PROOF OF π AND e IS IRRATIONAL NUMBER BY ITS TRANSCENDENCE
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Abstract. I will show how to prove π and e are irrational numbers with the fact that they are transcendental numbers.

Lemma 1. All transcendental numbers are irrational numbers.
Proof: Let x be a transcendental number. Assume x is a rational number.

\[ x = \frac{a}{b}, \text{ where } a \text{ and } b \text{ are integers with } b \neq 0. \]

However, \( bx - a = 0 \) and so x is an algebraic number, which contradicts the assumption. Therefore, x is an irrational number.

Lemma 2. Product of two algebraic numbers is also algebraic.
This can easily be proved by the resultant of the polynomial.

Theorem. (Lindemann–Weierstrass theorem) If \( a_1, a_2, \ldots, a_n \) are distinct algebraic numbers, then \( e^{a_1}, e^{a_2}, \ldots, e^{a_n} \) are linearly independent over the algebraic numbers. [1]

Corollary 1. \( e^a \) is a transcendental number if a is a nonzero algebraic number.
This can easily be proved by Lindemann–Weierstrass theorem.

Proposition. π is an irrational number.
Proof. First, we want to show π is a transcendental number.
Assume π is an algebraic number. Since \( i \) is an algebraic number, from lemma 2, \( \pi i \) is an algebraic number. From corollary 1, \( e^{\pi i} \) is a transcendental number. However, by Euler’s identity, \( e^{\pi i} = -1 \) which clearly is an algebraic number. Therefore, the assumption leads to a contradiction. Therefore, π is a transcendental number. By lemma 1, π is an irrational number.

Proposition. e is an irrational number.
Proof: Put \( a = 1 \) into corollary 1. We can immediately get e is a transcendental number.
By lemma 1, we can deduce that e is an irrational number.

REFERENCE