The connection between the Wilson’s theorem and the Lenstra-Pomerance-Wagstaff conjecture

Daoudi R.*

University of Caen Normandie 14 000 FRANCE

E-mail: red.daoudi@laposte.net

Abstract

A Mersenne number $M_n$ is defined such as $M_n = 2^n - 1$ and a Mersenne prime is the form of $M_p = 2^p - 1$ where $p$ is a prime number.

Lenstra, Pomerance and Wagstaff (called LPW conjecture) have conjectured that there are infinite many Mersenne primes. According to them there are many infinite Mersenne primes of the form $M_p = 2^p - 1$ for some prime $p$.

In this paper I try to reformulate the LPW conjecture using the Wilson’s theorem.

The reformulation

Let $n$ to be an integer $\geq 1$ and $\sigma(n)$ is the sum of the divisors of $n$ and $k$ is an integer $\geq 1$.

$$\frac{1}{(n - \sigma(n))^2} - \frac{(\sigma(n) - n)!}{(n - \sigma(n))^3} = k$$
I compute the formula using Python, Wolframalpha for small numbers and Sage.

1. When n is prime then $k=2$

2. With some values of n, $k \neq 2$

Now, we focus our study for $k \neq 2$

Sometimes $k \neq 2$ for different values of n but with the same values of $\sigma(n)$

For example with n=27 and n=35 we have: $k = 2834329$.

**Computation with Python**

We use the following code to check whether there is a counterexample.

```python
from math import *
from fractions import Fraction

def div(n):
    ll = []
    for i in range(1, n+1):
        if n % i == 0:
            ll.append(i)
    return sum(ll)

for i in range(2, 10000):
    divi = div(i)-i
    res = Fraction(1, divi**2) + Fraction(factorial(divi), divi**3)
    if res.denominator==1 and res.numerator!=2:
        print(i, res)
```
Note that $div(n)$ should be replaced by $mu(n)$ to reduce the computation time.

**Proof**

**Conjecture:** Let $p$ to be a prime number such that $p \geq 3$, $n$ a composite number and $k$ an integer such that $k \neq 2$. For some values of $n$ there exists an integer $k$ such that:

$$\frac{1}{(n - \sigma(n))^2} - \frac{(\sigma(n) - n)!}{(n - \sigma(n))^3} = k$$

where $\sigma(n) - n = p$

**Proof:** I tried to find all integers $n$ such that $\sigma(n) - n$ is prime.

It is trivial that if $p$ is a prime then $\sigma(p) - p = 1$.

If $\sigma(n) - n$ is not prime, I don’t focus on the values for which $n$ is prime.

Because $n$ can be a big number and tends to infinity we have:

If $n$ is even whe have: $\sigma(n) - n = 1 + 2 + ...$

If $n$ is odd and composite we have: $\sigma(n) - n = 1 + ...$

According to the Wilson’s theorem : $(1 + 2 + ...)$ is prime

if and only if $((1 + 2 + ...) - 1)! \equiv -1 \mod (1 + 2 + ...)$. In other words:(2 + ...)

$\equiv -1 \mod (1 + 2 + ...)$

It means $(2 + ...) = k(1 + 2 + ...) - 1$

We can simplify:

$\sigma(n) - n = (1 + 2 + ...) \text{ and } \sigma(n) - (n + 1) = (2 + ...)$, then we replace and we have:

$(\sigma(n) - (n + 1))! = k(\sigma(n) - n) - 1$

Then $((\sigma(n) - (n + 1))! + 1) = k(\sigma(n) - n)$

Then $\frac{((\sigma(n) - (n+1))!+1)}{(\sigma(n) - n)} = k$

Finally, $k$ must be an integer such that $k \neq 2$ and in this case $\sigma(n) - n$ is a prime number.
The connection between Wilson’s theorem and LPW conjecture

I wondered whether there is an infinity integer $n$ such that $\sigma(n) - n$ is prime.

We can reformulate this: there is an infinity integer $n$ such that:

$$\frac{((\sigma(n) - (n+1))! + 1)}{(\sigma(n) - n)} = k$$

$k$ is an integer $\neq 2$

Unfortunately it seems that is a subproblem of LPW conjecture and more precisely it is conjectured that there are infinitely many Mersenne primes.