Complex Ginzburg-Landau equation (CGLE) is a universal model of nonequilibrium dynamical systems. Focusing on the primordial stages of cosmological evolution, this work points out that the connection between CGLE and the Navier-Stokes (NS) equation bridges the gap between fluid flows and the mathematics of General Relativity (GR).

**Key words:** Complex Ginzburg-Landau equation, Navier-Stokes equation, gauge-gravity duality, dimensional reduction, continuous dimensions.

### 1. Introduction

CGLE is considered a paradigm of non-equilibrium statistical physics and dynamic critical phenomena. It encodes many key properties of collective
phenomena with space-time dependence, and it models the generic onset of chaos, turbulence, and spatiotemporal patterns in extended systems [1-3]. We recently argued that applying CGLE to the chaotic dynamics of interacting fields yields unforeseen solutions to the challenges raised by high-energy theory [4-5]. The goal of this work is to expand our findings to the possible link between CGLE and the high temperature / long wavelength limit of GR.

Let’s begin with the observation that there are (at least) four distinct routes leading from nonequilibrium dynamics to GR:

1. The emergence of a nonvanishing K-entropy in the unstable sector of gravitational dynamics, the N-body problem (N > 2) of cosmology in near or non-equilibrium conditions [6-8].

2. The emergence of a spacetime equipped with continuous dimensionality above the Fermi scale follows from several premises, one of them being the onset of Hamiltonian chaos and fractional dynamics [9-10]. Along the same
lines, it can be argued that fractional dynamics in flat spacetime is formally equivalent to classical dynamics on curved manifolds [11].

3. The geometry of Hamiltonian systems is dual to geometry on curved manifolds [12].

4. Thermodynamics of Black Holes lends support for the multifractal interpretation of horizon dynamics [13].

We believe that, besides 1) - 4), a scenario worthy of investigation is the fluid-gravity correspondence inspired by the gauge-gravity duality of string theory [14]. A drawback of this duality is that it operates with a negative cosmological constant, clearly at odds with current astrophysical observations. It was found in [15] that, applying the gauge-gravity conjecture to a 1+1 spacetime endowed with continuous dimensions leads to a positive cosmological constant. Besides leading to a positive cosmological constant, setting the fluid-gravity duality in 1+1 dimensions brings up two attractive features, namely, a) a low dimensional metric is compatible with
the framework of *dimensional reduction* (DR) applied to the primordial stages of Universe evolution [16], b) the *duality* of hydrodynamics and high-temperature/long-wavelength gravitational dynamics in $1+1$ dimensions necessarily turns into an *identity*, as continuous dimensions automatically overlap within an infinite range of positive values.

The paper is divided into four sections. Section 2 lists the main couple of assumptions underlying the approach, while section 3 and 4 delve into the route connecting CGLE, relativistic NS equation and General Relativity, following the straightforward diagram shown below:

$$\text{CGL Equation} \Rightarrow \text{NS Equation} \Rightarrow \text{General Relativity}$$

**2. Assumptions**

A1) The DR conjecture asserts that the number of spacetime dimensions monotonically drops with the boost in the observation scale. In a nutshell, the DR expectation is that spacetime becomes two-dimensional near the Big-Bang singularity. This conjecture is backed up by several cosmological
scenarios, including the BKL ansatz and the Kasnerian regime of metric fluctuations in the primordial Universe [17-18].

A2) When applied to the fluid-gravity correspondence, DR yields a positive cosmological constant in 1+1 dimensions [15]. It is conceivable that the cosmological constant stays unchanged as the Universe expands and cools off, on account of inherent memory effects attributed to nonequilibrium dynamics.

3. From CGLE to the NS equation

The standard form of CGLE is given by,

$$\partial_t z = az + (1+ic_1)\Delta z - (1-ic_3)z|z|^{2\sigma}$$

in which $z$ is a complex-valued field, the parameters $a$ and $\sigma$ are positive and the coefficients $c_1$ and $c_3$ are real [1-2]. The nonlinear Schrödinger equation (NSE) is a particular embodiment of the CGLE in the limit $a \to 0$, namely [19]
\(-i \partial_t z = c_1 \Delta z + c_3 z|z|^{2\sigma}\) \hspace{1cm} (2a)

In what follows we set \(\sigma = 1\). In natural units (\(\hbar = 1\)), the quantum-mechanical version of (2a) reads,

\[ i \frac{\partial}{\partial t} z(x,t) = \left[ -\frac{1}{2m} \nabla^2 + V(x,t) \right] z(x,t) \] \hspace{1cm} (2b)

where \(V(x,t)\) is the potential function. The Madelung transformation enables one to turn (2b) into the quantum Euler equation for compressible potential flows [20]. To this end, taking the complex-valued field in the canonical form,

\[ z(x,t) = \sqrt{\frac{\rho(x,t)}{m}} \exp[i S(x,t)] \] \hspace{1cm} (3)

and substituting it into (1)-(2) leads to

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \] \hspace{1cm} (4)

\[ \frac{du}{dt} = \partial_t u + u \cdot \nabla u = -\frac{1}{m} \nabla(Q + V) \] \hspace{1cm} (5)
Here, $u(x,t)$ denotes the flow velocity, $\rho = m|z|^2$ stands for the mass density and

$$Q = -\frac{1}{2m} \nabla^2 (\sqrt{\rho})$$

is the Bohm potential. The flow velocity and its associated probability current are given by, respectively,

$$u(x,t) = \frac{1}{m} \nabla S = -\frac{i}{m} \frac{\nabla z}{z}$$

$$j = \rho u = \frac{1}{2mi} [z^*(\nabla z) - z(\nabla z^*)]$$

Since the Schrödinger equation is conservative, the Madelung transformation naturally leads to the Euler equation, which is exclusively valid for inviscid flows. To account for fluid viscosity and arrive at the NS equation, one needs to either appeal to an extended version of the NS equation containing non-conservative terms or bring up the concept of kinematic viscosity – a concept linked to the mass of quantum particles as in
\[ \nu = \frac{1}{2m} \]  \hfill (9)

By (9) and for incompressible flows, the NS equation that mirrors (5) can be written as,

\[ \frac{du}{dt} = \frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u \]  \hfill (10)

where \( p \) denotes the pressure.

4. From the NS equation to gravitational dynamics

According to the gauge-gravity duality, Einstein’s equations in \( D=d+1 \) spacetime dimensions contain a negative cosmological constant \( \Lambda \) and are written as [14-15]:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \]  \hfill (11)

in which,

\[ \Lambda = -\frac{d(d-1)}{2R_{AdS}^{2}}; \quad d=0,1,2,... \]  \hfill (12)
with $R_{AdS}$ denoting the AdS curvature radius, a parameter that can be conveniently set to unity. On a minimal fractal spacetime defined in 1+1 dimensions ($\mu, \nu = 0, 1$), the spatial dimension flows with the Renormalization Group (RG) scale and spans a continuous range of values as in

$$d(\mu_{RG}) = 1 - \varepsilon(\mu_{RG}) \propto 1 - O\left\{ \frac{m^2(\mu_{RG})}{\Lambda_{UV}^2} \right\}; \quad \varepsilon \ll 1 \quad (13)$$

where $\mu_{RG}$ stands for the RG scale and $\Lambda_{UV}$ is the ultraviolet cutoff. In contrast with the conventional gauge-gravity duality, it follows from (13) that (12) turns into a positive cosmological constant, that is,

$$\Lambda = \Lambda R_{AdS}^2 = O(\varepsilon) > 0 \quad (14)$$

Following (13) and [14], in the high temperature / long wavelength limit of gravitational dynamics, Einstein’s equations reduce to the NS equations (10) in one-dimensional space ($d = 1$).
References


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