The connection between the Wilson's theorem and the Lenstra-Pomerance-Wagstaff conjecture

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Abstract

A Mersenne number M_n is defined such as $M_n = 2^n - 1$ and a Mersenne prime is the form of $M_p = 2^p - 1$ where p is a prime number.
Lenstra, Pomerance and Wagstaff (called LPW conjecture) have conjectured that there are infinite many Mersenne primes. According to them there are many infinite Mersenne primes of the form $M_p = 2^p - 1$ for some prime p.
In this paper I try to reformulate the LPW conjecture using the Wilson's theorem.

The reformulation

Let n to be an integer ≥ 1 and $\sigma(n)$ is the sum of the divisors of n and k is an integer≥ 1.

$$\frac{1}{(n-\sigma(n))^2} - \frac{(\sigma(n)-n)!}{(n-\sigma(n))^3} = k$$

I compute the formula using Python, Wolframalpha for small numbers and Sage.

1. When n is prime then k=2
2. With some values of n, $k \neq 2$

Now, we focus our study for $k \neq 2$

Sometimes $k \neq 2$ for different values of n but with the same values of $\sigma(n) - n$

For example with n=27 and n=35 we have: $k = 2834329$.

Computation with Python

We use the following code to check whether there is a counterexample.

```
from math import *
from fractions import Fraction


def div(n):
ll = []
for i in range(1, n+1):
 if n % i == 0:
 ll.append(i)
return sum(ll)


for i in range(2, 10000):
divi = div(i)-i
res = Fraction(1, divi**2) - Fraction(factorial(divi), divi**3)
if res.denominator==1 and res.numerator!=2:
print(i, res)
```

Note that $div(n)$ should be replaced by $mu(n)$ to reduce the computation time.

Proof

Conjecture: Let p to be a prime number such that $p \geq 3$, n a composite number and k an integer such that $k \neq 2$. For some values of n there exists an integer k such that:

$$\frac{1}{(n-\sigma(n))^2} - \frac{(\sigma(n)-n)!}{(n-\sigma(n))^3} = k$$

where $\sigma(n) - n = p$

Proof: I tried to find all integers n such that $\sigma(n) - n$ is prime.

It is trivial that if p is a prime then $\sigma(p) - p = 1$.

If $\sigma(n) - n$ is not prime, I don't focus on the values for which n is prime.

Lemma: I conjecture that

$$\lim_{n \to +\infty} \sigma(n) - n = +\infty$$

and consequently $\sigma(n) - n$ may be a big number and may tend to infinity. In fact, the limit is indeterminate.

If $\sigma(n) - n$ is even we have: $\sigma(n) - n = 1 + 2 + ...$

If $\sigma(n) - n$ is odd and composite we have: $\sigma(n) - n = 1 + ...$

According to the Wilson's theorem : $(1 + 2 + ...)$ is prime

if and only if $((1 + 2 + ...) - 1)! \equiv -1 \mod (1 + 2 + ...)$. In other words:$(2 + ...)$

$\equiv -1 \mod (1 + 2 + ...)$

It means $(2 + ...) = k(1 + 2 + ...) - 1$

We can simplify:

$\sigma(n) - n = (1 + 2 + ...)$ and $\sigma(n) - (n + 1) = (2 + ...)$, then we replace and we have:

$(\sigma(n) - (n+1))! = k(\sigma(n) - n) - 1$

Then $((\sigma(n) - (n+1))! + 1) = k(\sigma(n) - n)$

Then $\frac{((\sigma(n)-(n+1))!+1)}{(\sigma(n)-n)} = k$

Finally, k must be an integer such that $k \neq 2$ and in this case $\sigma(n) - n$ is a prime number.

The connection between Wilson's theorem and LPW conjecture

I wondered whether there is an infinity integer n such that $\sigma(n) - n$ is prime.

We can reformulate this: there is an infinity integer n such that: $\frac{((\sigma(n)-(n+1))!+1)}{(\sigma(n)-n)} = k$ where k is an integer $\neq 2$

Unfortunately it seems that is a subproblem of LPW conjecture and more precisely it is conjectured that there are infinitely many Mersenne primes.