Interpretation of superposition of eigen states and measurement problem concerning statistical ensemble

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Abstract

The measurement problem is an important open question for the interpretation of the foundations of quantum mechanics. For the purpose of solving this problem, we focus from a new angle on the interpretation of the superposition principle that is the origin of the measurement problem. As a result, we show that the measurement problem at issue cannot arise, provided the mathematical and physical aspects of the superposition principle are considered correctly. Our work demonstrates that since any mathematical superposition of eigenstates is never a new eigenstate, the superposed state at issue, if any, should be interpreted simply as a statistical ensemble of possible states that occur sequentially, instead of a mixed state indicative of simultaneous occurrence. Actually, this view leads to the conclusion that the concept of the currently accepted state vector and the motivation of the measurement problem have no perfect ground from both mathematical and physical aspects. Furthermore, using mainly mathematical method rather than thought experiments, we offer a realistic interpretation of the superposition of eigenstates based on ensemble of quantum states, thereby helping to capture the essence of the measurement problem which actually is not implicated in Apparatus and Observer.

Key words: Eigenstate, Quantum ensemble theory, Quantum mechanics, Superposition principle, Measurement problem

1. Introduction

One of the fundamental principles of quantum mechanics is the superposition principle, i.e., the principle of linear superposition of eigenstates. The superposition principle states that a quantum-mechanical system which can take on discrete eigenstates $\{\phi_n : (n \in \mathbb{N})\}$ is also able to occupy the superposed state

$$\psi = \sum_n a_n \phi_n, \tag{1}$$

where $\mathbb{N}$ denotes a set of integers [1]. Nevertheless, the measurement by Observer provides only eigenvalues without the occurrence of a likely state smeared by superposition. On that account, the measurement for quantum systems seems to cause the reduction of the superposed wave function to an eigenstate wave function [1, 2]. In this connection, von Neumann first assumed a dichotomy between two different types of evolutions: the unitary evolution and the measurement process [3]. The famous Projection Postulate proposed by him became one of the fundamental axioms in the standard theory of quantum mechanics. Hence the measurement problem arises that leads to the conception of Apparatus and Observer. The conception of the measurement process gave rise to dissenting views concerning the quantum phenomena.

A remarkable position is the attempt to ascribe a fundamental function of the measuring process to human consciousness [2, 3, 4, 5, 6].

The position of the statistical interpretation, or the many-world interpretation rejecting the Projection Postulate pursues the object to eliminate the jump-like features of the measurement [7, 8, 9, 10, 11, 12, 13, 14, 15].

Some of physicists direct their efforts to explaining the measurement problem in terms of the intervention of macroscopic system [16, 17].

Another method of the research into the measurement problem is characterized by the attempt to find some *ad hoc* parameters in order to explain the reduction [18].

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Their approach appears to be a theory of the relation between microscopic physics and macroscopic physics.

Some physicists have developed the theory of decoherence that describes non-unitary evolution by using the specific nature of open system [19, 20, 21, 22].

It should be noted the assertion that a mathematical superposition of states means an ensemble of states and not a mixed state [23].

Analyzing the previous allegations concerning the measurement problem, we can show the following classification of problems and contestations: (I) the problem of Observer, (II) the limits of the Projection Postulate, (III) the instantaneous character of the measurement process, (IV) the constraints imposed by conservation laws [1, 2, 24, 25, 26].

The major problem arises from the fact that while it is possible to observe eigenstates $\{\phi_n\}$ of a given quantum operator $\hat{L}$, a superposed state $\sum_n a_n \psi_n$ that is seen as being mathematically possible is not found. The history and the present situation of the measurement problem show that at the present stage any interpretation approach cannot give satisfactory answers to the measurement problem.

In this work, we newly review the superposition principle from the point of view distinct from the present ones. Our work focuses on the mathematical structure of the very superposition principle rather than the measurement problem, thus returning to the origin of the contestation. Starting from this, we demonstrate the self-evident mathematical fact that the superposition of eigenstates cannot yield a new eigenstate distinguished from a complete set of eigenstates. On the basis of this proposition we explain that the superposition of eigenstates indicates the sum of events characterized by an ensemble of eigenstates.

2. Mathematical impossibility of superposition of eigenstates as a mixture

The measurement problem shows a subtle inner inconsistency within the standard framework of quantum mechanics [2, 25]. In this work, we aim to reveal the inner inconsistency at issue. With this purpose we begin with analyzing the mathematical interpretation of the superposed state. It is obvious that the superposition of eigenstates is not an alternative eigenstate. If the superposition is mathematically possible, then we inevitably encounter the measurement problem. However if not so, the measurement problem should be assessed as having been a misled one from the outset. It is necessary to show for the purpose of systematic consideration that a state vector does not satisfy the eigenvalue equation which gives a complete set of eigenfunctions, although its proof is trivial and simple for experts. Actually, the discussion of this topic becomes the starting point for airtight demonstration of the measurement problem.

Let $\hat{L}$ be an operator and L, the corresponding eigenvalue. Then the eigenvalue equation reads as follows.

$$\hat{L}\phi = L\phi. \tag{2}$$

The above eigenvalue equation provides a set of eigenvalues $\{L_n : (n \in \mathbb{N})\}$ and a set of eigenfunctions corresponding to the eigenvalues, $\{\phi_n : (n \in \mathbb{N})\}$. Let us assume that a superposition of eigenfunctions $\psi = \sum_n a_n \phi_n$ is valid. Then $\psi = \sum_n a_n \phi_n$ should satisfy Eq. (2). Thus, we have

$$\hat{L} \sum_n a_n \phi_n = L \sum_n a_n \phi_n. \tag{3}$$

Multiplying the both sides of Eq. (3) by a particular eigenfunction ϕ_m^* and integrating with respect to position variables, we obtain

$$\sum_n a_n \left\langle \phi_m \middle| \hat{L} \middle| \phi_n \right\rangle = L \sum_n a_n \left\langle \phi_m | \phi_n \right\rangle. \tag{4}$$

In view of the orthonormality of eigenfunctions, we get

$$a_m L_m = L a_m. \tag{5}$$

Consequently, we arrive at

$$L_m = L. \tag{6}$$

Eq. (6) indicates that eigenvalues L_m are all identical. This is incompatible with the fact that we began with a set of different eigenfunctions. In order for Eq. (6) to be valid, all $\{a_n\}$ except for a particular a_m should be zero, which implies nothing but a particular eigenstate. In the end, we reach the conclusion that $\psi = \sum_n a_n \phi_n$ cannot be another eigenstate. From this, it follows that the state vector does not correspond to a physical reality, i. e., an observable.

This conclusion also can be confirmed by the following simple consideration. Should $\sum_n a_n \phi_n$ be an eigenstate of operator $\hat{L}$, it would be orthogonal to $\{\phi_n : (n \in \mathbb{N})\}$. However, we have

$$\left\langle \phi_m \middle| \sum_n a_n \phi_n \right\rangle = a_m \neq 0. \tag{7}$$

Accordingly, $\sum_n a_n \phi_n$ is not orthogonal to $\{\phi_n : (n \in \mathbb{N})\}$.

Meanwhile, this fact also is evident from the completeness of a system of eigenfunctions. Since a system of eigenfunctions is a complete system, any function different from the system of eigenfunctions cannot satisfy the eigenvalue equation.

It is common knowledge to be always possible to form an arbitrary function with the help of $\{a_n : (n \in \mathbb{N})\}$ and
$\{\phi_n : (n \in \mathbb{N})\}$ because of the completeness of eigenfunctions. This fact tells us that any functions can be expanded by eigenfunctions, but it is not an eigenfunction satisfying the eigenvalue equation. Thus, we arrive at the conclusion that $\sum_n a_n \phi_n$ is not an eigenstate of $\hat{L}$ and therefore the superposition of eigenstates is impossible.

On the other hand, it is necessary to consider the state vector in connection with eigenvalue equation. In fact, an eigenvalue equation involves a set of linear equations. Namely,

$$\left\{ \hat{L}\phi = L_n \phi : (n \in \mathbb{N}) \right\}. \tag{8}$$

For this reason, these equations distinguished by eigenvalues are all clearly different equations and therefore the linear combination of eigenfunctions loses mathematical meaning. In fact, $\sum_n a_n \psi_n$ amounts to a linear combination of solutions of different equations.

For example, for the two linear equations that are distinguished by eigenvalues L_n and L_m:

$$\hat{L}\phi_n = L_n \phi_n \tag{9}$$

and

$$\hat{L}\phi_m = L_m \phi_m, \tag{10}$$

it is mathematically meaningless to make a linear combination by use of the solutions of the different equations, ϕ_n and ϕ_m. Thus, it is clarified the fact that prior to measurement, a quantum system exists not in a superposed state, but in a particular eigenstate.

The final examination is directed towards the matter of whether the time-dependent Schrödinger equation ensures the application of the superposition principle in terms of eigenstates. If the superposition principle holds in the sense of eigenfunction, then for the expansion of a wave function in terms of eigenfunctions of $\hat{L}$, namely, $\psi = \sum_n a_n \phi_n$, the time-dependent Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \tag{11}$$

should be able to be written as

$$\sum_n a_n \left(\mathrm{i}\hbar \frac{\partial}{\partial t} - \hat{H} \right) \phi_n = 0. \tag{12}$$

However, the case is impossible.

In fact, if a_n is time-independent, the above equation loses mathematical meaning because ϕ_ns are only position-dependent and thus the wave function is time-independent. Accordingly, a_n should be necessarily time dependent. For this reason, Eq. (12) does not hold in the case of time-independent a_n. This is self-evident because eigenfunctions are not a solution of the time-dependent Schrödinger equation.

Now, we can examine the superposition of eigenstates in view of the time-dependency of a_n. Without loss of generality, we can expand the time-dependent wave function as

$$\psi(x,t) = \sum_n a_n(t)\, \phi_n(x). \tag{13}$$

In this case, the expansion coefficients $\{a_n(t)\}$ are determined by

$$\langle \phi_m(x) | \psi(x,t) \rangle = \left\langle \phi_m(x) \middle| \sum_n a_n(t)\, \phi_n(x) \right\rangle. \tag{14}$$

By the orthonormality of $\{\psi_m\}$, we get

$$a_m(t) = \langle \phi_m(x) | \psi(x,t) \rangle \ . \tag{15}$$

Inserting Eq. (13) into Eq. (11) enables us to go through the following manipulation:

$$\mathrm{i}\hbar \frac{\partial \sum_n a_n(t)\, \phi_n(x)}{\partial t} = \hat{H} \sum_n a_n(t)\, \phi_n(x), \tag{16}$$

$$\mathrm{i}\hbar \sum_n \frac{\mathrm{d}a_n(t)}{\mathrm{d}t} \phi_n(x) = \sum_n \hat{H} \left[a_n(t)\, \phi_n(x) \right], \tag{17}$$

$$\sum_n a_n(t) \left[\mathrm{i}\hbar \frac{1}{a_n(t)} \frac{\mathrm{d}a_n(t)}{\mathrm{d}t} \right] \phi_n(x) = \sum_n a_n(t)\, \hat{H}\phi_n(x). \tag{18}$$

Finally, we arrive at

$$\sum_n a_n(t) \left[\mathrm{i}\hbar \frac{\mathrm{d}\ \ln a_n(t)}{\mathrm{d}t} - \hat{H} \right] \phi_n(x) = 0. \tag{19}$$

By comparing Eq. (19) with Eq. (12), we find that the two wave equations with respect to eigenfunction take on different forms. The disagreement between them shows that the superposition principle cannot be satisfied by the superposition of eigenstates of an arbitrary Hermitian operator $\hat{L}$ taking a discontinuous eigenvalue spectrum, although a wave function can be expanded in terms of a set of eigenfunctions. Of course, the superposition of eigenstates does not fulfill the eigenvalue equation, and therefore it is not a possible state in the sense of eigenstate. Naturally, we arrive at the conclusion that the superposition of eigenstates implies an ensemble of possible states.

The impossibility of the superposition of eigenstates as a mixture needs to reassess the validity of the measurement problem. It should be considered that eigenstates $\{\phi_n : (n \in \mathbb{N})\}$ exhibit only their inherent property, irrespective of Apparatus and Observer because quantum operators possess attributes irrelative to Apparatus and Observer. The fact that results of measurement reflect purely eigenstates and not a superposed state requires the review of the mathematical foundation of the superposition principle rather than the measurement problem. The aforementioned description has been done in compliance with this requirement. Thus, the mathematical result is not inconsistent but compatible with physical result of measurement. As a consequence, it is useless to debate upon the measurement problem. It can be concluded that there does not exist the superposition principle for eigenstates, and therefore we are in a position to withdraw the measurement problem originating from the superposition of eigenstates.

3. Possibility of superposition in the sense of ensemble of states

It is necessary to discuss in more detail what the superposition of eigenstates, $\sum_n a_n\phi_n$ that is not an eigenstate really means.

First, let us consider a closed system described by a wave function ψ. When measuring a variable L represented by the Hermitian operator $\hat{L}$, the eigenvalues L_n of $\hat{L}$ are the possible results of measuring that variable. If a Hermitian operator $\hat{L}$ does not commute with the Hamiltonian operator $\hat{H}$, then the wave function is not the eigenfunction of $\hat{L}$. In this case, the wave function may be expanded in terms of the set of eigenfunctions ϕ_n of $\hat{L}$:

$$\psi = \sum_n a_n\phi_n. \tag{20}$$

By definition, the expectation value of $\hat{L}$ is obtained through

$$\begin{aligned}\left\langle \hat{L} \right\rangle &= \int \sum_{n,m} (a_m^*\phi_m^*)\,\hat{L}\,(a_n\phi_n)\,\mathrm{d}x \\ &= \sum_{n,m} a_m^* a_n \int \phi_m^* \hat{L}\phi_n \mathrm{d}x \\ &= \sum_{n,m} a_m^* a_n L_n \delta_{mn} \\ &= \sum_n |a_n|^2 L_n. \end{aligned} \tag{21}$$

In order to interpret the physical meaning of the above result, it should be taken into account that for a single measurement the values of measurement always belongs to $\{L_n : (n \in \mathbb{N})\}$ and the probability of a result L_n is given by $|a_n|^2$. In addition, it should be taken into consideration that as already mentioned, $\psi = \sum_n a_n\phi_n$ is not an eigenfunction of $\hat{L}$, while it represents the sum of events described by $\{\phi_n\}$. With these two facts in mind, we can interpret $\left\langle \hat{L} \right\rangle$ as the expectation value for the ensemble of eigenstates, but not for the mixture of eigenstates. What is concluded here is that the superposition principle is established in *the context of ensemble of eigenstates*.

For example, a wave function can be represented as a linear combination of the spin-up wave function and the spin-down one. Obviously, the superposition of the two spin wave functions cannot satisfy the eigenvalue equation with respect to spin operator. In this sense, the superposition is impossible. Due to the character of wave function, this superposition reflects an ensemble of states in the context of spin rather than a mixed state. With Eq. (21) in mind, it is obvious that this superposition merely represents an ensemble in regard to spin. In this sense, the superposition is possible.

On the other hand, it is necessary to pay attention to the fact that an eigenfunction of a certain operator $\hat{A}$ can be expressed as a linear combination of eigenfunctions of another operator $\hat{B}$. This fact implies by no means the superposition in terms of eigenfunction and does not bear physical meaning. It is due purely to the mathematical possibility of expanding an arbitrary function in terms of a complete system of eigenfunctions of a certain Hermitian operator. In fact, it has nothing to do with the superposition principle because eigenfunctions of operator $\hat{B}$ do not satisfy the eigenvalue equation of $\hat{A}$.

Let us consider the case when eigenvalues of $\hat{L}$ form a continuous spectrum. For example, let us examine expectation $\langle \hat{p} \rangle$ for a momentum component operator $\hat{p}$. If we expand the wave function in terms of eigenfunctions of the momentum operator, $\left\{\phi_p : (p \in \mathbb{R})\right\}$ we have

$$\psi = \int_p a_p \phi_p \mathrm{d}p, \tag{22}$$

where $\mathbb{R}$ denotes a set of real numbers.

Hence, the expectation value of $\hat{p}$ is written as

$$\begin{aligned} \langle \hat{p} \rangle &= \int \left(\left(\int_p a_p^* \phi_p^* \mathrm{d}p \right) \hat{p} \left(\int_{p'} a_{p'} \phi_{p'} \mathrm{d}p' \right) \mathrm{d}x \right. \\ &= \int_p \int_{p'} a_p^* a_{p'} \mathrm{d}p \mathrm{d}p' \int \phi_p^* \hat{p} \phi_{p'} \mathrm{d}x \\ &= \int_p \int_{p'} a_p^* a_{p'} p' \delta \left(p - p' \right) \mathrm{d}p \mathrm{d}p' \\ &= \int_p \left| a_p \right|^2 p \mathrm{d}p. \end{aligned} \tag{23}$$

With Eq. (21) in mind, we can interpret $\langle \hat{p} \rangle$ as the mean value of an ensemble of the momenta, but not that of a momentum mixture. It can be seen that also in the case of continuous spectrum of eigenvalues the superposition principle is established in the context of ensemble of eigenstates.

Next, let us consider an open system described by a density matrix

$$\rho = \sum_n |\psi_n \rangle W_n \langle \psi_n| \ . \tag{24}$$

Here, $\{\psi_n : n \in \mathbb{N}\}$ are determined by

$$\hat{H} \psi_n = E_n \psi_n. \tag{25}$$

The wave function for the mixed state can be written as

$$\psi = \sum_n a_n \psi_n. \tag{26}$$

In this case, the physical quantity of $\hat{L}$ for a mixed state is represented by averaging over all the pure states in consideration of their weights as

$$\left\langle \hat{L} \right\rangle = \sum_n W_n \left\langle \psi_n (t) \middle| \hat{L} \middle| \psi_n (t) \right\rangle. \tag{27}$$

This also shows that for open microscopic systems, $\psi = \sum_n a_n \psi_n$ as a mixed state merely reflects the ensemble of alternating states and not the mixture of eigenstates because $\psi = \sum_n a_n \psi_n$ is not an eigenfunction of $\hat{H}$.

Let us consider the case of degenerate eigenvalue. In this case, there are several eigenfunctions corresponding to an eigenvalue. For example, for the Hamiltonian operator $\hat{H}$ we can suppose an equation corresponding to at once an energy eigenvalue and several eigenfunctions:

$$\left\{\hat{H} \psi_n = E \psi_n : \quad (n \in \mathbb{N})\right\}. \tag{28}$$

Since the solutions ψ_n satisfy an equation, the mathematical superposition of the degenerate eigenfunctions is possible. Namely, for the superposed wave function

$$\psi = \sum_n a_n \psi_n, \tag{29}$$

we have

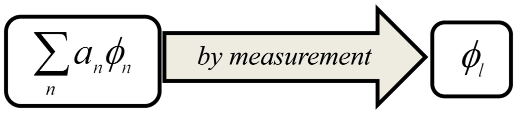
$$\hat{H} \sum_n a_n \psi_n = E \sum_n a_n \psi_n, \tag{30}$$

by further arrangement,

$$\sum_n a_n \left(\hat{H} - E \right) \psi_n = 0. \tag{31}$$

Consequently, the superposition principle holds for the degenerate eigenfunctions. The matter is whether $\sum_n a_n \psi_n$ is a mixed state or an ensemble of states. $\{\psi_n\}$ are orthogonal to one another. This implies that different states $\{\psi_n\}$ of a certain operator $\hat{L}$ entering $\hat{H}$ correspond to the equal E. In this case, the eigenvalues $\{L_n\}$ and eigenfunctions $\{\psi_n\}$ constitute a discontinuous spectrum. As mentioned above, for this discontinuous spectrum $\{\psi_n\}$, a superposed state $\sum_n a_n \psi_n$ assumes not a mixed state but an ensemble of eigenstates. Therefore, $\sum_n a_n \psi_n$ also represents the probabilistic superposition.

The most of physicists believe that the wave function of a system in the beginning, given by $\psi = \sum_n a_n \phi_n$ has changed by measurement to be reduced to a particular eigenstate, ϕ_l. This context can be expressed by the following diagram [1].



Essential here is that while the measurement gives rise to the reduction of the superposed state to a definite eigenstate, it does not change the eigenstates themselves.

For a closed system, it is possible to give from Eqs. (21, 23) obvious answer to the question about why the repeat of measurement provides eigenvalues of an operator with a definite probability. A closed system should be in a definite eigenstate of $\hat{H}$. Therefore, the results of measurement of a conservative physical quantity which corresponds to an operator commuting with $\hat{H}$ always are a particular eigenvalue. On the contrary, if an operator $\hat{L}$ does not commute with $\hat{H}$, namely, the physical quantity corresponding to $\hat{L}$ is non-conservative, the measurement yields an ensemble of eigenstates. This fact is obvious from Eqs. (21) and (23). For an open system, it is necessary to consider that any quantum system can transfer from an eigenstate of $\hat{H}$ to another due to the interaction with the surrounding system. Of course, the surrounding system may contain Apparatus as well. For open systems, there is no conservative physical quantity including energy, and therefore we are to obtain a series of eigenvalues instead of a particular eigenvalue. Actually, there is no ideally closed microscopic system. A surrounding system is to remarkably influence a microscopic system unlike a macroscopic system. Therefore, the surrounding system plays the role that makes the system under consideration transfer between eigenstates in sequence. In this case, the results of measurement are given as a series of eigenvaules subject to a definite probability rule. In the end, the measurement yields the ensemble of possible states the open system can take. If a microscopic system is considered to be a closed one, it is impossible to explain why the state of the system changes alternately despite the requirement for the unconditional observance of conservation laws. If a certain phenomenon violates conservation laws, then we naturally have to consider the system as an open one.

It is farfetched to imagine the measurement problem in order to account for the disagreement between the mathematical and physical results. In fact, it makes the situation of discussion more difficult and complicated rather than solvable and improved. The present situation characterized by the long and serious debates about the measurement problem shows that it is difficult to find the final solution to the queer problem, so far as we do not veer the direction of the ongoing research.

Eventually, we can unravel the mystery of the measurement problem. In essence, the strange measurement problem originated from the superposition of eingenfunctions. As mentioned above, the superposition principle does not work in the sense of simultaneous mixture of eigenstates.

In the case of open systems, $\sum_n a_n\psi_n$ represents the sum of events, since the wave function assumes a probabilistic property. Since the sum of events means sequential occurrence of events and not the simultaneous occurrence of events, the superposed wave function does not imply a mixed state, but an ensemble of possible states. It is necessary to recall that in reality, the wave function stands for an ensemble of positions of particles as a sum of events. Similarly, $\sum_n a_n\psi_n$ describes an ensemble of eigenstates. Based on the standard theory of quantum mechanics, most of physicists have used the idea that quantum states can exist in a superposition, although such a physical circumstance seems odd. It is a usual position to be accustomed, for example, to particles being in several places at once or being in a superposition of different polarizations in EPR experiments that test local realism. If anything, although the wave function appears to write that a particle exists in several places at once, we really observe the particle at a point. Since the wave function reflects all possible events, it exactly represents an ensemble of positions as a quantum state. Indeed, it is reasonable to describe such a physical situation by use of probability as in statistical mechanics.

It is no wonder that the superposition principle holds true for time-dependent process. In fact, in the case of the time-dependent Schrödinger equation we can apply the superposition principle because it is a linear differential equation. However, it is not superposition with respect to eigenstates. Suppose we can find independent solutions $\{\psi_n : (n \in \mathbb{N})\}$ for the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi_n}{\partial t} = \hat{H}\psi_n. \tag{32}$$

In this case, we can readily make certain that the linear combination of these independent solutions also fulfills the time-dependent Schrödinger equation. Namely,

$$\mathrm{i}\hbar\frac{\partial\sum_n a_n\psi_n}{\partial t} = \hat{H}\sum_n a_n\psi_n, \tag{33}$$

$$\sum_n a_n\left(\mathrm{i}\hbar\frac{\partial\psi_n}{\partial t}\right) = \sum_n a_n\hat{H}\psi_n, \tag{34}$$

$$\sum_n a_n\left(\mathrm{i}\hbar\frac{\partial\psi_n}{\partial t} - \hat{H}\psi_n\right) = 0. \tag{35}$$

After all, the time-dependent Schrödinger equation does not pose any inconsistent problem with the superposition of independent solutions. Of course, it is impossible to endow the independent solutions with a certain physical meaning. Therefore, the superposed wave function in terms of the independent solutions should be regarded as a wave function. It should be emphasized that in this case the superposition is described in terms of independent solutions instead of eigenfunctions.

The physical meaning of the superposition can be explained by means of eigenfunctions of operator $\hat{L}$. It is possible to expand a time-dependent wave function in terms of eigenfunctions of operator $\hat{L}$ as

$$\psi(x,t) = \sum_n a_n(t)\,\phi_n(x). \tag{36}$$

In view of Eq. 21, it can be seen that the expectation value of $\hat{L}$ is represented as

$$\left\langle \hat{L} \right\rangle = \sum_n |a_n(t)|^2 L_n. \tag{37}$$

Of course, $\psi(x,t) = \sum_n a(t)\,\phi_n(x)$ is not an eigenfunction of $\hat{L}$ and merely reflects a statistical state, namely, an ensemble of eigenstates. Accordingly, $\left\langle \hat{L} \right\rangle$ also indicates the time-dependent expectation value for the ensemble of eigenstates.

Thus, we come to the conclusion that the superposition assumes statistical ensemble and there is not the measurement problem for time-dependent processes.

4. Impossibility of intervention of Apparatus and Observer: Quantum mechanics without Apparatus and Observer

From a different angle, we can examine whether it is justifiable to include Apparatus and Observer to solve the measurement problem. The eigenvalue equation for a system under consideration is represented as

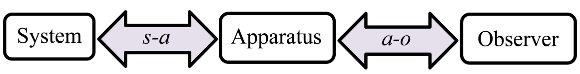
$$\hat{L}_s \psi = L\psi. \tag{38}$$

If based on the measurement theory, the intervention of Apparatus and Observe is accepted, the system is not a closed system but an open system. Therefore, the operator of whole system should be constituted by including Apparatus and Observer. Let us denote the subscript indicating System by "s", the subscript representing Apparatus by "a" and the subscript expressing Observer by "o".

For example, "s-a-o" denotes an entangled system formed by System, Apparatus and Observer. Similarly, "s-a" and "a-o" refer to entangled partial systems formed by System plus Apparatus and Apparatus plus Observer, respectively. Since the entangled system: System, Apparatus and Observer, and interactions between them should be taken into consideration, the operator for the whole system can be written as

$$\hat{L}_{s-a-o} = \hat{L}_s + \hat{L}_a + \hat{L}_o + \hat{L}_{s-a} + \hat{L}_{a-o}. \tag{39}$$

The above correlation between System, Apparatus and Observer can be represented by the following diagram.



The eigenvalue equation for superposed state is represented as

$$\hat{L}_{s-a-o}\psi_{sup} = L\psi_{sup}, \tag{40}$$

where L should give a set of the same eigenvalues as $\{L_n : (n \in \mathbb{N})\}$ and ψ_{sup} is determined by $\{\phi_n : (n \in \mathbb{N})\}$ and different sets of $\{a_n : (n \in \mathbb{N})\}$ corresponding to every eigenvalue. Namely,

$$\psi_{sup} = \sum_n a_n \phi_n. \tag{41}$$

In particular, in order to distinguish between the system under consideration and the surrounding system we use the representation of

$$\hat{L}_{s-a-o} = \hat{L}_s + \hat{L}^s_{s-a-o}. \tag{42}$$

For a superposed state, if eigenvalue equation

$$\left(\hat{L}_s + \hat{L}^s_{s-a-o}\right)\psi_{sup} = L\psi_{sup}. \tag{43}$$

is projected into a particular basis vector of the Hilbert space ψ_m, we have

$$\left\langle \phi_m \middle| \left(\hat{L}_s + \hat{L}^s_{s-a-o}\right) \middle| \psi_{sup} \right\rangle = L \left\langle \phi_m \middle| \psi_{sup} \right\rangle. \tag{44}$$

By adopting the Projection Postulation, we get

$$L = L_m. \tag{45}$$

Consequently, we have

$$a_m L_m + \left\langle \phi_m \middle| \hat{L}^s_{s-a-o} \middle| \psi_{sup} \right\rangle = a_m L_m. \tag{46}$$

Hence, we get

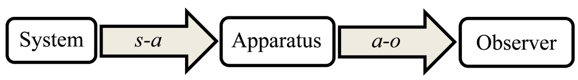
$$\left\langle \phi_m \middle| \hat{L}^s_{s-a-o} \middle| \psi_{sup} \right\rangle = 0. \tag{47}$$

Considering that $\{\phi_n : (n \in \mathbb{N})\}$ constitute a complete system of eigenfunctions and that if all projection components of a function equal zero, the function itself also equals zero, we reach the conclusion that $\hat{L}^s_{s-a-o}\psi_{sup} = 0$.

Finally, from Eq. (43) we arrive at the reduced equation:

$$\hat{L}_s \psi_{sup} = L\psi_{sup}. \tag{48}$$

Unlike in the previous description, the system of System, Apparatus and Observer can be represented by the following diagram as well.



In this case, the reduction can be represented as eigenvalue equation

$$\hat{L}_o\, \hat{L}_a\, \hat{L}_s \sum_n a_n \phi_n = L \sum_n a_n \phi_n. \tag{49}$$

If we would get the result; $\hat{L}_o\, \hat{L}_a = 1$, there could not exist Apparatus and Observer. As a result of the projection of Eq.(49) into ϕ_m, we get

$$\left\langle \phi_m \middle| \hat{L}_o\, \hat{L}_a\, \hat{L}_s \middle| \sum_n a_n \phi_n \right\rangle = L_m a_m. \tag{50}$$

By further arrangement, we have

$$\sum_n a_n L_n \left\langle \phi_m \middle| \hat{L}_o\, \hat{L}_a \middle| \phi_n \right\rangle = L_m a_m. \tag{51}$$

From this it follows that

$$\left\langle \phi_m \middle| \hat{L}_o\, \hat{L}_a \middle| \phi_n \right\rangle = \delta_{mn} = \langle \phi_m | \phi_n \rangle . \tag{52}$$

In the end, we reach the conclusion that

$$\hat{L}_o\, \hat{L}_a = 1. \tag{53}$$

Eventually, Eqs. (48) and (53) tell us that there cannot be the intervention of Apparatus and Observer, as far as we believe in the objectiveness of quantum operators as uninfluenced by Apparatus and Observer.

5. Results and discussion

The results of our work can be summarized as follows.

First, we have demonstrated that the mathematical superposition of eigenstates is impossible in the currently accepted sense. This is an undeniable mathematical conclusion. As a result, it turns out that there cannot be justified both the superposition principle as the superposition of eigenstates and the resultant measurement problem based on the interpretation of the standard theory of quantum mechanics.

Second, we have offered an alternative interpretation of the superposition principle which does not cause the conceptual problem of Apparatus and Observer. In our view, the superposition of eigenstates reflects the probabilistic evolution of a microscopic system from one eigenstate to another which is forced since either the physical quantity in consideration is non-conservative, or the microscopic system as an open system is affected by the surrounding system. Accordingly, the superposition of eigenstates, $\{\psi_n : (n \in \mathbb{N})\}$ stands for an ensemble of quantum states. For this reason, Measurement necessarily yields a statistical ensemble of physical quantity for a given quantum system. From this, von Neumann's assumption about the existence of the different types of evolutions [3] involving the unitary evolution and the measurement process should be assessed as being insignificant. Moreover, it should be assessed as being inconsistent to associate the measurement problem with human consciousness [4, 5, 6].

It is quite surprising that from the beginning of quantum mechanics, the measurement problem has been a problematic area of quantum mechanics and has spawned a diversity of different viewpoints. This physical circumstance results from the mathematically simple fact that an arbitrary wave function is expanded in terms of a complete set of orthonormal eigenfunctions to yield a superposition of eigenstates, while the wave function satisfies the time-dependent wave equation. Originally, the mathematical logic of the superposition principle can be understood by the fact that if ψ_n are the solutions of a linear equation, a linear combination of the solutions $\sum_n a_n \psi_n$ also becomes a solution because of the linearity of the equation. Rigorously to assess, the superposition of eigenfunctions differs from the linear combination of independent solutions. It is due to the fact that eigenfunctions are not independent solutions of the time-dependent Schrödinger equation.

Our explanation of impossibility of the eigenstate superposition has demonstrated that all arguments about the measurement problem are useless. A key aspect of the measurement problem is whether the state vector given by the linear combination of eigenfunctions is able to satisfy the eigenvalue equation which gives the eigenfunctions. In other words, it is identified with the question as to whether an eigenvalue equation can be considered as a linear equation. According to the previous demonstration, the arguments about the superposition of eigenstates are not needed anymore.

However, it is necessary to illustrate in more detail how to resolve open questions pertaining to the measurement problem from our perspective. Actually, the concept and meaning of the measurement have always held a central position in the discussions of the foundations of quantum mechanics. Obviously, despite the privileged place of the measurement problem in quantum theory, the studies of this problem have not offered good prospect.

The measurement problem arises from the basic proposition of quantum mechanics that the real state of a quantum system is represented as a state vector which is constituted by eigenstates whereas measurement always yields only eigenvalues corresponding to eigenstates. Since a smeared-out measurement result as a result of superposition cannot be found out in experiments, the measurement problem becomes an important conceptual problem for the standard quantum theory.

Although the motive of the measurement problem is likely to be plausible, it is untenable to devise several thought experiments for solving the measurement problem. This is because the problem has the wrong starting point relevant to the superposition principle, thereby affording a multitude of unsolvable paradoxes, and conflicts with natural epistemology. The standard quantum theory cannot solve the measurement problem but merely circumvents it by applying the Projection Postulate: a measurement instantaneously projects the superposition into an eigenstate with a definite probability. The Projection Postulate then refers to the collapse of a wave function by measurement. At this stage, we can conclude that according to the impossibility of the superposition of eigenstates producing a new eigenstate, the state vector is meaningless and therefore the Projection Postulate should be rejected.

The paradoxes arising from the superposition principle and the measurement problem thereof make the foundations of quantum mechanics unreliable. As a well-known quantum paradox, we can illustrate Schrödinger's cat problem on the basis of the impossibility of eigenstate superposition. In the standard theory of quantum mechanics it is alleged that prior to the observation, the cat's state vector is in a superposition corresponding to the cat being both alive and dead and not being either alive or dead. Although a microscopic quantum entity described by superposition may be palatable to most physicists, such a result is unacceptable for macroscopic objects, such as cats [27]. Should the state vector give a complete description of the state of the cat, the observation, i. e., measurement would project the cat's state vector into either of the two eigenstates: the eigenstate where the cat is alive and the eigenstate where the cat is dead. However, we actually cannot experience the fact that according to observation a cat may be either alive or dead. Obviously, it is not reasonable that a simple act of observation could so stupendously alter the state of the cat. Practically, we cannot witness such a mysterious reality.

Now, we examine this paradox from our perspective. If it were not for the superposition principle, the cat could not exist in the superposed state of both life and death at once. Since the superposition of eigenstates is impossible, it is evident that there cannot be the cat's mixed state of life and death. Meanwhile, if the transition between eigenstates of life and death were possible (it means the superposition in the sense of the ensemble of states), then it could exist alternately in either life or death state. It is necessary to consider that the transition from an eigenstate to another can be realized by the action of the surrounding system. The radioactive decay as an action makes the living cat die, while the reverse process is forbidden since the radioactive decay cannot make the dead cat be brought back to life: the biological process of transition from life to death is irreversible. Eventually, the paradox of Schrödinger's cat comes to be unraveled. Finally, it can be concluded that the Schrödinger's cat state cannot exist because it indicates an ensemble of irreversible states which cannot be transferred to each other by quantum action.

Once it is clarified that an alternative superposed eigenstate in terms of the superposition of eigenstates is impossible, it is (I) the problem of Apparatus and Observer, (II) the limits of the Projection Postulate and (III) the instantaneous character of the measurement process that have no significance at all in the researches into the foundation of quantum mechanics. Since usually microscopic systems should be regarded as open systems, (IV) the constraints imposed by conservation laws should be considered by taking both microscopic system in consideration and surrounding system as a whole.

The two-slit experiment is at the core of the mysteries surrounding quantum mechanics. This experiment exemplifies a subtle point of quantum phenomena governed by the superposition principle. Our view on this phenomenon is that since the wave field surrounding a particle is non-local, it passes through two slits and then the field disturbed by the two slits affects the movement of the particle afterward[28]. All told, the particle passes through either of the two slits, while the wave field passes through both of the slits. Naturally, particles are self-interferential because particle and field are inseparably unified. In essence, the diffraction of a particle through two slits is identified with the scattering problem. Since a microscopic particle is non-local, two slits can be regarded as a single unified object scattering an incoming particle. To explain the two-slit experiment, we start with the wave equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{54}$$

Dividing the Hamiltonian operator into the kinetic energy and potential energy part, we have

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left(\hat{K} + U\right)\psi, \tag{55}$$

where $\hat{K}$ is the kinetic energy operator, U the potential organized by two slits. To unravel the two-slit diffraction in a fundamental way, it is necessary to find the sophisticated technique for determining U.

Nevertheless, it is possible to treat U in a simple way. It is justifiable to consider that U approximates to U_1 or U_2 associated only with either of the two slits in the

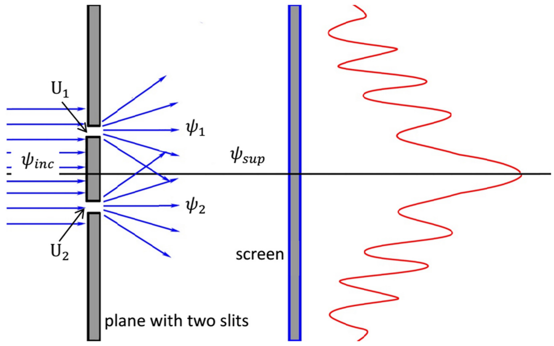


Figure 1: Two-slit experiment: ψ_{inc} denotes the wave function of an incoming particle. ψ_1 is the wave function of the particle scattered by the potential U_1 and ψ_2, that by the potential U_2. The superposition of ψ_1 and ψ_2 furnishes the wave function of the scattered particle, ψ_{sup}.

vicinity of it, respectively. Therefore, solving Eq. (55), we obtain two independent solutions, ψ_1 and ψ_2 for U_1 and U_2, respectively. Since these are all the independent solutions of Eq. (55), we can apply the superposition principle. Thus, the scattered wave can be represented as the linear combination of ψ_1 and ψ_2. Namely,

$$\psi_{sup}(\mathbf{q},t) = a\psi_1(\mathbf{q},t) + b\psi_2(\mathbf{q},t). \tag{56}$$

The probability density is represented as

$$\begin{aligned}\rho(\mathbf{q},t) &= \left|\psi_{sup}(\mathbf{q},,t)\right|^2 \\ &= |a|^2 |\psi_1(\mathbf{q},t)|^2 + |b|^2 |\psi_2(\mathbf{q},t)|^2 \\ &\quad + 2\mathrm{Re}\left[ab^*\psi_1(\mathbf{q},t)\psi_2^*(\mathbf{q},t)\right].\end{aligned} \tag{57}$$

The term, $2\mathrm{Re}\left[ab^*\psi_1(\mathbf{q},t)\psi_2^*(\mathbf{q},t)\right]$ stands for the interference via the two slits.

Here, we merely showed essentials for dealing with the two-slit interference applying the superposition principle. The aim of this description is to show an example of applying the superposition principle to the study of time-dependent processes. Obviously, the superposition is related only to independent solutions and not eigenstates. It is necessary to analyze the state vector from the point of view of probability theory. The probability in quantum mechanics, represented by the wave function, takes on the meaning of mutually exclusive and jointly exhaustive possibilities. This means that at most and at least one of possible events happens. In this case, we describe such an event as a sum of events. On the other hand, the superposition of eigenstates can be identified with a sum of weighed events, since the wave function assumes probabilistic property and thus is represented as the linear combination of eigenfunctions. Here, we should keep in our mind the fact that the coefficients of the linear combination are time dependent. Of course, in the perfect sense, the probability in quantum mechanics is determined by probability density and directly not wave function. In this work we do not deal with why as an open problem, the relation between wave function and probability density should be defined as

$$\rho = |\psi|^2, \tag{58}$$

since it is beyond the scope of this paper. However it is necessary to stress that this relation leads to the entanglement of eigenstates in a quantum statistical process. As far as the entanglement is concerned, it implies correlation between members of a given ensemble of events indicative of eigenstates. It should be considered that every eigenstate is forbidden to occur simultaneously with the rest due to the mutually exclusive property of sum of events and a quantum process continues with a definite probability through eigenstates that happen in a sequence due to the jointly exhaustive property of sum of events. Evidently, if the conventional interpretation of the state vector were accepted, the superposition of eigenstates should have to be represented as a product of events, i.e., a product of eigenfunctions rather than a sum of events, which cannot be both physically and mathematically justifiable because it contradicts the physical causality.

6. Conclusion

In this work, we have discussed the superposition principle and the measurement problem mainly from the mathematical point of view to reach an important conclusion capable of improving the conventional understanding. Our attention was paid importantly to the mathematical structure of the superposition principle. Our work has shown that the superposition principle cannot be established on the basis of the notion of the standard theory implicated in Observer and Apparatus. In fact, a superposed state does not satisfy the time-dependent wave equation as well as an eigenvalue equation and an individual eigenstate constituting a superposed state also does not fulfill the wave equation. This fact raises the question as to what the superposition means in the physical sense.

Without examining the superposition of eigenstates starting with the origin of the problem at issue, it is impossible to find out a key for solving the measurement problem. For that reason, we newly examined whether the superposition of eigenstates can yield an eigenstate and what the mathematical superposition of eigenstates means. Our investigation has led to the important conclusion that the superposition of eigenstates should be interpreted based on the notion of ensemble of eigenstates.

It remains to be seen whether on the whole, the aim to resolve the measurement problem shrouded in mystery has been attained. For all that, our research gives a clear reason that the superposition principle and the measurement problem should be interpreted based on the superposition in the sense of the ensemble of states. We believe that the wrong understanding of the mixed state caused the measurement problem which led to a conceptual confusion of interpretations of quantum mechanics.

Finally, our work has offered a possibility of resolving some intractable open questions of quantum mechanics concerning the superposition of states and the measurement problem from a realistic viewpoint based on the conception of ensemble of quantum states which is irrelevant to Observer and Apparatus.

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