Interpretation of correspondence principle based on examination of existence of isomorphic mapping between observables and operators

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Abstract

It still remains an important open question of the interpretation of the foundations of quantum mechanics to thoroughly elucidate the essence and significance of the correspondence principle. We focus from a new mathematical aspect on the review of the correspondence principle to gain the correct understanding of the principle. As a result, we show that there does not exist the algebraic isomorphism between the algebra of the observables and that of the quantum operators, and therefore the previous interpretation of the correspondence principle aiming to provide all the operators corresponding to physical quantities is inconsistent from the mathematical view point. Furthermore, it is demonstrated that the correspondence between physical quantities and quantum operators is possible within canonically conjugate observables constituting the action, while classical and quantum quantities satisfy one and the same dynamical relation. Moreover, it is shown that the classical limit of quantum mechanics can be explained not by the correspondence principle but by the de Broglie relation and the operator equations.

Key words: Correspondence principle, Observable, Operator, Isomorphic mapping, de Broglie relation

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1. Introduction

The history of the argument about the correspondence principle dates from Planck's study. Already in 1906, Planck proposed the assumption that classical mechanics is a limit case of microphysics when $h \to 0$ [1]. Inspired by the idea for this assumption, Bohr could formulate a kind of principle that specifies the correspondence between classical and quantum mechanics, which supposes that the elements of quantum theory should correspond to analogous counterparts in classical physics. This reflects the standpoint that classical mechanics is a complete theory within its realm, while quantum mechanics is a mechanics restricted by assumptions about some additional conditions ascribed to quantum nature. Therefore, classical mechanics could not but acknowledged to be required for quantum mechanics [2, 3].

To clarify the relationship between classical and quantum physics is most important for establishing the consistent foundations of quantum mechanics [4, 5, 6]. From the outset of the development of quantum mechanics, the correspondence principle has been considered the most significant instrument for obtaining formulae of quantum mechanics and elucidating the classical limit of quantum mechanics, but its interpretation is not unique as yet [7]. Apparently, quantum mechanics is substantially different from classical mechanics judging from the fact that states of quantum systems are to be represented by the state vectors instead of trajectories of classical mechanics and physical quantities are to be described by operators in place of scalar functions of q and p.

According to Petersen, this principle makes formal analogy between classical and quantum theory, so that the principle has been the most important instrument in formulating the quantum theory [8]. His opinion reflects the standpoint referred to as the "Copenhagen spirit of quantum theory". The idea that Bohr proposed to solve the atom problem highlighted the significance of the correspondence principle in constructing the mathematical formalism of quantum mechanics.

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The question on the correspondence between classical and quantum mechanics has two different aspects. The first aspect is the question of how to find valid quantum-mechanical formulae starting with classical formulae. The second aspect is the question of how to determine the classical limit of quantum mechanics.

The first question of a heuristic character leads to the common understanding that quantum mechanics cannot be derived from classical mechanics; at most, the classical formulae can imply what forms the quantum-mechanical ones can take. It is not always that the simple prescription of the correspondence principle "Replace q and p in the expression of classical quantities by operators $\hat{q}$ and $\hat{p}$ to obtain the expression of the operators corresponding to observables" is reliable. It is even for simple cases that one can observe that the routine application of the principle may lead to wrong results. The correspondence principle played an important role in the early days of the development of quantum mechanics because it seemed to provide fundamental methods to describe the atomic phenomena [7]. For the standard theory of quantum mechanics, the correspondence principle has been the most important recipe for construing the mathematical structure of quantum mechanics [8]. If it had not been for the correspondence principle, it could not be possible to imagine even the Hamiltonian operator.

Remarkably, Heisenberg extended this idea to arbitrary observable quantities [9] and therefore the principle has been deemed to be a general principle of quantum mechanics. It is possible to view Ehrenfests theorem as an apotheosis of the correspondence principle. Especially, Bohr intended to endow the correspondence principle with more far-reaching meaning, according to which the applicability of the correspondence principle was considered to hold even beyond the domain of quantum mechanics, too. Since then, this principle has been qualified as a guiding principle in finding the correct expressions for quantum-mechanical quantities and equations, starting from the classical ones. This view on the correspondence principle is referred to as the strong form of the correspondence principle [10].

However, nowadays, the correspondence principle above all is appreciated because of its heuristic value and no attention is paid to the correspondence principle except for a short historical remark or a limited definition in the weak sense concerning the classical limit of quantum mechanics[7]. Nevertheless, in relation to the category and tenor, the problem of the correspondence principle still remains unsolved, so this question requires the final solution.

The question on the classical limit of quantum mechanics is whether quantum mechanics is applicable also to macroscopic objects [11]. When evaluating the principle in the sense that quantum-mechanical predictions of large quantum numbers must agree with the results of classical mechanics, it is referred to as the weak form of the correspondence principle. If it would be the case, then it should be possible to find the classical equations by applying the quantum-mechanical ones to macroscopic systems and then by making a certain approximation [7, 12]. From the point of view of the correspondence principle, it should be noted that as an autonomous formalism of quantum mechanics, quantum mechanics in phase space has made a considerable contribution to elucidating the classical limit of quantum mechanics [13, 14]. This formalism of quantum mechanics is based on Wigner's quasi-distribution function and Weyl's correspondence between quantum-mechanical operators in Hilbert space and ordinary complex-valued functions in phase space [15, 16, 17]. It has been shown that in the case of limit $h \to 0$, the formulae of observables obtained from quantum mechanics via the Weyl transformation turn to the formulae of classical quantities and the Weyl transform of the density operator becomes a true probability density [18]. This formulation should be assessed as an attempt to form the intuitive quantum theory capable of expounding the connection between classical and quantum mechanics. However, the weak form of the correspondence principle too still remains unsolved.

In this paper, we explain that due to the impossibility of the isomorphic mapping between observables and quantum operators, the correspondence principle cannot be established in the sense of both the weak and strong form. As a solution, based on the concept of quantum operator issuing from the action, we propose an alternative interpretation of the correspondence relation between classical and quantum mechanics, and on the classical limit of quantum mechanics.

2. Interpretation of correspondence principle in terms of isomorphic mapping between observables and quantum operators

2.1. Impossibility of isomorphic mapping between observables and quantum operators

Dirac is the first who emphasized that the quantum-mechanical equation which the Ehrenfest theorem offers was formally identified with the evolution equation of a classical quantity, i.e., the Hamilton equation [19]. The classical time-evolution equation writes

$$\frac{dA}{dt}=\sum_i\left(\frac{\partial A}{\partial q_i}\frac{dq_i}{dt}+\frac{\partial A}{\partial p_i}\frac{dp_i}{dt}\right)+\frac{\partial A}{\partial t}=\{A,H\}_{cl}+\frac{\partial A}{\partial t}, \tag{1}$$

where $\{A,H\}_{cl}$ is the classical Poisson bracket of quantity A and Hamiltonian function H. In particular, the time-evolution equation for classical quantities, q and p yields the Hamiltonian equations. Elegantly, the time-evolution equation for a quantum-mechanical operator can be obtained from that for a classical quantity by means of the routine substitution,

$$\{A,H\}_{cl}+\frac{\partial A}{\partial t}\rightarrow\frac{1}{\mathrm{i}\hbar}\left[\hat{A},\hat{H}\right]+\frac{\partial\hat{A}}{\partial t}, \tag{2}$$

where the classical Poisson bracket is replaced by the quantum-mechanical commutator. Apparently, this correspondence seems to provide the possibility of establishing a close relation between classical mechanics and quantum mechanics in a general way, although these two sciences are essentially distinguished. The quantization procedure established by this commutation relation between $\hat{A}$ and $\hat{H}$ dictates that for an arbitrary classical phase-space function, the assumption about the correspondences between the fundamental operators and observables:

$$q_i\rightarrow q_i,\qquad p_i\rightarrow-\mathrm{i}\hbar\frac{\partial}{\partial q_i} \tag{3}$$

should provide a definite quantum-mechanical operator, since all physical quantities are represented as a function with respect to q and p. Of course, this assumption is not proven rigorously. According to Eq. (3), the Hamiltonian operator $\hat{H}=\sum_i\hat{p}_i^2/2m+V(q)$ can be easily obtained purely in a formal way. However, this plausible proposition of quantum mechanics actually does not hold even for the Hamiltonian operator for fairly simple quantum systems.

It is evident that we cannot verify that the correspondence principle is possessed of generality from the purely mathematical point of view. In fact, it is not difficult to demonstrate that there is not isomorphic mapping M from an arbitrary classical phase-space functions to a quantum-mechanical operator. In other words, there does not exist an algebraic isomorphism between the algebraic relation of the classical quantities and that of Hermitian operators. In order for the correspondence principle to hold, it is necessary that the algebra of operators is isomorphic to that of physical quantities, while in reality, mathematical relations between the operators of quantum mechanics and those between physical quantities of classical mechanics do not coincide. The case is obvious when we note the fact that as an example, it is impossible to imagine the division by differential operators or the root of the operators. Therefore, it is not always justified to assume the isomorphic mapping between physical quantities and quantum operators.

Now, we shall review this matter in detail. Let us denote observables by $a,b,c,\cdots$ and the corresponding operators of quantum mechanics by $\hat{A},\hat{B},\hat{C},\cdots$. Then the isomorphic relations between physical quantities and operators can be represented as

$$\begin{aligned}&a=M\left(\hat{A}\right),\\&b=M\left(\hat{B}\right),\\&c=M\left(\hat{C}\right),\\&\cdots\\&F(a,b,c,\cdots)=M\left[F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\right],\end{aligned} \tag{4}$$

where F refers to the algebraic expression for observables or operators and M, the mapping from operator to observable. It is currently accepted that the isomorphic mapping M via the wave function should satisfy

$$\begin{aligned}&a\psi=\hat{A}\psi,\\&b\psi=\hat{B}\psi,\\&c\psi=\hat{C}\psi,\\&\cdots\\&F(a,b,c,\cdots)\psi=F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi.\end{aligned} \tag{5}$$

From this, the isomorphic mapping is represented as

$$\begin{aligned}
&a = M\left(\hat{A}\right) = \frac{\hat{A}\psi}{\psi},\\
&b = M\left(\hat{B}\right) = \frac{\hat{B}\psi}{\psi},\\
&c = M\left(\hat{C}\right) = \frac{\hat{C}\psi}{\psi},\\
&\cdots\\
&F\left(a, b, c, \cdots\right) = M\left[F\left(\hat{A}, \hat{B}, \hat{C}, \cdots\right)\right] = \frac{F\left(\hat{A}, \hat{B}, \hat{C}, \cdots\right)\psi}{\psi}.
\end{aligned} \tag{6}$$

Hence, it follows that for the algebra and mapping between observables and operators, we should adopt as a general rule

$$F\left[M(\hat{A}), M(\hat{B}), M(\hat{C}), \cdots\right] = M\left[F\left(\hat{A}, \hat{B}, \hat{C}, \cdots\right)\right] \tag{7}$$

or

$$F\left(\frac{\hat{A}\psi}{\psi}, \frac{\hat{B}\psi}{\psi}, \frac{\hat{C}\psi}{\psi}\right) = \frac{F\left(\hat{A}, \hat{B}, \hat{C}, \cdots\right)\psi}{\psi}. \tag{8}$$

It is obvious that for arbitrary operators including multiplication and differential operators, and the wave function, the mathematical relation such as Eq. (8) in general does not hold. Therefore, it is concluded that there is not the mathematically isomorphic relation between observables and operators.

Now, let us examine the algebraic isomorphism between observables and operators from a different aspect. There is no doubt for the fact that the definition of the mean value for quantum mechanics is true, since it is rudimentary knowledge of quantum mechanics.

By definition, for an arbitrary quantum operator $\hat{L}$, we can represent its mean value as

$$\left\langle \hat{L} \right\rangle = \int \psi^* \hat{L}\psi \mathrm{d}v = \int \psi^* \psi \frac{\hat{L}\psi}{\psi} \mathrm{d}v == \int \rho \frac{\hat{L}\psi}{\psi} \mathrm{d}v.$$

On the other hand, since $\hat{L}$ is a Hermitian operator, the mean value always is real. Therefore, the meaningful part of the integrand in the above integration is $\mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right)$. Thus, we represent the observable L corresponding to an operator $\hat{L}$ as

$$L = \mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right). \tag{9}$$

As an example, the observable of momentum is written by the corresponding operator as

$$\mathbf{p} = \mathrm{Re}\left(\frac{\hat{\mathbf{p}}\psi}{\psi}\right).$$

Therefore, according to Eq. (9) the mapping from operators to observables can be adopted as

$$L = M\left(\hat{L}\right) = \mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right).$$

Here is not any assumption. Actually, the above relation expresses the mapping relation between observable and operator. According to Eq. (9), if the correspondence principle is justified, then the mapping relations must be written as

$$\begin{aligned}
&a = M\left(\hat{A}\right) = \mathrm{Re}\left(\frac{\hat{A}\psi}{\psi}\right),\\
&b = M\left(\hat{B}\right) = \mathrm{Re}\left(\frac{\hat{B}\psi}{\psi}\right),\\
&c = M\left(\hat{C}\right) = \mathrm{Re}\left(\frac{\hat{C}\psi}{\psi}\right),\\
&\cdots\\
&F\left(a,b,c,\cdots\right) = \mathrm{Re}\left[\frac{F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi}{\psi}\right] = M\left[F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\right].
\end{aligned} \tag{10}$$

The comparison of Eq. (6) with Eq. (10) leads to the conclusion that these mapping relations are not identical. Therefore, we must abandon either of two mapping rules. The mapping relations in terms of Eq. (6) should be rejected because Eq. (10) is more tenable than Eq. (6) according to Eq. (9).

From Eq. (10), we have

$$F\left(a,b,c,\cdots\right) = F\left[M\left(\hat{A}\right),M\left(\hat{B}\right),M\left(\hat{C}\right)),\cdots\right] = F\left[\mathrm{Re}\left(\frac{\hat{A}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{B}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{C}\psi}{\psi}\right),\cdots\right], \tag{11}$$

and at the same time

$$F\left(a,b,c\right) = M\left[F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\right] = \mathrm{Re}\left[\frac{F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi}{\psi}\right]. \tag{12}$$

From Eq. (11) and Eq. (12) follows

$$\mathrm{Re}\left[\frac{F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi}{\psi}\right] = F\left[\mathrm{Re}\left(\frac{\hat{A}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{B}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{C}\psi}{\psi}\right),\cdots\right]. \tag{13}$$

Clearly, Eq. (13) is not possessed of generality. Of course, for multiplication operators, Eq. (13) holds. Actually, we can verify purely in a simple examination that for differential operators and arbitrary wave functions, Eq. (13) in general is not valid. Thus, we arrive at the conclusion that the correspondence principle is not justified, so far as we review the relation of isomorphic mapping merely from the mathematical point of view.

Meanwhile, it is necessary to scrutinize whether there necessarily exists a definite operator corresponding to every observable. The correspondence principle is of important significance for the standard theory of quantum mechanics. In fact, it is not too much to say that the correspondence principle had formulated the mathematical structure of quantum mechanics because all the operators of quantum mechanics were obtained with the help of the correspondence principle. We would be justified in saying that this principle actually has commanded quantum mathematical operations.

As an example, the Hamiltonian operator is an exemplary result of the correspondence principle. Without the Hamiltonian operator, it is impossible to imagine even the fundamental equation of quantum mechanics, viz. the Schrödinger equation. According to the ordinary understanding of the correspondence principle, a relation between quantities of classical mechanics and that between the corresponding operators of quantum mechanics are in formal accordance. So the correspondence principle has been adopted as an important principle up to date.

Now, let us consider what contradiction the principle involves. In order for the correspondence relation to hold, operators should produce all mathematical rules of calculation that physical quantities satisfy. As an example, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ should yield operator relation $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ according to the correspondence principle. It is right to write the angular momentum also as $\mathbf{L} = -\mathbf{p} \times \mathbf{r}$, but the application of the correspondence principle to this leads to inconsistent result. In fact, in this case, $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ should be identified with $\hat{\mathbf{L}} = -\hat{\mathbf{p}} \times \hat{\mathbf{r}}$, but in view of the relation between the operator and wave function, it follows that they are not identical. For $\hat{\mathbf{L}} = -\hat{\mathbf{p}} \times \hat{\mathbf{r}}$, operator ∇ should act as the rotation on the product of $\mathbf{r}$ and the wave function to be mathematically valid, while for $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, operator ∇ should act as the gradient on the wave function. In the end, the role of the same operator varies with cases. The same goes for the Hamiltonian operator.

Let us consider the Laplace operator $\Delta = \nabla^2$ entering the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(r).$$

Obviously, ∇ entering the momentum operator does not play the mathematically identical role in constituting the Laplace operator because it first acts as the gradient, while next, it does as the divergence. This fact implies that the correspondence principle has mathematically inconsistent aspect. Now, we are at the stage to devise from what to start the discussion about quantum-mechanical operators. We think that undoubtedly, the definition of the quantum-mechanical mean value is just the trustable starting point. This is because it does not contain any assumptions.

2.2. Correspondence principle and Hamiltonian operator

The Ehrenfest theorem provided a hopeful result for the question of the correspondence between the classical and quantum formulation. This theorem applies to expectation values $\langle \hat{q}_i \rangle$ and $\langle \hat{p}_i \rangle$ of the quantum-mechanical position and momentum observables. It states that the evolution equations of these operators are formally identical to the classical Hamilton equations of the corresponding classical quantities, i.e.,

$$\frac{d}{dt}\langle \hat{q}_i \rangle = \left\langle \frac{\partial \hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{p}_i} \right\rangle, \qquad \frac{d}{dt}\langle \hat{p}_i \rangle = -\left\langle \frac{\partial \hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{q}_i} \right\rangle. \tag{14}$$

Of course, the above equations are meaningful only if the differentiations with respect to the operators $\hat{q}_i$ and $\hat{p}_i$ can be valid in the mathematical sense and the Hamiltonian operator is justifiable. It is commonly accepted that this is the case only if we restrict ourselves to systems with simple Hamiltonians of the form $\hat{H} = \sum_i \hat{p}_i^2/2m + V(q)$. As far as the Hamiltonian operator is valid, we can easily verify that relations Eq. (14) hold. To confirm the validity or invalidity of the Hamiltonian operator, it is worth considering a central force problem (e.g. the hydrogen atom) with the potential

$$V(r) = -\frac{e^2}{r}.$$

The Hamiltonian function in the Cartesian coordinates takes the form $H = \mathbf{p}^2/2m + V(r)$. Replacing the momentum by the operator $-i\hbar\nabla$ yields the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r) = -\frac{\hbar^2}{2m}\Delta + V(r).$$

Now, it is necessary to represent operator Δ with respect to the spherical coordinates r, ϑ, φ because the problem is a central force problem. The outcome of this computation is well known to lead to the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial \psi}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 \psi}{\partial \varphi^2}\right] + V(r)\psi = E\psi. \tag{15}$$

On the other hand, we can imagine another way. In fact, we may well start with the classical Hamiltonian function represented in the system of spherical coordinates as far as the correspondence principle in general is valid. Thus, we have the Hamiltonian function:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\vartheta^2 + \frac{1}{r^2\sin^2\vartheta}p_\varphi^2\right) + V(r). \tag{16}$$

Now, assuming according to the correspondence principle the following operators and commutation relations:

$$\begin{aligned}
\hat{p}_r &= -\mathrm{i}\hbar\frac{\partial}{\partial r} &&\rightarrow && [\hat{p}_r, r] = -\mathrm{i}\hbar,\\
\hat{p}_\vartheta &= -\mathrm{i}\hbar\frac{\partial}{\partial \vartheta} &&\rightarrow && [\hat{p}_\vartheta, \vartheta] = -\mathrm{i}\hbar,\\
\hat{p}_\varphi &= -\mathrm{i}\hbar\frac{\partial}{\partial \varphi} &&\rightarrow && \left[\hat{p}_\varphi, \varphi\right] = -\mathrm{i}\hbar,
\end{aligned} \tag{17}$$

from Eq. (16), we obtain an alternative form of the Hamiltonian operator different from Eq. (15)

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right) + V(r). \tag{18}$$

We immediately come to recognize that the outcomes of the two procedures differ. Evidently, in the second case, Eq. (18) has lost some terms as compared to Eq. (15) due to Eqs. (17). Eventually, we reach the disagreeable conclusion that the validity of the Hamiltonian operator depends on which coordinate system is used. The result shows that it is impossible to represent the Hamiltonian operator in any curvilinear coordinate system based on the correspondence principle to obtain one and the same result. Therefore, we come to conclude that operators including the Hamiltonian operator are meaningful only for the Cartesian coordinate system. There may be coordinate systems in which it is impossible to represent the kinetic energy operator because quantum operators with regard to generalized coordinates are not known. In this case, we cannot but rely on the transformation from a curvilinear coordinate system to the Cartesian coordinate system to represent the kinetic energy operator. This situation appears to falsify the generality of the theory on quantum operators. Thus, the question of how to explain the generality of conception of quantum operator inevitably arises. This argument implies that applying the correspondence principle by rote to every case leads to unreasonable results. Hence, we arrive at the conclusion that the Hamiltonian operator in general is not justifiable. It is worthwhile to examine the Hamiltonian operator with the help of relations (6) and (9). To ensure the correspondence principle, according to the fourth equation of relations (6) and Eq. (7) it is necessary that the following relation holds

$$\left[\mathrm{Re}\left(\frac{\hat{p}_x\psi}{\psi}\right)\right]^2 = \frac{\hat{p}_x^2\psi}{\psi}. \tag{19}$$

Obviously, for arbitrary wave functions, this relation does not hold. Therefore, it turns out that the Hamiltonian operator is not consistent. It seemed as if the Ehrenfest theorem corroborated the formal correspondence between the classical and quantum-mechanical description. On that account, the expectation values of observables q_i and p_i are regarded as being behaving classically, in the sense of satisfying the "classical" equations. It is necessary to give correct solution to whether the Ehrenfest theorem is an exact result of quantum mechanics or not. In fact, it has no bearing on the classical limit: in general the solutions of Eq. (14) differ from those of the corresponding classical equation. It is known that for systems having a less simple Hamiltonian operator, Eq. (14) is not ensured mathematically. In the end, since the Ehrenfest theorem is based on the Hamiltonian operator that is not mathematically exact, the theorem is not possessed of generality.

The fact that relation (19) is not satisfied implies that in the exact sense, the Hamiltonian operator violates the correspondence principle, and thus is an approximation. This result impels our argument to an acute phase. In fact, if the Hamiltonian operator would not be justifiable, we could not but examine and renounce not a few results obtained with the help of the Hamiltonian operator, which contain the Schrödinger equation as well. As mentioned above, using the routine representation of operator based on the correspondence principle results in such inaccurate results as the Hamiltonian operator encounters in relation to the correspondence principle. Therefore, the Schrödinger equation has to be assessed as an approximate equation in that the equation is built on the basis of the Hamiltonian operator. Such a circumstance emphasizes that the correspondence principle is not airtight from the point of view of mathematics and in not a few case may confuse solutions of quantum problems.

2.3. Approximate relevance of the Schrödinger equation

The Schrödinger equation can be considered to be a typical result that the correspondence principle yields. On that account, the point in question concerning the Hamiltonian operator naturally cannot but make us review the exactitude of the Schrödinger equation. It is obvious that the correspondence principle naturally provides the Hamiltonian operator, so that the operator leads to the Schrödinger equation.

It is a provocative examination to review the problem of whether the Schrödinger equation is mathematically rigorous. If we examine the problem starting with the definition of mean value, we can make certain that the Schrödinger equation makes some approximations besides non-relativistic one.

For convenience, we consider the Schrödinger equation for one particle. By definition, the mean value of momentum component p_x reads

$$\langle p_x\rangle = \int \psi^*\hat{p}_x\psi\mathrm{d}v = \int \psi^*\left(\frac{\hat{p}_x\psi}{\psi}\right)\psi\mathrm{d}v. \tag{20}$$

Obviously, the real part of $\frac{\hat{p}_x\psi}{\psi}$ is $x-$component of momentum. Therefore we can write $\frac{\hat{p}_x\psi}{\psi}$ as

$$\frac{\hat{p}_x\psi}{\psi} = \tilde{p}_x = p_{x-real} + \mathrm{i}p_{x-imag}, \tag{21}$$

where p_{x-real} and p_{x-imag} are the real and imaginary part of $\frac{\hat{p}_x\psi}{\psi}$, respectively. Generally, $\tilde{p}_x = \frac{\hat{p}_x\psi}{\psi}$ is a complex function dependent on coordinates, since ψ is not the eigen function of $\hat{p}_x$. As a result, we easily arrive at

$$\begin{aligned}\hat{p}_x^2\psi &= \hat{p}_x\hat{p}_x\psi = \hat{p}_x(\tilde{p}_x\psi) = \tilde{p}_x\hat{p}_x\psi + \psi\hat{p}_x\tilde{p}_x \\ &= \tilde{p}_x{}^2\psi + \psi\hat{p}_x\tilde{p}_x \neq (\mathrm{Re}\tilde{p}_x)^2\psi = p_{x-real}^2\psi,\end{aligned} \tag{22}$$

where Re denotes the real part of complex number. On the other hand, the Schrödinger equation is obtained in terms of the energy relation

$$E = \frac{\mathbf{p}^2}{2m} + U. \tag{23}$$

The operator relation corresponding to Eq. (23) reads

$$\hat{E} = \frac{\hat{\mathbf{p}}^2}{2m} + U. \tag{24}$$

Consequently, the wave equation for this operator is written as

$$\hat{E}\psi = \left(\frac{\hat{\mathbf{p}}^2}{2m} + U\right)\psi. \tag{25}$$

This equation is nothing but the Schrödinger equation. According to Eq. (22), it is well-grounded that Eq. (25), i. e., the Schrödinger equation is assessed as adding some terms violating the correspondence principle. It is obvious that only when the terms except for $p_{x-real}^2\psi$ in Eq.(22) are negligible, the wave equation takes the form of the Schrödinger equation. Consequently, the requirement for approximation to the Schrödinger equation is that $\tilde{p}_x$ approximates to a real constant. From the above argument, it follows that the double application of differential operator such as the momentum operator to wave function violates the exact correspondence relation between operator and observable.

If we assume the validity of the correspondence principle, then it is possible to obtain, by substituting $\mathbf{p} = \mathrm{Re}\left(\frac{\hat{\mathbf{p}}\psi}{\psi}\right)$ into Eq. (23), the following equation:

$$\mathrm{Re}\left(\frac{\hat{E}\psi}{\psi}\right) = \frac{1}{2m}\left[\mathrm{Re}\left(\frac{\hat{\mathbf{p}}\psi}{\psi}\right)\right]^2 + U.$$

Unfortunately, this equation is nonlinear, so this disparate equation alien to superposition of wave cannot be consider a possible wave equation of quantum mechanics. Nevertheless, this equation is more strict and reliable than the Schrödinger equation from the point of view of use of as few assumptions as possible in that it makes only use of the correspondence principle. It shows that the correspondence principle is not logically rigorous, while the Schödinger equation is approximate in several respects.

Especially, for the case of real-valued wave function, we encounter an intractable problem. In this case, $\frac{\hat{p}_x\psi}{\psi}$ via a single application of $\hat{p}_x$ to the wave equation becomes a purely imaginary number. This indicates that the momentum vanishes. On the other hand, the calculation of $\frac{1}{2m}\frac{\hat{p}_x^2\psi}{\psi}$ via a double application of $\hat{p}_x$ to the wave function gives a purely real number which means nonzero kinetic energy. This result shows that in despite of zero momentum, the corresponding kinetic energy may have a nonzero value. It is this fact that demonstrates the approximate property of Schrödinger equation.

The aforementioned argument shows that the correspondence principle generally does not hold for arbitrary operators. For the purpose of understanding this context, it is necessary to recall the fact that there does not exist the isomorphic mapping between physical quantities and the corresponding quantum-mechanical operators. Therefore, it is not mathematically rigorous to apply in general the correspondence principle to obtain quantum-mechanical operators. If we would consider the wave equation based on the isomorphic mapping between momentum and the corresponding operator,

$$p_x = \mathrm{Re}\left(\frac{\hat{p}_x\psi}{\psi}\right), \tag{26}$$

we could not obtain such a simple wave equation as Eq. (25). What is worse, we are faced with the fact that the wave equation is a non-linear equation. Such a situation significantly emphasizes the necessity of accepting our phase-space formalism not involving the abovementioned inconsistency [20]. Actually, the Hamiltonian operator entering the fundamental equation of our formulation is not implicated in any approximation, and thus the mathematical foundation of the present formalism is ensured ab initio.

3. Classical limit of quantum mechanics

3.1. de Broglie relation and space-time nonlocality of quantum state

The form of the wave function is closely associated with the de Broglie relation.

The explanation runs as follows. It is obvious that the de Broglie's relation defines the frequency and the wave vector of the de Broglie wave. Using this relation, we can determine the phase of the wave without loss of generality by

$$\Phi = \int_0^{\mathbf{q}} \mathbf{k}\left(\mathbf{q}'\right)\, \mathrm{d}\mathbf{q}' - \int_0^t \omega\left(t'\right) \mathrm{d}t' \tag{27}$$

For a period, the phase relation reads

$$\int_0^{\mathbf{q}} \mathbf{k}\left(\mathbf{q}'\right)\, \mathrm{d}\mathbf{q}' - \int_0^t \omega\left(t'\right) \mathrm{d}t' = 2\pi. \tag{28}$$

The above relation indicates the minimal condition necessary for a wave. According to the de Broglie relation, we get from Eq. (28)

$$\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\, \mathrm{d}\mathbf{q}' - \int_0^t E\left(t'\right) \mathrm{d}t' = 2\pi\hbar = h. \tag{29}$$

Here, we suppose that even though the momentum of a particle changes, the de Broglie relation is applicable as in the case of a free particle.

Consequently, we in general can write the phase of the de Broglie wave as

$$\Phi = 2\pi \frac{\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right) \mathrm{d}\mathbf{q}' - \int_0^t E\left(t'\right) \mathrm{d}t'}{h}. \tag{30}$$

Here, $\int \mathbf{p}\, \mathrm{d}\mathbf{q} - \int E \mathrm{d}t$ is recast as the action which is represented as

$$S\left(\mathbf{q}, \mathbf{p}, t\right) = \int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\, \mathrm{d}\mathbf{q}' - \int_0^t H\left(t'\right) \mathrm{d}t'. \tag{31}$$

In view of Eq. (29), we get as the general condition for the periodicity of whole quantum process

$$\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\, \mathrm{d}\mathbf{q}' - \int_0^t H\left(t'\right) \mathrm{d}t = nh. \tag{32}$$

From Eq. (30), the phase of the probability wave is written in terms of the action as

$$\Phi = \frac{S\left(\mathbf{q}, \mathbf{p}, t\right)}{\hbar}. \tag{33}$$

Accordingly, the de Broglie wave, without loss of generality, can be represented as

$$\psi\left(\mathbf{q},\mathbf{p},t\right)=\varphi(\mathbf{q},\mathbf{p},t)\exp\left(\mathrm{i}\frac{S\left(\mathbf{q},\mathbf{p},t\right)}{\hbar}\right), \tag{34}$$

where $\varphi\left(\mathbf{q},\mathbf{p},t\right)$ is a real-valued function.

We can therefore conclude that the de Broglie relation enables us to determine the form of the wave function. Meanwhile, Eq. (29) shows the essential content of the uncertainty relation reflecting ensemble in phase space and the broad context of classical mechanics and quantum mechanics. Eqs. (29) and (30) tell us that the quantum of the action is h. In addition, it shows the necessity and validity of the simultaneous determination of position and momentum, and of time and energy. This is because if it were not to be possible, we could not imagine even the phase of a wave necessary for the study of wave.

Of course, this assumption cannot be regarded to be new, since such a form of wave function has already been used in the preceding formulations. It is necessary to recall the fact that the Schrödinger equation was obtained, implicitly employing this assumption. In fact, for the Schrödinger equation, the phase part of the wave function assumed for a free particle is in accordance with this assumption. Such an understanding serves as the basis for establishing the present formalism of quantum mechanics in phase space.

We can presuppose from Eq. (29) that the quantum in phase space should be represented as

$$h=\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\,\mathrm{d}\mathbf{q}', \tag{35}$$

while the quantum in energy-time space should be represented as

$$h=\int_0^{t}H(t')\mathrm{d}t'. \tag{36}$$

These relations really shed light on the nature of space-time quantization. It is natural to interpret these relations as characterizing an ensemble that consists of pairs of position and momentum, and those of time and energy. Hence, we can adopt the idea for the quantization of space and time. Relations (35), (36) show that the greater the momentum is, the more the space is localized, while the greater the energy is, the shorter the time of quantum process. It can be considered as the content of the quantization reflecting nonlocality. The statistical formalism of quantum mechanics [20] enables us to regard operators as the tool for deriving dynamical quantities from the wave function. The application of a differential operator to the wave function should yield the corresponding dynamical quantity. By inference, we arrive at finding out the operator relations equal or analogous to ones in the Schrödinger equation.

What should be stressed here is that the obtained results are due to the form of wave function, Eq. (34). Of course, from the point of view of dynamics, the assumption about the form of the wave function is explained in some degree by Ref. [20]. To begin with, we calculate the derivatives of the action in the extended phase space with respect to q, p, t. It is ascribed to the introduction of ensemble of paths that momenta are regarded as an independent variable.

To advance further, an essential step has to be ventured as follows.

$$\begin{aligned}\frac{\partial S\left(\mathbf{q},\mathbf{p},t\right)}{\partial q_i}&=\frac{\partial}{\partial q_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^{t}H(t')\mathrm{d}t'\right)\\&=\frac{\partial}{\partial q_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'\right)=\frac{\partial}{\partial q_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'|_0^{\mathbf{q}}-\int_0^{\mathbf{q}}\mathbf{q}'\frac{\partial\mathbf{p}(\mathbf{q}')}{\partial\mathbf{q}'}\mathrm{d}\mathbf{q}'\right)\\&=\frac{\partial}{\partial q_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'|_0^{\mathbf{q}}\right)=\frac{\partial}{\partial q_i}\left(\mathbf{pq}\right)=p_i,\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial S\left(\mathbf{q},\mathbf{p},t\right)}{\partial p_i}&=\frac{\partial}{\partial p_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^{t}H(t')\mathrm{d}t'\right)\\&=\frac{\partial}{\partial p_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'\right)=\frac{\partial}{\partial p_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'|_0^{\mathbf{q}}-\int_0^{\mathbf{q}}\mathbf{q}'\frac{\partial\mathbf{p}(\mathbf{q}')}{\partial\mathbf{q}'}\mathrm{d}\mathbf{q}'\right)\\&=\frac{\partial}{\partial p_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'|_0^{\mathbf{q}}\right)=\frac{\partial}{\partial p_i}\left(\mathbf{pq}\right)=q_i,\end{aligned} \tag{38}$$

$$\frac{\partial S\left(\mathbf{q},\mathbf{p},t\right)}{\partial t}=\frac{\partial}{\partial t}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^{t}H(t')\mathrm{d}t'\right)=-H. \tag{39}$$

Next, let us review $\mathrm{i}\hbar\frac{\partial\psi}{\partial t}$. The application of $\mathrm{i}\hbar\frac{\partial}{\partial t}$ to the wave function yields

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t}=\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial t}\psi+\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial t}\psi=H\psi+\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial t}\psi. \tag{40}$$

From the above expression, we can interpret $\mathrm{i}\hbar\frac{\partial}{\partial t}$ as the operator relative to total energy, since it makes H be derived from the wave function.

Furthermore, let us examine $-\mathrm{i}\hbar\frac{\partial\psi}{\partial p_i}$. The application of this operator to the wave function produces

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial p_i}=-\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial p_i}\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial p_i}\psi=q_i\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial p_i}\psi. \tag{41}$$

Since this operation gives q_i in the first term of Eq. (41), we can regard $-\mathrm{i}\hbar\frac{\partial}{\partial p_i}$ as the position operator.

Similarly, we have

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial q_i}=-\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial q_i}\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial q_i}\psi=p_i\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial q_i}\psi. \tag{42}$$

We hereafter shall refer to Eq. (40), (41), (42) as the operator equation. As a consequence, $-\mathrm{i}\hbar\frac{\partial}{\partial q_i}$ is adopted as the momentum operator. The obtained results naturally bring the idea for operator to us. From the operator equations, we can interpret the meaning of the relation between an observable, L and the corresponding operator, $\hat{L}$ as

$$L=\mathrm{Re}\left(\frac{1}{\psi}\hat{L}\psi\right). \tag{43}$$

In fact, this relation naturally comes from the definition of mean value. By definition, the mean value with respect to $\hat{L}$ is written as

$$\bar{L}=\int\psi^*\hat{L}\psi\mathrm{d}\tau=\int\psi^*\psi\frac{\hat{L}\psi}{\psi}\mathrm{d}\tau=\int\rho\frac{\hat{L}\psi}{\psi}\mathrm{d}\tau. \tag{44}$$

Accordingly, $\mathrm{Re}\left(\frac{1}{\psi}\hat{L}\psi\right)$ should be regarded as the observable with respect to operator $\hat{L}$.

By inference, we find the operators corresponding to kinetic energy and potential. Altogether, the operators corresponding to fundamental observables are represented as

$$\hat{E}=\mathrm{i}\hbar\frac{\partial}{\partial t}, \tag{45a}$$

$$\hat{p}_i=-\mathrm{i}\hbar\frac{\partial}{\partial q_i}, \tag{45b}$$

$$\hat{q}_i=-\mathrm{i}\hbar\frac{\partial}{\partial p_i}, \tag{45c}$$

$$\hat{U}=-\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{f}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}=\frac{1}{2}\sum_{i=1}^{f}\left(\dot{p}_{group}\right)_i\hat{q}_i, \tag{45d}$$

$$\hat{T}=-\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{f}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}=\frac{1}{2}\sum_{i=1}^{f}\left(\dot{q}_{group}\right)_i\hat{p}_i, \tag{45e}$$

where $\left(\dot{q}_{group}\right)_i$ and $\left(\dot{p}_{group}\right)_i$ denote the ith components of the group velocity with respect to position and momentum, respectively [20]. The successive three operators correspond to energy, momentum and position, respectively, which become basic dynamical quantities. The fourth operator should be considered as the potential energy operator, since it corresponds to a potential energy function. This operator suggests nothing but the virial theorem in statistical mechanics. Thus, we can arrive at the important conclusion that in quantum mechanics the potential energy should be represented as the virial of the system under consideration. Meanwhile, the fifth operator should be considered as the kinetic energy operator, since it corresponds to kinetic energy. It is remarkable that in our formulation the kinetic energy operator is represented as a first-order expression with respect to the momentum operator and thus is not involved in the abovementioned approximate relevance of the kinetic operator.

Since the action, Eq. (32) is represented by generalized coordinates and generalized momenta, naturally the angular momentum operator should have the following form.

$$\hat{L}_\varphi = -\mathrm{i}\hbar\frac{\partial}{\partial\varphi}. \tag{46}$$

In this connection, it should be emphasized that for the configuration formulation of quantum mechanics, it is not possible to imagine operators represented by use of generalized coordinates.

The difference of the operators from ones in the Schrödinger equation consists in the fact that the wave functions applied by them are defined in phase space. For the Schrödinger equation, the wave function is the state function defined in configuration space, whereas for the fundamental equation of quantum mechanics in phase space, the wave function is the state function defined in phase space [20]. It is important to note that the dynamical quantities which are obtained with the help of operators and wave function are not the same as classical ones, and get quantal.

Such an interpretation of quantum observables naturally leads to adopting time as an ordinary quantum observable. Extending the phase space furthermore, we can take the action as

$$S\left(\mathbf{q},\mathbf{p},H,t\right) = \int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\,\mathrm{d}\mathbf{q}' - \int_0^t H\left(t'\right)\mathrm{d}t'. \tag{47}$$

Then we get

$$\begin{aligned}\frac{\partial S\left(\mathbf{q},\mathbf{p},H,t\right)}{\partial H} &= \frac{\partial}{\partial H}\left(\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\,\mathrm{d}\mathbf{q}' - \int_0^t H(t')\mathrm{d}t'\right)\\ &= -\frac{\partial}{\partial H}\left(\int_0^t H(t')\mathrm{d}t'\right) = -\frac{\partial}{\partial H}\left(H(t')t'|_0^t - \int_0^t t'\frac{\partial H(t')}{\partial t'}\mathrm{d}t'\right)\\ &= -\frac{\partial}{\partial H}\left(H(t')t'|_0^t\right) = -\frac{\partial}{\partial H}\left(Ht\right) = -t.\end{aligned} \tag{48}$$

The time operator therefore becomes

$$\hat{t} = \mathrm{i}\hbar\frac{\partial}{\partial H}. \tag{49}$$

Consequently, the operator equation for the time operator can be written as

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial H} = \mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial H}\psi + \mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial H}\psi = t\psi + \mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial H}\psi.$$

The time operator should be considered to apply to the wave function in phase space as follows.

$$\hat{t} = \mathrm{i}\hbar\frac{\partial}{\partial H} = \mathrm{i}\hbar\sum_i\left[\left(\frac{\partial H}{\partial q_i}\right)^{-1}\frac{\partial}{\partial q_i} + \left(\frac{\partial H}{\partial p_i}\right)^{-1}\frac{\partial}{\partial p_i}\right] + \left(\frac{\partial H}{\partial t}\right)^{-1}\frac{\partial}{\partial t}. \tag{50}$$

Hereafter, we shall usually represent the time operator as

$$\hat{t} = \mathrm{i}\hbar\frac{\partial}{\partial E}. \tag{51}$$

Thus, the special status of time as an exceptional observable without the corresponding operator comes to be lost and the system of basic operators of quantum mechanics becomes complete.

The introduction of these operators helps to clarify the relations of this formalism with the others. Using the above operators, we can write the fundamental equation of quantum mechanics in phase space [20] as

$$\hat{E}\psi = \left\{ \sum_{i=1}^{f} \left[\left(\dot{q}_{phase} \right)_i \hat{p}_i + \left(\dot{p}_{phase} \right)_i \hat{q}_i \right] \right\} \psi. \tag{52}$$

In the non-relativistic case, with the help of the relations between the phase and group velocity, we have

$$\hat{E}\psi = \frac{1}{2} \left\{ \sum_{i=1}^{f} \left[\left(\dot{q}_{group} \right)_i \hat{p}_i + \left(\dot{p}_{group} \right)_i \hat{q}_i \right] \right\} \psi \tag{53}$$

or

$$\hat{E}\psi = \frac{1}{2} \left\{ \sum_{i=1}^{f} \left(\frac{\partial H}{\partial p_i} \hat{p}_i - \frac{\partial H}{\partial q_i} \hat{q}_i \right) \right\} \psi.$$

In more compact form, we write

$$\hat{E}\psi = \hat{H}\psi, \tag{54}$$

where in view of Eq. (45d) and (45e), the Hamiltonian operator takes the following form:

$$\hat{H} = \hat{T} + \hat{U}.$$

The fundamental equation of this formalism is distinguished from the Schrödinger equation because the wave function is defined not in configuration space but in phase space. This formalism is expected to be useful to elucidate the essential contents of the correspondence principle.

For quantum mechanics, it is important to give answer to the question as to whether all observables should correspond to quantum operator. For that purpose, it is necessary to consider in terms of what the operators were conceived and derived. The foundation is nothing but the action. As mentioned above, we inferred quantum operators from the action,

If the action did not possess physically profound meanings relevant to quantum observables, we could not conceive the idea for quantum operator. In fact, it is possible to find out quantum operators because dynamical quantities of quantum mechanics are determined from the action constituting the phase part of wave function. Really, the relation between the action and quantum operators can be written as

$$L = \hat{L}\left(\frac{\mathrm{i}}{\hbar} S \right), \tag{55}$$

where L is an observable and $\hat{L}$ a quantum operator, and S the action. We have already confirmed that the general form of the wave function has the phase part determined by the action and it yields the dynamical quantities via the application of operators to the wave function. Judging from these relations, we can arrive at the conclusion that it is not reasonable to conceive a certain general rule of the correspondence between classical phase-space functions and quantum operators, irrespective of the action. In fact, it is not necessary to make every physical quantity correspond to a quantum operator. This is because that it is possible and significant to find out only the operators corresponding to main canonically conjugate variables constituting the action such as position and momentum, and time and energy. Within the framework of our formulation, the operators for position, momentum, time and energy are entirely determined from the action. These operators determine the observables of position, momentum, time and energy. On the other hand, these observables are enough to constitute all the dynamical quantities. For that reason, we can say that this formalism provides a complete system of quantum operators.

What do we mean by the correspondence between operators and quantum observables?

Obviously, the correspondence indicates the correspondence between the main canonically conjugate variables entering the action and the corresponding quantum operators. It is reasonable to confine the correspondence only to canonically conjugate variables (quantum observables) constituting the action, so that it is no use extending the correspondence beyond the category of action.

On the other hand, what do we mean by the correspondence between classical and quantum relations?

As for this subject, we maintain that quantum observables obtained with the help of operators and classical quantities should satisfy one and the same dynamical formulae given by classical mechanics in that classical formulae were used just to obtain the fundamental equation of quantum mechanics and its validity had been already verified, and, moreover, any substantially distinct force of quantum origin still has not been discovered. Evidently, for classical mechanics and quantum mechanics, dynamical relations are identical, while classical and quantum quantities are different. In fact, quantum quantities are to be determined from the wave function with the help of fundamental operator relations, i.e., Eqs. (45a), (45b), (45c) and (49). These relations can be considered to represent constraints on microscopic particles due to quantum correlation created by the wave field. On account of these constraints, quantum observables naturally are distinct from classical quantities, but formulae for physical quantities of two physics should be identical. If one adopts the correspondence principle in this context, then one can successfully overcome difficulties arising concerning the interpretation and applications of the principle.

3.2. Nonlocality and Classical limit of quantum mechanics

It is an open question of quantum mechanics to reveal the classical limit of quantum mechanics: under what condition the behavior of a quantum system approximates to that of classical mechanics. This is the main aim of the correspondence principle in the weak sense. In order for the correspondence principle to give the correct answer to this question, it should elucidate particular conditions for transition from quantum mechanics to classical mechanics in relation to dynamical quantities, e.g., position and momentum. In this regard, it is interesting to recall the fact that the Ehrenfest theorem became an apotheosis, which seemed to demonstrate the validity of the correspondence principle. According to this theorem, classical mechanics and quantum mechanics seem to have similarity in a formal relation. However, in essence this theorem too could not demonstrate under what physical condition the behavior of quantum system reduces to classical one. Based on our work, we can state that the problem of classical limit of quantum mechanics is ascribed to the de Broglie relation which according to our views, reflects the non-local characteristic of quantum entities, thus determining the minimal size of space and time necessary for a given quantum system. According to the Broglie relation, it is reasonable to consider that the localization of a particle is limited at least to the de Broglie wavelength and the duration of process characteristic of an ensemble of quantum state:

$$\lambda = \frac{h}{p}, \qquad \tau = \frac{h}{E}, \tag{56}$$

although the particle itself is regarded as a point-like body. In essence, the de Broglie relation expresses the domain of space and time required for a quantum state of particle. Specifically, in order for a quantum state to be realized, at least the space and time determined by Eq. (56) must be assigned to a particle. Therefore, if the magnitude of the whole space in which a particle moves (the mean magnitude of space which is assigned to a particle) is much greater than the quantum volume defined by the de Broglie wavelength, i.e., given as

$$V_q = \lambda^3 = \left(\frac{h}{p}\right)^3, \tag{57}$$

then the particle becomes classical, since under the abovementioned condition the particle is not to be influenced by interference of itself or other origin. This shows that the de Broglie relation, rather than the contents of the correspondence principle that we have argued so far, becomes the cornerstone for solving the classical limit of quantum mechanics. The existence of the condensation temperature in statistical mechanics is enough to understand this conclusion. It is well known that the transition from quantum statistics to classical statistics requires the condition that the de Broglie wavelength of a particle should be much shorter than the constants of crystal lattice. It is in this condition that particles are free of quantum influence.

In order to understand the classical limit of quantum mechanics, it is necessary to consider that the fundamental equation of quantum mechanics in phase space (QMPS) contains both the energy relation of particles and the relation of probability wave. Starting with the form of wave function, we can study the structure of the fundamental equation of QMPS.

Substituting the wave function,

$$\psi(\mathbf{q}, \mathbf{p}, t) = \varphi(\mathbf{q}, \mathbf{p}, t) \exp\left(\frac{\mathrm{i}S}{\hbar}\right)$$

into the fundamental equation of QMPS and separating the equation into real and imaginary part, we get

$$\sum_{i=1}^{f} \left(\dot{q}_i \frac{\partial \varphi^2}{\partial q_i} + \dot{p}_i \frac{\partial \varphi^2}{\partial p_i} \right) = \frac{\partial \varphi^2}{\partial t}, \tag{58}$$

i.e.,

$$\sum_{i=1}^{f} \left(\dot{q}_i \frac{\partial \rho}{\partial q_i} + \dot{p}_i \frac{\partial \rho}{\partial p_i} \right) = \frac{\partial \rho}{\partial t}, \tag{59}$$

and

$$\sum_{i=1}^{f} \left(\dot{q}_i \frac{\partial S}{\partial q_i} + \dot{p}_i \frac{\partial S}{\partial p_i} \right) + \frac{\partial S}{\partial t} = 0, \tag{60}$$

i.e.,

$$\sum_{i=1}^{f} \left(\frac{p_i^2}{2m} - \frac{1}{2} q_i \frac{\partial H}{\partial q_i} \right) = H. \tag{61}$$

Here, φ as the probability amplitude is a real-valued function.

Eq. (59) is the probability continuity equation, while Eq. (61) represents the energy relation of particles. Eq. (60) pertaining to the phase relation implies that there is no change in phase of probability wave with respect to the reference system moving at the phase velocity. The wave equation thus involves the duality, i.e., the particlelike and wavelike property. Exactly, the wave equation contains the relations of not only probability wave but also energy of particles.

On the other hand, starting with the operator equation,

$$\hat{L}\psi = L\psi + \frac{\hat{L}\varphi}{\varphi}\psi, \tag{62}$$

we can treat the matter for the classical limit of quantum mechanics. Operator $\hat{L}$ stands for the fundamental operators originating from the action such as position, momentum, time and energy operator. As seen from the operator equations, $\frac{\hat{L}\varphi}{\varphi}$ is a purely imaginary number. If term $\frac{\hat{L}\varphi}{\varphi}$ is able to be neglected compared with L, the the wavelike behavior of quantum systems tends to disappear, since the quantum fluctuation as the wave-like property is due to $\frac{\hat{L}\varphi}{\varphi}$ [20]. From this, in order for a system to be classical, it is necessary and sufficient that $\frac{\hat{L}\varphi}{\varphi}$ vanishes. Thus, it would be more exact to explain that when $\frac{\hat{L}\varphi}{\varphi}$ approaches zero rather than when $\hbar$ approaches zero, then quantum system transits to classical system, since $\hbar$ always is a definite constant and not a variable. The condition for the classical limit can be represented as

$$\frac{\partial\varphi(x,p,t)/\partial x}{\varphi(x,p,t)} \to 0, \quad \frac{\partial\varphi(x,p,t)/\partial p}{\varphi(x,p,t)} \to 0, \quad \frac{\partial\varphi(x,p,t)/\partial t}{\varphi(x,p,t)} \to 0. \tag{63}$$

If $\partial\varphi/\partial x \neq 0$, it means that a position has a spectrum of momenta, while if $\partial\varphi/\partial p \neq 0$, it indicates that a momentum has a spectrum of positions. Clearly, such spectra represent a state of a quantum system characterized by stochastic processes.

If $\partial\varphi/\partial x = 0$, $\partial\varphi/\partial p = 0$ and $\partial\varphi/\partial t = 0$, then there is not such a spectrum. In this case, the wave equation, Eq. (54) reduces solely to the equation of classical mechanics, i.e., the Hamilton-Jacobi equation $\frac{\partial S}{\partial t} + H = 0$ irrelevant to the probability wave. This means that the motion of a particle is subject only to a definite trajectory determined by the Hamilton-Jacobi equation.

If not so, then the dynamical relation which is represented by Eq. (54) becomes the Hamilton-Jacobi equation $\dfrac{\partial S}{\partial t} + H = 0$ subject to the probability wave, exactly, the probability amplitude φ. This means that the motion of a particle is associated with a family of trajectories determined by the Hamilton-Jacobi equations subject to $\varphi(\mathbf{q}, \mathbf{p})$. That is, a quantum state turns out to be represented by an ensemble of fluctuating trajectories constrained by $\varphi(\mathbf{q}, \mathbf{p})$. The very fluctuating trajectories stand for the probabilistic behavior of microscopic particles. Understanding like this actually is identified with Feynman's idea for the path integral.

From Eq. (63), we can see that in order for an ensemble in phase space not to have wavelike property, it is necessary that the probability density should be constant, i.e., $\rho = const$, since the probability amplitude, φ is irrelevant to x, p, t. Eqs. (58) and (63) explain this fact alike. Clearly, Eq. (63) becomes the condition for a system under consideration to be classical.

On the other hand, it is useful to consider the quantal force. If the quantal potential were the same as classical one, one could not find any other behaviors of microscopic particles than classical ones. The second term of Eq. (61),

$$U_{c+q} = -\frac{1}{2}\sum_{i=1}^{f} q_i \frac{\partial H}{\partial q_i} \tag{64}$$

reflects the fact that the potential of a quantal entity is different from the pure classical potential to involve both classical and quantum causalities. Here, subscript $c + q$ refers to the classical and quantum causalities. The potential of microscopic system depends on the probability wave. In fact, this potential is represented by the product of the force acting on particles and the position vector determined from the action as the phase of wave function. Therefore, the dynamical relation of a quantum system is regulated to be subject to the wave function. In this connection, we can consider the wave function to be a constraint imposed on the motion of a microscopic particle by quantum causality.

It is possible to explain the quantum potential in terms of QMPS as in Bohmian mechanics. For the fundamental equation of QMPS, the potential operator becomes a composite operator composed of differentiation and multiplicand. This composite operator should be considered as the potential operator that reflects both classical and quantal causality. Therefore, the potential displaying the pure quantal causality can be denoted by

$$U_q = \sum_{i=1}^{f} \mathrm{Re}\left(\frac{\mathrm{i}\hbar}{2\psi}\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\psi\right) - U = \sum_{i=1}^{f} \mathrm{Re}\left(\frac{\mathrm{i}\hbar}{2}\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\ln\psi\right) - U, \tag{65}$$

where subscript q refers to *quantum*. Hence, the quantal force, essentially as a constraining force of quantum origin can be determined by

$$\mathbf{F}_q = -\nabla U_q. \tag{66}$$

It is conceivable that this quantal force causes some quantal fluctuation in the classical paths of microscopic particles.

In the case of $\varphi = const$, taking into consideration that potential energy of statistical system should be represented as $-\frac{1}{2}\sum_{i=1}^{f} q_i \partial H/\partial q_i$ according to the Liouville theorem, the quantal potential according to Eq. (65) vanishes.

The above argument shows that the present perspective is expected to be useful for expounding the classical limit of quantum mechanics.

4. Results and discussion

In the present work, we have demonstrated that there does not exist the isomorphic mapping between observables and operators in the general sense. It is natural that this result requires the reappraisal of the significance and role of the correspondence principle maintained in constructing quantum mechanics. As for the correspondence, we can say only the correspondence between the formulae for classical quantities and those for quantum observables, and it is superfluous to discuss the correspondence between formulae for quantum observables and those for operators. The consideration in terms of algebraic isomorphism gives this conclusion.

The correspondence principle played a key role in solving the atom problem but many ambiguous questions relevant to this principle still remain unsolved. In despite of the successes of the approach on the basis of the correspondence principle, it is fair to say that it did not require, nor did it produce, fundamental understanding of the quantum theory itself [21].

In our view, the correspondence principle is of value neither in the sense of the strong form nor in the sense of the weak form. As a matter of fact, the universally accepted correspondence principle cannot provide the reasonable explanation both for how to find out quantum-mechanical formulae on the basis of formulae of classical mechanics and for how to determine the classical limit of quantum mechanics, even though it seems to be of heuristic value. It should be noted that there are some indications that it is improbable that the canonical quantization procedure can offer an automatic recipe for a derivation of quantum mechanics from classical mechanics [22].

Obviously, the study of the principle cannot reach the goal in the original sense, but makes only one fact clear. This is the fact that microscopic particles also are of nature of particle in itself, so quantum mechanics makes use of the dynamical relations of particles which classical mechanic has provided. The derivation of the Schrödinger equation is a straightforward example of showing that quantum mechanics makes use of relations of classical mechanics just as it is. If quantum mechanics has something other than classical mechanics, it is that quantum constraints given by Eqs. (40), (41), (42) are imposed on microscopic particles. In this sense, the correspondence principle is significant. It is undoubted that in the realm of quantum world too, the concepts of all dynamical quantities such as position and momentum and the rest inevitably should be in common use as in macroscopic world. Microscopic particles can possess dynamical quantities as it is, though they undergo quantum fluctuation due to the wave field. Thus, we arrive at the conclusion that microscopic particles should be considered to be in all respects the particles fluctuating according to quantum laws representing, in essence, constraints. As a consequence, in order to delve in depth into the essence of quantum phenomena, we must deal with concrete dynamical quantities for microscopic particle.

What is the correspondence in the true sense?

It would be correct to consider that for quantum mechanics, the fundamental canonical variables such as position, momentum, time and energy are derived from the wave function by means of quantum operators, and these have the same relationships as those of classical mechanics. Therefore, physical relations of two mechanics are identical, while in the two cases, the dynamical quantities are different. The fundamental operators corresponding to canonically conjugate quantities can constitute all dynamical relations for quantum mechanics as classical mechanics produces. In this sense, the correspondence principle is significant.

It is unreasonable to imagine the operators corresponding to every dynamical quantity on the basis of the assumption about operator relations isomorphic to the classical relations. This description can be depicted in a schematic way by using Figure 1.

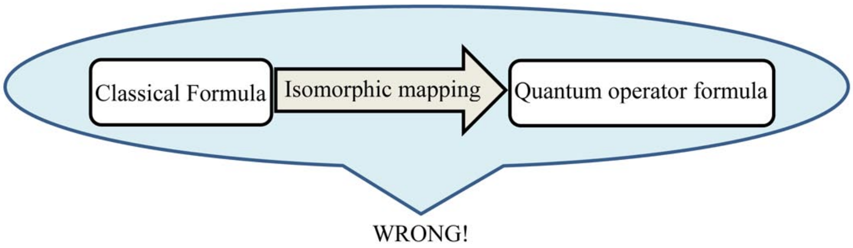

Figure 1: Relations of classical quantities are not isomorphic to relations of quantum operators.

The fundamental operators such as position, momentum, time and energy operator are determined from the action. It is neither possible nor necessary to imagine other operators beyond the fundamental operators, since they do not come from the action. On the other hand, the fundamental operators are enough to investigate quantum theory at the perfect level, since the other dynamical quantities are the function of the fundamental canonical variables given by the fundamental operators and wave function. The rest of quantum observables do not take the corresponding operators but can be calculated by fundamental canonical variables entering the action with the help of classical formulae. On the other hand, the determined quantum observables differ from classical quantities on account of quantum constraints. These relations can be represented in a schematic way by using Figures 2 and 3.

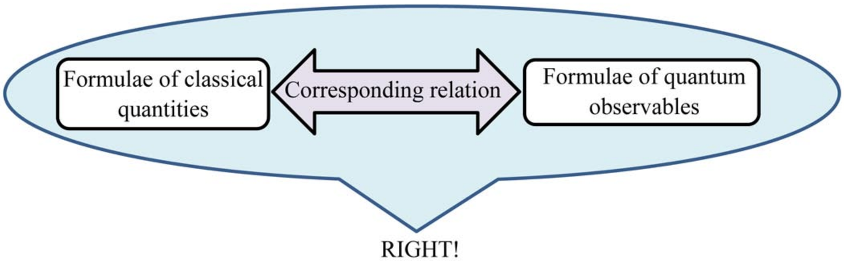

Figure 2: Formulae of classical quantities correspond to formulae of quantum observables.

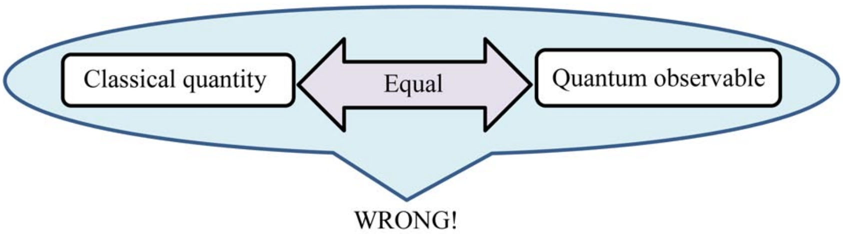

Figure 3: Quantum observables differ from corresponding classical quantities because of quantum constraints.

Meanwhile, it is impossible to determine the classical limit of quantum mechanics from some approximation of corresponding relations between classical and quantum mechanics. As mentioned above, it is reasonable to elucidate the classical limit of quantum mechanics in terms of the de Broglie relations and the operator equations.

An important result of our reappraisal of the correspondence principle is that even the Hamiltonian operator is not exact, and thus the Schrödinger equation naturally is possessed of approximate relevance. Therefore, all arguments based on the premise that the Hamiltonian operator is exact, for example, the Ehrenfest theorem and the like cannot be regarded as being justifiable, provided that we represent the Hamiltonian operator in configuration space.

5. Conclusion

In this work, we have investigated the essentials of the correspondence principle based on the consideration of the possibility of the isomorphic mapping between observables and quantum operators.

Our work has demonstrated that the formalism of quantum mechanics in phase space gives the complete system of quantum operators and reveals the origin and essence of operators. With the help of the action, we were able to obtain the complete system of operators and as a result have established the solid foundations for reassessing the essence and significance of the correspondence principle.

We have examined the correspondence principle from the point of view of algebraic isomorphism to gain a correct understanding of the principle. Thus, we arrived at the conclusion that there is not the mathematically isomorphic relation between physical quantities and quantum operators, and thus the correspondence principle is not substantiated by mathematics. The confirmation that the previous interpretation of the correspondence principle is not consistent in several respects, especially in the mathematical aspect leads to the clear understanding that even both the Hamiltonian operator and the Schrödinger equation cannot avoid approximate relevance.

Our work has shown that while for both classical and quantum mechanics, formulae of physical quantities are identical, quantum observables are different from the corresponding classical quantities as a consequence of the constraints of quantum origin. Specifically, it is possible to state that the correspondence does not hold for the relations between quantities of classical physics and those between quantum operators but does for the relations between quantities of classical mechanics and those between quantities of quantum mechanics.

Moreover, we have shown the methodology for treating the problem of the classical limit of quantum mechanics based on the de Broglie relation and the operator equations.

Ultimately, we do claim to have arrived at an alternative understanding of the correspondence principle, which may help to further our knowledge about the foundations and interpretation of quantum mechanics. We plan to extend our resarch to the interpretation of the foundations of quantum mechanics such as the general proof of the uncertainty relations and the explanation of the relation between the wave function and the probability density.

Acknowledgments

This work was supported partially from the Committee of Education, Democratic People's Republic of Korea, under the project entitled "Statistical Formalism of Quantum Mechanics". Profs. Ji-Min Pak, Song-Uk Ri and Won-Chol Ri from Kim Chaek University of Technology are appreciated for valuable discussion and advice.

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