{3.1}$$

$\hat{p}(x)$ persists constant on the value p_n. The length of this interval will be denoted as d_n. It is depending on the distance between successive primes. Since they can only be even, we have with $a=2,4,6,\ldots$

$$\begin{aligned}d_n &= \frac{(p_n+a)^2-1}{6}-\frac{p_n^2-1}{6}\\ &= \frac{2ap_n+a^2}{6}\\ &= \frac{a}{3}(p_n+\frac{a}{2})\\ &\ge \frac{2}{3}(p_n+1).\end{aligned} \tag{3.2}$$

On the other hand we obtain because of $p_{n+1}<2p_n$ (see [2], p. 188)

$$\begin{aligned}d_n &= \frac{p_{n+1}^2-1}{6}-\frac{p_n^2-1}{6}\\ &= \frac{p_{n+1}^2-p_n^2}{6}\\ &= \frac{(p_{n+1}+p_n)(p_{n+1}-p_n)}{6}\\ &< \frac{3p_n\cdot p_n}{6}=\frac{p_n^2}{2}.\end{aligned}$$

And since d_n is an integer and p_n^2 is odd it holds

$$d_n \le \frac{p_n^2-1}{2} = 3\xi_n \text{ and hence} \tag{3.3}$$

$$\xi_{n+1}\le 4\xi_n. \tag{3.4}$$

The congruences in (2.10)

$$x+\kappa(p_i)\equiv\tau(x,p_i)(\mathrm{mod}\,p_i),\quad 3\le i\le n \tag{3.5}$$

fulfill the requirements of the Chinese Remainder Theorem (see [1], p. 89). Therefore it is modulo $5 \cdot 7 \cdot \ldots \cdot p_n$ uniquely resolvable. With

$$p_n\#_5 := \prod_{i=3}^{n} p_i = 5 \cdot 7 \cdot \ldots \cdot p_n \tag{3.6}$$

it's $(\bmod\, p_n\#_5)$ [4] uniquely resolvable. Therefore the aggregate indicator function $\Psi(x, p_n)$ has the period length $p_n\#_5$ and it holds:

$$\Psi(x + a \cdot p_n\#_5, p_n) = \Psi(x, p_n) \mid a \in \mathbb{N}.$$

Definition 3.1. A positive integer x will be called as an ω_{p_n}–**number** if both $6x - 1$ and $6x + 1$ are coprime [5] to $p_n\#_5$. Then is $\Psi(x, p_n) > 0$.

Corollary 3.2. *Because of the periodicity of the aggregate indicator function $\Psi(x, p_n)$ in x there are infinitely many ω_{p_n}–numbers.*

Definition 3.3. Let be

$$\mathcal{P}_n := \{x \in \mathbb{N} \mid \xi_n \le x < \xi_n + p_n\#_5\}$$

the interval of one period of the aggregate indicator function $\Psi(x, p_n)$. We'll denote it henceforth as **period section**. Evidently is $\mathcal{A}_n \subset \mathcal{P}_n$ for all $n \ge 3$.

Proposition 3.4. *The period section $\mathcal{P}_n$ contains*

$$\phi(p_n) := \prod_{k=3}^{n} (p_k - 2) \tag{3.7}$$

ω_{p_n}–numbers.

Proof. Due to Definition 3.3 the period section $\mathcal{P}_n$ contains $p_n\#_5$ successive natural numbers $\xi_n, \xi_n + 1, \ldots, \xi_n + p_n\#_5 - 1$. Each sequence of p_k successive members of them for $3 \le k \le n$ contains due to (2.11) two members $x_\mp$ with the two τ–bad values

$$\tau(x_-, p_k) = 0 \text{ and } \tau(x_+, p_k) = 2\kappa(p_k)$$

and $p_k - 2$ members y with the $p_k - 2$ τ–good values. Each ω_{p_n}–number $y \in \mathcal{P}_n$ is represented exactly by one m–tuple with $m = n - 2$

$$(\tau_3, \tau_4, \ldots, \tau_n),$$

where all values $\tau_k = \tau(y, p_k)$ are τ–good values and since by virtue of (2.11) for only τ–good values holds

$$\Psi(y, p_n) = \prod_{k=3}^{n} \frac{\tau_k(\tau_k - 2\kappa(p_k) \operatorname{Mod} p_k}{p_k} > 0.$$

Because the primes $5, 7, \ldots, p_n$ are independent, all the m–tuples are different and their number is

$$\prod_{k=3}^{n} (p_k - 2).$$

[4] It is $p_n\#_5 = \frac{p_n\#}{6}$, with the primorial $p_n\#$.

[5] Then is $\gcd(36x^2 - 1, p_n\#_5) = 1$.

Since each such m–tuple represents exactly one different ω_{p_n}–number in $\mathcal{P}_n$ it holds

$$\phi(p_n) = \prod_{k=3}^{n} (p_k - 2).$$

□

Corollary 3.5. *Let x be an ω_{p_n}–number as a member of $\mathcal{A}_n$. Then x is a* ***twin prime generators*** *because by virtue of Definition 3.1 $6x-1$ as well as $6x+1$ are prime to $5, 7, \ldots, p_n$ and it holds $\Psi(x, p_n) = \widehat{\Psi}(x) > 0$.*

Vice versa an ω_{p_m}–number as member of $\mathcal{A}_n$ by $m < n$ is not necessarily a twin prime generator.

The ratio between (3.7) and the period length by virtue of (3.6) results in

$$\eta(p_n) := \frac{\phi(p_n)}{p_n\#_5} = \prod_{i=3}^{n} \frac{p_i - 2}{p_i}, \tag{3.8}$$

as the **average density** of the ω_{p_n}–numbers in $\mathcal{P}_n$. Obviously $\eta(p)$ is a strictly monotonic decreasing function.

Lemma 3.6. *The average density function $\eta(p_n)$ is double-sided bounded by*

$$\frac{3}{p_n} < \eta(p_n) < \frac{3}{\log p_n}.$$

Proof.

A) At first we prove the left inequality.
Because all primes > 2 are odd numbers, the count of primes $\leq p_n$ is less than the count of odd numbers in the same range. All factors of $\eta(p_n)$ are less than 1. Thus we get with $m = \frac{p_n+1}{2}$

$$\begin{aligned}\eta(p_n) = \prod_{k=3}^{n} \frac{p_k - 2}{p_k} &> \prod_{k=3}^{m} \frac{(2k-1)-2}{2k-1} \\ &= \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{p_n - 4}{p_n - 2} \cdot \frac{p_n - 2}{p_n} \\ &= \frac{3}{p_n}.\end{aligned}$$

B) It's well known that (see [1], p. 40 above)

$$\log x < \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{k=1}^{\pi(x)} \left(\frac{p_k - 1}{p_k}\right)^{-1}.$$

Therefore is

$$\begin{aligned}\frac{1}{\log p_n} &> \prod_{k=1}^{n} \frac{p_k - 1}{p_k} \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 1}{p_k} = \frac{1}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 1}{p_k} \\ &> \frac{1}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 2}{p_k} = \frac{\eta(p_n)}{3}.\end{aligned}$$

Hence we get for the right inequality $\eta(p_n) < \dfrac{3}{\log p_n}$.

With it the proof is completed. □

Corollary 3.7. *Since both bounds in Lemma 3.6 go to zero holds*

$$\lim_{n\to\infty} \eta(p_n) = 0.$$

This means that the ω_{p_n}–numbers have an asymptotic zero–density. Therefore the twin prime generators as subset of the ω_{p_n}–numbers have an asymptotic zero–density too. This result is in accordance with the the fact that also the set of the primes has an asymptotic zero–density.

3.1. The Symmetry of the ω_{p_n}–Numbers.

In the period section $\mathcal{P}_n$ the element

$$x_n^{(0)} := p_n\#_5$$

has a particular importance. Because $p_n\#_5$ is divisible by all primes between 5 and p_n it holds

$$p_n\#_5 \equiv 0 (\operatorname{mod} p_m) \mid 3 \le m \le n, p_m \in \mathbb{P}^*$$

and hence

$$x_n^{(0)} \not\equiv \pm\kappa(p_m)(\operatorname{mod} p_m) \mid 3 \le m \le n, p_m \in \mathbb{P}^*.$$

Hence $x_n^{(0)}$ is an ω_{p_n}–number and it holds

$$\Psi(p_n\#_5, p_n) > 0. \tag{3.9}$$

Because of $\xi_n < p_n\#_5 < \xi_n + p_n\#_5$ the number $x_n^{(0)}$ is in the inner of $\mathcal{P}_n$ but near to the end.

Theorem 3.8. *The ω_{p_n}–numbers are symmetrically distributed around the axis $x_n^{(0)}$*

$$\Psi(x_n^{(0)} - a, p_n) = \Psi(x_n^{(0)} + a, p_n)$$

for any positive integer $a < x_n^{(0)}$.

Proof. Since by virtue of (2.7) the aggregate indicator function $\Psi(x, p_n)$ consists of the product of the indicator functions $\psi(x, 5), \ldots, \psi(x, p_n)$, the ω_{p_n}–numbers are symmetrically arranged around the axis $x_n^{(0)}$ if and only if holds for $m = 3, \ldots, n$ [6]

$$\psi(x_n^{(0)} - a, p_m) = \psi(x_n^{(0)} + a, p_m) \tag{3.10}$$

with any number $a < x_n^{(0)}$. From (2.6) we have

$$\psi(x, p_m) = \left(x^2 - \kappa(p_m)^2\right) \operatorname{Mod} p_m.$$

With $x_n^{(0)} \pm a$ for x we get

$$\begin{aligned}
\psi(x_n^{(0)} \pm a, p_m) &= \Big((x_n^{(0)} \pm a)^2 - \kappa(p_m)^2\Big) \operatorname{Mod} p_m \\
&= \Big(x_n^{(0)}(x_n^{(0)} \pm 2a) + a^2 - \kappa(p_m)^2\Big) \operatorname{Mod} p_m \\
&\quad \text{and since } x_n^{(0)} \equiv 0 (\operatorname{mod} p_m) \\
&= \left(a^2 - \kappa(p_m)^2\right) \operatorname{Mod} p_m.
\end{aligned} \tag{3.11}$$

[6] Unless otherwise specified the use of the variable p_m means below $p_m \in \mathbb{P}^* \mid 3 \le m \le n$.

Since $\psi(x_n^{(0)} \pm a, p_m)$ result in a common value it follows (3.10). □

Additionally from (3.11) we see that

$$\psi(x_n^{(0)} \pm a, p_m) = 0 \text{ for } a = \kappa(p_m).$$

This means that at these positions there cannot be ω_{p_n}–numbers. Around the axis $x_n^{(0)}$ there are ω_{p_n}–gaps with the length $\kappa(p_n)$ but an ω_{p_n}–gap cannot reach over $x_n^{(0)}$.

If we limit our consideration to the period section $\mathcal{P}_n$ then there is a section of symmetry with a length of $2\xi_n$ at the end of $\mathcal{P}_n$ around on $x_n^{(0)} = p_n\#_5$. But in the remaining of the period section there is symmetry too. Let

$$x_n^{(1)} := \frac{p_n\#_5}{2}$$

be a rational number as the middle between the integers

$$x_n^{(1-)} := \frac{p_n\#_5 - 1}{2} \text{ and } x_n^{(1+)} := \frac{p_n\#_5 + 1}{2}. \tag{3.12}$$

Theorem 3.9. *The ω_{p_n}–numbers are symmetrically distributed around the axis $x_n^{(1)}$*

$$\Psi(x_n^{(1-)} - a, p_n) = \Psi(x_n^{(1+)} + a, p_n)$$

for any positive integer $a < x_n^{(1)}$.

Proof. Because of the periodicity of the aggregate indicator function we have

$$\begin{aligned}\Psi(\xi_n + a, p_n) &= \Psi(p_n\#_5 + \xi_n + a, p_n) \text{ and because of Theorem 3.8} \\ &= \Psi(p_n\#_5 - \xi_n - a, p_n).\end{aligned} \tag{3.13}$$

Therefore there is symmetry around

$$\frac{\xi_n + a + p_n\#_5 - \xi_n - a}{2} = \frac{p_n\#_5}{2} = x_n^{(1)}.$$

We set $x_n^{(1+)}$ instead of ξ_n in (3.13) and get

$$\begin{aligned}\Psi\left(x_n^{(1+)} + a, p_m\right) &= \Psi\left(p_n\#_5 - (x_n^{(1+)} + a), p_m\right) \\ &= \Psi\left(p_n\#_5 - \frac{p_n\#_5 + 1}{2} - a, p_m\right) \\ &= \Psi\left(\frac{p_n\#_5 - 1}{2} - a, p_m\right) \\ &= \Psi\left(x_n^{(1-)} - a, p_m\right)\end{aligned}$$

□

The section around $x_n^{(1)}$ has a length of $p_n\#_5 - 2\xi_n$. It increases more quickly than the length $2\xi_n$ of the section around $x_n^{(0)}$ because $p_n\#_5$ increases more quickly than ξ_n.

With the transition $p_n \to p_{n+1}$ the symmetry of the ω_{p_n}–numbers in their period sections $\mathcal{P}_n$ repeats p_{n+1}–times oneself in $\mathcal{P}_{n+1}$, disturbed by the

$$\begin{aligned} p_{n+1} \cdot \phi(p_n) - \phi(p_{n+1}) &= p_{n+1} \cdot \phi(p_n) - \phi(p_n) \cdot (p_{n+1} - 2) \\ &= 2\phi(p_n) \end{aligned} \tag{3.14}$$

excludings by the indicator function $\psi(x, p_{n+1})$. How the $2\phi(p_n)$ positions are distributed in $\mathcal{P}_{n+1}$ is uncertainly. Nevertheless in $\mathcal{P}_{n+1}$ we have symmetry again (see Corollary 3.10), but only on the whole.

Corollary 3.10. *Since by virtue of Theorems 3.8 and 3.9 the ω_{p_n}–numbers are symmetrically distributed around the axes $x_n^{(0)} = p_n\#_5$ as well as $x_n^{(1)} = \frac{p_n\#_5}{2}$ therefore the ω_{p_n}–gaps are symmetrically distributed with respect to their sizes.*

Lemma 3.11. *At the positions $x_n^{(1-)}$ and $x_n^{(1+)}$ by virtue of (3.12) are ω_{p_n}–numbers.*

Proof. Since $4 \not\equiv 0 (\operatorname{mod} p_n\#_5)$ we can multiply (2.6) by 4 and have for $3 \le m \le n$

$$4\psi(x, p_m) \operatorname{Mod} p_m = \big((2x)^2 - (2\kappa(p_m))^2\big) \operatorname{Mod} p_m$$

and get for $x = \frac{p_n\#_5 \pm 1}{2}$

$$\begin{aligned} 4\psi\Big(\frac{p_n\#_5 \pm 1}{2}, p_m\Big) \operatorname{Mod} p_m &= \big((p_n\#_5 \pm 1)^2 - 4\kappa(p_m)^2\big) \operatorname{Mod} p_m \\ &= \big(p_n\#_5(p_n\#_5 \pm 2) + 1 - 4\kappa(p_m)^2\big) \operatorname{Mod} p_m \\ &= \big(1 - 4\kappa(p_m)^2\big) \operatorname{Mod} p_m. \end{aligned}$$

Because this never can be zero therefore at the positions $x_n^{(1-)}$ and $x_n^{(1+)}$ are ω_{p_n}–numbers. □

Hence no ω_{p_n}–gap can reach over the symmetry axis $x_n^{(1)}$. Each ω_{p_n}–gap has in $\mathcal{P}_n$ always a symmetry partner, each ω_{p_n}–gap occurs twice in $\mathcal{P}_n$.

3.2. **The Overlapping of the period sections.**

The intervals $\mathcal{A}_n, n \ge 3$ defined by (3.1) cover the positive integers ≥ 4 gapless and densely. It is

$$\mathbb{N} = \{1, 2, 3\} \cup \bigcup_{n=3}^{\infty} \mathcal{A}_n \text{ and } \bigcap_{n=3}^{\infty} \mathcal{A}_n = \emptyset.$$

They are the beginnings of the period sections $\mathcal{P}_n$ of the ω_{p_n}–numbers. Hereafter let's say **A-sections** to the intervals $\mathcal{A}_n$. Every ω_{p_n}–number that lies in an A-section is a twin prime generator (see Corollary 3.5). In contrast to the A-sections the period sections $\mathcal{P}_n$ overlap each other very closely. So the period section $\mathcal{P}_9$ reachs over 1739 A-sections up to the beginning of the period section $\mathcal{P}_{1748}$ and the next $\mathcal{P}_{10}$ over 7863 A-sections up to the beginning of $\mathcal{P}_{7873}$.

Lemma 3.12. *Each origin ξ_n cannot be located at the beginning $\xi_m + a \cdot p_m\#_5$ of any period of the aggregate indicator function $\Psi(x, p_m)$ for $n > m$ and $a \in \mathbb{N}$. Therefore it holds for $n > m$*

$$\xi_n \not\equiv \xi_m (\operatorname{mod} p_m\#_5).$$

Proof. The equation

$$\frac{p_m^2-1}{6}+a\cdot p_m\#_5=\frac{p_n^2-1}{6} \text{ and hence } p_m(p_m+a\cdot p_{m-1}\#)=p_n^2$$

is for no prime $p_n > p_m$ solvable since holds $\gcd(p_n, p_m) = 1$. □

Vice versa every period section $\mathcal{P}_{n+1}$ starts always inside of the previous period section $\mathcal{P}_n$ nearby to its origin because (see (3.3) too)

$$\xi_{n+1} = \xi_n + d_n \text{ and}$$

$$d_n < \frac{p_n^2}{2} \ll \frac{p_n\#_5}{2}.$$

Lemma 3.13. *Even with modul p_m instead of $p_m\#_5$ holds*

$$\xi_n \not\equiv \xi_m (\operatorname{mod} p_m) \text{ for all } n > m.$$

Proof. We assume contrarily $\xi_n = \xi_m + a \cdot p_m$ for any $a \in \mathbb{N}$. Analogously to the proof of Lemma 3.12 we multiply by 6 and get finally

$$p_n^2 = p_m(p_m + 6a).$$

Also this equation is since $\gcd(p_m, p_n) = 1$ for no $p_m \neq p_n \in \mathbb{P}$ solvable. □

4. The ω_{p_n}–Gaps

Definition 4.1. The inverse of $\eta(p_n)$ means the **average distance** between two immediately successive ω_{p_n}–numbers in the period section $\mathcal{P}_n$

$$\bar{\delta}(p_n) := \frac{1}{\eta(p_n)}, \tag{4.1}$$

the average size of the so called ω_{p_n}–**gaps**.

Corollary 4.2. *From Lemma 3.6 follows*

$$\lim_{n\to\infty} \bar{\delta}(p_n) = \infty.$$

Theorem 4.3. *For $p_n > 200$ the square of the average size of the ω_{p_n}–gaps is less than the length of the interval $\mathcal{A}_n$*

$$\bar{\delta}(p_n)^2 < d_n \text{ for } p_n > 200.$$

Proof. At first we prove that

$$u(p_n) := p_n \eta(p_n)^2$$

is an increasing function by trend. We consider their properties for two cases:

A) $p_{n+1} \geq p_n + 4$:

$$\begin{aligned} u(p_{n+1}) - u(p_n) &= \eta(p_n)^2 \left(p_{n+1} \frac{(p_{n+1}-2)^2}{p_{n+1}^2} - p_n \right) \\ &= \eta(p_n)^2 \left(\frac{p_{n+1}(p_{n+1}-4)+4}{p_{n+1}} - p_n \right) \\ &= \eta(p_n)^2 \left(p_{n+1} - 4 - p_n + \frac{4}{p_{n+1}} \right) \\ &\geq \frac{4\eta(p_n)^2}{p_{n+1}} > 0. \end{aligned}$$

Hence it holds in this case $u(p_{n+1}) > u(p_n)$.

B) $p_{n+1} = p_n + 2$:

$$u(p_{n+1}) - u(p_n) = \eta(p_n)^2 \left(p_{n+1} \frac{(p_{n+1}-2)^2}{p_{n+1}^2} - p_n \right)$$

and since $p_n = p_{n+1} - 2$

$$= p_n \eta(p_n)^2 \left(\frac{p_n}{p_{n+1}} - 1 \right)$$

$$= p_n \eta(p_n)^2 \cdot \left(\frac{p_n - p_{n+1}}{p_{n+1}} \right)$$

$$= -\frac{2p_n}{p_n+2} \eta(p_n)^2 < 0.$$

Hence holds $u(p_{n+1}) < u(p_n)$. Now we set $u(p_{n+1}) = u(p_n) - v(p_n)$ with the "*loss function*"

$$v(p_n) := \frac{2p_n}{p_n+2} \eta(p_n)^2.$$

At first we'll look for the behavior of $u(p_{n+2})$ depending on the prime distance

$$a := p_{n+2} - p_{n+1} = p_{n+2} - p_n - 2 \text{ for } a = 4, 6, 10, 12, 16, \ldots$$

With it we get

$$\begin{aligned} u(p_{n+2}) &= p_{n+2} \cdot \eta(p_{n+2})^2 \\ &= (p_n + a + 2) \cdot \eta(p_n + a + 2)^2 \\ &= (p_n + a + 2) \cdot \frac{(p_n + a)^2}{(p_n + a + 2)^2} \cdot \frac{p_n^2}{(p_n + 2)^2} \cdot \eta(p_n)^2 \\ &= u(p_n) \cdot \frac{(p_n + a)^2 \cdot p_n}{(p_n + a + 2)(p_n + 2)^2}. \end{aligned}$$

We consider the difference between nominator and denominator of the fraction

$$\begin{aligned} &(p_n + a)^2 \cdot p_n - (p_n + a + 2)(p_n + 2)^2 \\ &= (p_n^2 + 2ap_n + a^2)p_n - (p_n + a + 2)(p_n^2 + 4p_n + 4) \\ &= p_n^3 + 2ap_n^2 + a^2 p_n - p_n^3 - 4p_n^2 - 4p_n - ap_n^2 - \\ &- 4ap_n - 4a - 2p_n^2 - 8p_n - 8 \\ &= (a - 6)p_n^2 + (a^2 - 4a - 12)p_n - 4a - 8 \end{aligned}$$

and get for

$$\begin{aligned} a = 4 &\rightarrow -2p^2 - 12p - 24 < 0 \\ a = 6 &\rightarrow -32 < 0 \\ a = 10 &\rightarrow 4p^2 + 48p - 48 > 0 \mid p \geq 2. \end{aligned}$$

This means that $u(p_{n+2}) < u(p_n)$ for $a = 4, 6$ and $u(p_{n+2}) > u(p_n)$ for $a \geq 10$.

With an analogous procedure we can demonstrate even for the case $p_{n+3} = p_{n+2} + 2$ that also holds

$$u(p_{n+3}) > u(p_n)$$

if $a \geq 10$. In the case $p_{n+3} > p_{n+2} + 2$ it holds $u(p_{n+3}) > u(p_n)$ because of A). It seems to be important to emphasize that all these results hold for the case $p_{n+1} - p_n = 2$.

The loss function $v(p_n)$ is strictly monotonic decreasing, because for two twin primes $p_n, p_n + 2$ und $p_n + 2 + a, p_n + 4 + a$ with the distance $a \geq 4$ holds [7]

$$\begin{aligned} v(p_n + 2 + a) &= \frac{2(p_n + 2 + a)}{p_n + 4 + a}\eta(p_n + 2 + a)^2 \\ &\quad \text{and since } \eta(p) \text{ is a decreasing function} \\ &\leq \frac{2}{p_n + 4 + a} \cdot \frac{(p_n + a)^2}{p_n + 2 + a}\eta(p_n + 2)^2 \\ &= \frac{2}{p_n + 4 + a} \cdot \frac{(p_n + a)^2}{p_n + 2 + a} \cdot \frac{p_n^2}{(p_n + 2)^2}\eta(p_n)^2 \\ &= v(p_n) \cdot \frac{p_n}{p_n + 2} \cdot \frac{p_n + a}{p_n + 2 + a} \cdot \frac{p_n + a}{p_n + 4 + a} < v(p_n). \end{aligned}$$

From twin prime to twin prime the loss function $v(p_n)$ monotonicly decreases for each twin distance a.

As upshot we see that in the majority of cases $u(p)$ is an increasing function while the loss function $v(p)$ from twin prime to twin prime decreases. The function $u(p)$ tends to result in an increasing function.

The greatest twin prime < 200 is $(197, 199)$. For the next prime number 211 holds $u(211) > 1.5159 = \frac{3}{2} + 0.0159$. Since the next prime after 211 follows only at 223 therefore for no prime $p_n > 211$ is $u(p_n) < u(211)$. On the other hand since $v(p_n)$ is monotonicly decreasing, is $v(197) < 0.0148 < 0.0159$ and all further $v(p_{n+k})$ are even less. Therefore we get for $p_n > 200$ with (3.2) and (4.1)

$$u(p_n) = p_n\eta(p_n)^2 > \frac{3}{2} \longrightarrow \bar{\delta}(p_n)^2 < \frac{2}{3}p_n < \frac{2}{3}(p_n + 1) \leq d_n.$$

This completes the proof. □

Corollary 4.4. *Since $u(p)$ is an increasing function by trend there is always a prime p and a number $c_p > 1$ such that holds*

$$c_p \cdot \bar{\delta}(p_n)^2 < d_n \textit{ for } p_n > p.$$

For instance we get for

$$\begin{aligned} p_n > 1,277 &\to 2\bar{\delta}(p_n)^2 < d_n, \\ p_n > 25,561 &\to 10\bar{\delta}(p_n)^2 < d_n, \\ p_n > 77,291 &\to 20\bar{\delta}(p_n)^2 < d_n \text{ or for} \\ p_n > 830,293 &\to 100\bar{\delta}(p_n)^2 < d_n. \end{aligned}$$

[7]For $a \geq 10$ there can be further single primes between $p_n + 2$ and $p_n + 2 + a$.

5. Generic Extensions

Definition 5.1. Let $x \in \mathbb{N}$. We will denote with

$$\gamma_n^{(m)}(x) := x + m \cdot p_n\#_5$$

the m^{th} **generic extension** of x of the order n.

In order to study the distribution of the ω_{p_n}–numbers in their period section $\mathcal{P}_n$ we will it partition in the following subsections

$$\mathcal{P}_n = \bigcup_{m=0}^{p_n-1} \mathcal{P}_n^{(m)}$$
$$\text{with} \tag{5.1}$$
$$\mathcal{P}_n^{(m)} := \{x \in \mathbb{N} \mid \gamma_{n-1}^{(m)}(\xi_n) \leq x < \gamma_{n-1}^{(m+1)}(\xi_n)\}, 0 \leq m \leq p_n - 1.$$

Now we shift these subsections to the left by $\xi_n - 1$ and obtain

$$\mathcal{G}_n = \mathcal{P}_n - (\xi_n - 1) = \bigcup_{m=0}^{p_n-1} \mathcal{G}_n^{(m)} = \{x \in \mathbb{N} \mid 1 \leq x \leq p_n\#_5\}$$
$$\text{with} \tag{5.2}$$
$$\mathcal{G}_n^{(m)} := \mathcal{P}_n^{(m)} - (\xi_n - 1) = \{x \in \mathbb{N} \mid \gamma_{n-1}^{(m)}(1) \leq x < \gamma_{n-1}^{(m+1)}(1)\}$$

for $0 \leq m \leq p_n - 1$. Obviously holds $\mathcal{G}_n^{(0)} \equiv \mathcal{G}_{n-1}$.

Lemma 5.2. *The sections $\mathcal{P}_n$ and $\mathcal{G}_n$ are equivalent with respect to the count of the ω_{p_n}–numbers and their relative positions shifted by $\xi_n - 1$.*

Proof. Due to the periodicity of the aggregate indicator function $\Psi(x, p_n)$ and the setting $\psi(x, p_n) = 0 \mid x \leq \kappa(p_n)$ holds for all $x \mid 1 \leq x \leq \xi_n - 1$

$$\Psi(x, p_n) = 0 \text{ if and only if } \Psi(x + p_n\#_5, p_n) = 0$$
$$\text{and}$$
$$\Psi(x, p_n) > 0 \text{ if and only if } \Psi(x + p_n\#_5, p_n) > 0$$

for each $n \geq 3$. Hence the sets

$$A_n = \{x \in \mathbb{N} \mid 1 \leq x \leq \xi_n - 1\}$$
$$\text{and}$$
$$B_n = \{x \in \mathbb{N} \mid p_n\#_5 + 1 \leq x \leq p_n\#_5 + \xi_n - 1\}$$

are equivalent with respect to the relative positions to the ω_{p_n}–numbers and have the same count of them.

On the other hand is

$$\mathcal{G}_n \cap \mathcal{P}_n = \{x \in \mathbb{N} \mid \xi_n \leq x \leq p_n\#_5\}$$

and

$$\mathcal{G}_n = A_n \cup (\mathcal{G}_n \cap \mathcal{P}_n) \tag{5.3}$$

as well as

$$\mathcal{P}_n = (\mathcal{G}_n \cap \mathcal{P}_n) \cup B_n. \tag{5.4}$$

Hence the sets $\mathcal{P}_n$ and $\mathcal{G}_n$ are also equivalent with respect to the count of ω_{p_n}–numbers. Due to the equivalence with respect to their relative positions between A_n and B_n and the positions of them in (5.3) resp. (5.4) the sets $\mathcal{G}_n$ and $\mathcal{P}_n$ are equivalent with respect to the positions of the ω_{p_n}–numbers shifted by $\xi_n - 1$, the size of A_n. This completes the proof. □

With Lemma 5.2 holds

$$\phi(p_n) = |\{x \in \mathcal{P}_n \mid \Psi(x, p_n) > 0\}| = |\{x \in \mathcal{G}_n \mid \Psi(x, p_n) > 0\}|. \tag{5.5}$$

Since it holds $\mathcal{G}_{n+1}^{(0)} \equiv \mathcal{G}_n$, the relations above with respect to the ω_{p_n}–numbers are valid for all $\mathcal{G}_{n+1}^{(0)}$ and $\mathcal{P}_{n+1}^{(0)}$ and because of the periodicity also for all $\mathcal{G}_{n+1}^{(m)}$ and $\mathcal{P}_{n+1}^{(m)}$ with $1 \le m \le p_{n+1} - 1$. Obviously they are in all subsections $\mathcal{P}_{n+1}^{(m)}$ symmetrically distributed. Due to Lemma 5.2 this is valid for $\mathcal{G}_{n+1}^{(m)}$ too. For the $\omega_{p_{n+1}}$–numbers this all is not valid.

Definition 5.3. Analogously to (3.7) we denote with

$$\phi^{(m)}(p_n) := |\{x \in \mathcal{G}_n^{(m)} \mid \Psi(x, p_n) > 0\}|$$

the number of ω_{p_n}–numbers in the subsection $\mathcal{G}_n^{(m)}$ for $0 \le m \le p_n - 1$.

Lemma 5.4. *Let x be a fixed member of $\mathcal{G}_n$, p a prime number, m an integer varying between 0 and $p-1$ and $\gamma_n^{(m)}(x)$ the m^{th} generic extension of order n. Then for all $p > p_n$ the value of*

$$\tau(\gamma_n^{(m)}(x), p) = \tau(x + m \cdot p_n\#_5, p)$$

is uniquely determined by m.

Proof. Contrarily we assume that two different values m_1, m_2 of m result in the same value of the position function

$$\tau(x + m_1 \cdot p_n\#_5, p) \overset{!}{=} \tau(x + m_2 \cdot p_n\#_5, p).$$

By virtue of (2.4) we have

$$\begin{aligned}\tau(x + m \cdot p_n\#_5, p) &= (x + m \cdot p_n\#_5 + \kappa(p)) \operatorname{Mod} p \\ &= (\tau(x, p) + m \cdot p_n\#_5) \operatorname{Mod} p.\end{aligned}$$

Since x is a fixed member of $\mathcal{G}_n$ it must hold

$$(m_1 \cdot p_n\#_5) \operatorname{Mod} p = (m_2 \cdot p_n\#_5) \operatorname{Mod} p.$$

Because of $m_1, m_2 < p$ and $\gcd(p_n\#_5, p) = 1$ this equation is only solvable with $m_1 = m_2$. □

Lemma 5.5. *Let x as a member of $\mathcal{G}_n$ be an ω_{p_n}–number. Then $p_{n+1} - 2$ generic extensions $\gamma_n^{(m)}(x)$ are $\omega_{p_{n+1}}$–numbers and two aren't.*

Proof. Because of the periodicity of $\Psi(x, p_n)$ it's evident that $\gamma_n^{(m)}(x)$ are ω_{p_n}–numbers for $0 \le m \le p_{n+1} - 1$, it holds

$$\Psi(\gamma_n^{(m)}(x), p_n) > 0 \text{ since } \Psi(x + m \cdot p_n\#_5, p_n) = \Psi(x, p_n) > 0.$$

We consider for a fixed x_o the position function with respect to p_{n+1}

$$\tau(\gamma_n^{(m)}(x_o), p_{n+1}) = \tau(x_o + m \cdot p_n\#_5, p_{n+1}).$$

It has p_{n+1} different values and by virtue of Lemma 5.4 exactly one value 0 for $m = m_-$ and one value $2\kappa(p_{n+1})$ for $m = m_+$

$$\tau(\gamma_n^{(m_-)}(x_o), p_{n+1}) = 0 \text{ and}$$
$$\tau(\gamma_n^{(m_+)}(x_o), p_{n+1}) = 2\kappa(p_{n+1}).$$

By virtue of (2.11) we get for the indicator function

$$\psi(\gamma_n^{(m_\pm)}(x_o), p_{n+1}) = 0$$

and since

$$\Psi(\gamma_n^{(m_\pm)}(x_o), p_{n+1}) = \Psi(\gamma_n^{(m_\pm)}(x_o), p_n) \cdot \frac{\psi(\gamma_n^{(m_\pm)}(x_o), p_{n+1})}{p_{n+1}}$$

we get finally for 2 values $m_\pm$

$$\Psi(\gamma_n^{(m_\pm)}(x_o), p_{n+1}) = 0,$$

while for $p_{n+1} - 2$ values of m holds $\psi(\gamma_n^{(m)}(x), p_{n+1}) > 0$, which results in

$$\Psi(\gamma_n^{(m)}(x), p_{n+1}) > 0.$$

Hence there are $p_{n+1} - 2$ generic extensions of the order n that are $\omega_{p_{n+1}}$–numbers. □

Theorem 5.6. *The number $\phi^{(m)}(p_n)$ of ω_{p_n}–numbers in any subsection $\mathcal{G}_n^{(m)}$ with $0 \le m \le p_n - 1$ is lower bounded by*

$$\phi^{(m)}(p_n) \ge (p_{n-1} - 4) \cdot \phi(p_{n-2}).$$

Proof. We consider a fixed $\omega_{p_{n-2}}$–number $x_o \in \mathcal{G}_{n-2}$ with $\Psi(x_o, p_{n-2}) > 0$ and their generic extensions of order $n - 2$

$$\gamma_{n-2}^{(m)}(x_o) \text{ for } 0 \le m \le p_{n-1} - 1.$$

By virtue of Lemma 5.5 we know that there are $p_{n-1} - 2$ generic extensions with $\Psi(\xi_{n-2}(x_o, m), p_{n-1}) > 0$, which means they are $\omega_{p_{n-1}}$–numbers. As generic extensions they are members of $\mathcal{G}_{n-1}$. Hence they are also members of $\mathcal{G}_n^{(0)}$. In what follows we will denote these $\omega_{p_{n-1}}$–numbers as x_o–**candidates**. Let y be any x_o–candidate. Then holds $\Psi(y, p_{n-1}) > 0$ and we can use the position function $\tau(y, p_n)$ to check whether the x_o–candidates are also ω_{p_n}–numbers. By virtue of Lemma 5.4 all the $p_{n-1} - 2$ x_o–candidates have different τ–values. But since the position function $\tau(y, p_n)$ has $p_n > p_{n-1} - 2$ values, the τ–bad values

$$\tau(y, p_n) = 0 \text{ or } \tau(y, p_n) = 2\kappa(p_n)$$

do not necessarily occur among the x_o–candidates, but they can occur single or both. Hence there are at least $(p_{n-1} - 2) - 2 = p_{n-1} - 4$ x_o–candidates that are also ω_{p_n}–numbers in $\mathcal{G}_n^{(0)}$. This holds for one (fixed) $\omega_{p_{n-2}}$–number x_o. But since there are $\phi(p_{n-2})$ different $\omega_{p_{n-2}}$–numbers in $\mathcal{G}_{n-2}$ we have

$$\phi^{(0)}(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}) \tag{5.6}$$

ω_{p_n}–numbers in the subsection $\mathcal{G}_{n-1} = \mathcal{G}_n^{(0)}$.

The above described situation related to the uniqueness of the τ–values holds also for the generic extensions of order $n-1$ of all $p_{n-1} - 2$ x_o–candidates y

$$\gamma_{n-1}^{(r)}(y) \mid \; 1 \leq r \leq p_n - 1$$

because if y_1, y_2 are two different x_o–candidates then holds by virtue of Lemma 5.4

$$\begin{aligned} \tau(y_1, p_n) &\neq \tau(y_2, p_n) \text{ and hence} \\ \tau(\gamma_{n-1}^{(r)}(y_1), p_n) &= (y_1 + r \cdot p_{n-1}\#_5 + \kappa(p_n)) \operatorname{Mod} p_n \\ &= (\tau(y_1, p_n) + r \cdot p_{n-1}\#_5) \operatorname{Mod} p_n \\ &\neq (\tau(y_2, p_n) + r \cdot p_{n-1}\#_5) \operatorname{Mod} p_n \\ &= \tau(\gamma_{n-1}^{(r)}(y_2), p_n) \end{aligned}$$

for $1 \leq r \leq p_n - 1$ and we have in all subsections $\mathcal{G}_n^{(1)}, \mathcal{G}_n^{(2)}, \ldots, \mathcal{G}_n^{(p_n-1)}$ with respect to the τ–values the same situation like in $\mathcal{G}_n^{(0)}$. Therefore and since the right side of (5.6) is independent from the number of a subsection it holds in all subsections

$$\phi^{(r)}(p_n) \geq (p_{n-1} - 4) \cdot \phi(p_{n-2}), \; 0 \leq r \leq p_n - 1.$$

□

Theorem 5.7. *The ω_{p_n}–numbers in $\mathcal{G}_n$ are over the subsections $\mathcal{G}_n^{(0)}, \ldots, \mathcal{G}_n^{(p_n-1)}$ asymptotically uniform distributed*

$$\phi^{(m)}(p_n) \sim \frac{\phi(p_n)}{p_n} \text{ for } 0 \leq m \leq p_n - 1.$$

Proof. We consider the following ratio and get by virtue of Theorem 5.6

$$\begin{aligned} \frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} &\geq \frac{p_n(p_{n-1} - 4) \cdot \phi(p_{n-2})}{\phi(p_n)} \\ &= \frac{p_n(p_{n-1} - 4)}{(p_{n-1} - 2)(p_n - 2)} \\ &\text{and with } p_{n-1} = p_n - d \\ &= \frac{p_n\,(p_n - (d+4))}{(p_n - (d+2))(p_n - 2)} \\ &= \frac{1 - \frac{d+4}{p_n}}{\left(1 - \frac{d+2}{p_n}\right)\left(1 - \frac{2}{p_n}\right)} \underset{p_n \to \infty}{\longrightarrow} 1. \end{aligned}$$

On the other hand holds $\mathcal{G}_n^{(0)} = \mathcal{G}_{n-1}$ and from

$$x \in \mathcal{G}_n^{(0)} \mid \Psi(x, p_{n-1}) > 0 \text{ follows } \Psi(\gamma_{n-1}^{(m)}(x), p_{n-1}) > 0 \text{ for } 1 \leq m \leq p_n - 1.$$

Hence we have

$$\begin{aligned}\phi(p_{n-1}) &= |\{x \in \mathcal{G}_n^{(0)} \mid \Psi(x, p_{n-1}) > 0\}| \\ &\text{and for } 1 \le m \le p_n - 1 \\ &= |\{x \in \mathcal{G}_n^{(m)} \mid \Psi(x, p_{n-1}) > 0\}|\end{aligned}$$

and therefore

$$\begin{aligned}\phi(p_{n-1}) &= |\{x \in \mathcal{G}_n^{(m)} \mid \Psi(x, p_{n-1}) > 0\}| \\ &\ge |\{x \in \mathcal{G}_n^{(m)} \mid \Psi(x, p_n) > 0\}| = \phi^{(m)}(p_n).\end{aligned}$$

Finally with

$$\frac{p_n \cdot \phi^{(m)}(p_n)}{\phi(p_n)} \le \frac{p_n \cdot \phi(p_{n-1})}{\phi(p_n)} = \frac{p_n}{p_n - 2} \underset{p_n \to \infty}{\longrightarrow} 1$$

holds

$$\phi^{(m)}(p_n) \sim \frac{\phi(p_n)}{p_n}.$$

□

Corollary 5.8. *The in the previous section shown symmetry of the ω_{p_n}–numbers in their period section $\mathcal{P}_n$ and the above demonstrated asymptotically uniform distribution of the ω_{p_n}–numbers over the subsections $\mathcal{G}_n^{(0)}, \ldots, \mathcal{G}_n^{(p_n-1)}$* ***avert*** *with Lemma 5.2 the formation of extreme configurations of the ω_{p_n}–gaps in their period section $\mathcal{P}_n$.*

6. Proof of the Twin Prime Conjecture

Proof. The proof will be done by contradiction. We assume contrarily that there is only a finite number of twin primes and therefore only a finite number of twin prime generators. Let y_o be the greatest one. It lies in the A-section $\mathcal{A}_{n_o}$ with $n_o = \pi(\hat{p}(y_o))$, the beginning of the period section $\mathcal{P}_{n_o}$. W.l.o.g. we can assume that $n_o > 200$. In the successive A-sections $\mathcal{A}_t$ with $t > n_o$ consequently there cannot be any twin prime generators and hence by virtue of Corollary 3.5 no ω_{p_t}–numbers. But then we have ω_{p_t}–gaps with sizes $> d_t$ in **all** (infinitely many) period sections $\mathcal{P}_t$ for $t > n_o$.
Because

- the squared average size of the ω_{p_t}–gaps $\bar{\delta}(p_t)^2$ is **less** than $\frac{d_t}{c_p}$ with a number c_p by virtue of Corollary 4.4,
- all period sections $\mathcal{P}_t$ are very closely overlapped and due to the symmetrical distribution of the ω_{p_t}–numbers around $\frac{p_t\#_5}{2}$ and $p_t\#_5$ and
- since the asymptotic uniform distribution of the ω_{p_t}–numbers over the subsections $\mathcal{G}_t^{(0)}, \ldots, \mathcal{G}_t^{(p_t-1)}$ extreme configurations of the ω_{p_t}–gaps in $\mathcal{P}_t$ cannot occur

therefore it is not possible to have for **all** $t > n_o$ only period sections $\mathcal{P}_t$ with ω_{p_t}–gaps at their beginnings that all are **greater** than d_t.

Therefore the proof assumption cannot be valid and hence the Twin Prime Conjecture must be true.

□

References

[1] *Bundschuh, P.*: Einführung in die Zahlentheorie , Springer, 2008.
[2] *Ribenboim, P.*: Die Welt der Primzahlen , Springer, 2011.
[3] *Brun, V.*: La série $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\ldots$ où les dénominateurs sont "nombres premiers jumeaux" est convergente où finie. Bull. Sci. Math. (2) 43 (1919), 100-104 and 124-128.
[4] *Erdös, P.*: The difference of consecutive primes. Duke Mathematical Journal, 6, 1940, p. 438-441
[5] *Bombieri, E., Davenport, H.*: Small differences between prime numbers. Proc. Roy. Soc. (A) 293 (1966), 1-18
[6] *Goldston, D.A., Motohashi, Y., Pintz, J., Yıldırım, C.Y.*: Small gaps between primes exist. In: Proceedings of the Japan Academy, Series A 82, 2006, 61-65
[7] *Green, B., Tao, T.*: The primes contain arbitrarily long arithmetic progressions. Annals of Math. (2) 167 (2008), 481-547
[8] *Zhang, Y.*: Bounded gaps between primes. Annals of Math. 179 (2014), 1121-1174
[9] *Tao, T.* Bounded gaps between primes (Polymath8) a progress report. https://terrytao.wordpress.com/2013/06/30/bounded-gaps-between-primes-polymath8-a-progress-report/
[10] *Maynard, J.*: Small gaps between primes. Annals of Math., Second Series, 181 (1), 383-413 (2015)
[11] *Gensel, B.*: An Elemenatary Proof of the Twin Prime Conjecture. Turkish Journal of Analysis and Number Theory, 2020, 8(3), 52-56

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