Collatz Conjecture

By: Gaurav Rudra Krishna

The problem

Conjecture: The following operation is applied on an arbitrary positive integer $n$

$$f(n) = \begin{cases} 
\frac{n}{2}, & \text{if } n \equiv 0 \text{ mod } 2 \\
3n + 1, & \text{if } n \equiv 1 \text{ mod } 2 
\end{cases}$$

The Collatz conjecture states: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

Abstract

We consider $n$ to have only odd values, and even values are written in the form $n \cdot 2^b$. We create a predefined function $r_b(n)$. Define, $g(n) = r_b(n) + r_{b-1}(n)$ and prove $g(n) = f(n)$. $g(n)$ being an identical function to Collatz transformations, we use the properties of said function to probe if some number $n$ can explode to infinity.

We study $n_x$ in detail, establish pattern for $n_x$ modulo 3. We use our understanding to probe if some number $n$, can loop to itself with more than one transformation.

Format of the solution: The solution does not adhere to the conventional framework of paragraphed proof writing, every piece of maths that is important (to conjecture) is tabular.

- The solution template is inspired from Leslie Lamport; how to write a 21st century proof
- The Solution is framed in a structured template with every argument followed its proof.
- All the subsections are tabulated to study, IF-THEN clause: for main case and sub cases.
- Tabulation should help the reader understand the larger picture in context to some specific case.

Current understanding: The heuristic and probabilistic arguments that support the conjecture are well known. The conjecture has been proven valid for numbers upto $2^{68}$ but hasn’t been proven yet for all numbers. There has been a lot of interesting work done in this problem by notable mathematicians. Few of the notable efforts have been by; Terras showing almost all values $n$ eventually iterated to a value less than $n$, Krasikov and Lagarias showed that for any large number $x$, there were at least $x^{0.84}$ initial values $n$ between 1 and $x$ whose Collatz iteration reached 1. Terrence Tao showed Almost all Collatz orbits attain almost bounded values.

The conjecture has been studied using Benford’s law, Markov’s chains, binary systems among other approaches. Variants of the Collatz function have been studied, John Conway invented a computer language called Fractran in which every program was a variant of the Collatz function, it turned out to be Turing complete.
There has been some interesting commentary by reputed names, regarding the problem; Paul Erdos said about the Collatz conjecture: "Mathematics may not be ready for such problems." Jeffery Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present-day mathematics. Richard K guy stated "Don't try to solve these problems! " Some call it the most dangerous problem in mathematics. All this commentary makes us more interested in looking into the problem.

For verbal explanation refer: https://www.youtube.com/watch?v=ZXK56OdwdrE

**Definition 0.1  Transformation:** Application of $3n+1$ followed by application of $n/2$ (one or more times) till we get odd number is termed as transformation. Application of $3n+1$ always results the form of $n'.2^b$ and we just need to divide $n'.2^b$ by 2, b number of times, to get n' which may go through transformation once again.

**Notation**

{ } : square brackets are used to represent sets. All the sets in the analysis are open ray sets, that is having a certain starting point and can be extended to infinity.

≡ : Equivalence is used for operations under the defined transformations in the problem, that is $3n+1 & n/2$. Example; $5 \equiv 1$. One may consider $\equiv$ as applying transformation on odd element and dividing it by max power of 2 with result being an integer.

n is defined to be only odd and we may apply $3n+1$ upon it. Any even entity shall be represented as $even = n_{odd}.2^b$

$\equiv$ : is used to describe congruence modulo some number.

**Definition 0.2**

$n_x$(before transformation; applying $3n+1$) 

≡ $n_s$(after transformation; applying $3n+1$ and dividing it by max power of 2)

$n_x & n_s$ are always odd

The co-application of $3n+1$ and $n/2$ shall be considered as a single step

$$3n_x + 1 = n_s.2^b \mid n_x & n_s = 2k + 1 & k, b \in \mathbb{Z^+}$$

<table>
<thead>
<tr>
<th>D0.2</th>
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</thead>
<tbody>
<tr>
<td>$3n_x + 1 = n_s.2^b$ is same as $n_x \equiv n_s$</td>
</tr>
</tbody>
</table>

Take the Universal set of all positive integers $\{U\}$

$\{U\} = \{1,2,3,4,5 \ldots \}$

On all even elements, apply map ($n/2$ till we get odd) on $\{U\}$, we get:

$$\frac{n}{2} \rightarrow \{U\}, \text{we get}\{U'\} = \{1,3,5,7,9 \ldots \}$$

We begin our study considering set $\{U'\}$ with only positive odd integers
Rooster Notation: \( \{U'\} = \{1, 3, 5, 7, 9, \ldots \} \) Set Builder Notation: \( \{U'\} = \{2k - 1\} | k \in \mathbb{Z}^+ \)

We define \( \{r_y\} \& \{r_b\} \), formulate expansion for \( \{r_b\} \) and establish the relationship between \( \{r_b\} \& n_x \)

**Definition 1:** \( \{r_y\} \) is a set of sets contains elements corresponding to values of \( \{U'\} \) based upon parity of \( y \) with the given definition;

<table>
<thead>
<tr>
<th>Condition</th>
<th>( r_y = \frac{r_{y-1} \pm 1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y \equiv 1 \text{ mod } 2 ) (y=odd)</td>
<td>( {r_0} = {U'} \Rightarrow r_0 = n_x \text{ and } r, y \in {\mathbb{Z}^+} \cup {0} )</td>
</tr>
<tr>
<td>( y \equiv 0 \text{ mod } 2 ) (y=even)</td>
<td>( r_y = \frac{r_{y-1} + 1}{2} )</td>
</tr>
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</table>

\( r_{y-1} \pm 1 \) implies, we add or subtract 1 to the value of \( r \) for any given subset \( (y-1) \)

\( r_{y-1} \) is mapped to \( r_y \) if and only if value of \( r \) in \( r_{y-1} \) is odd. The mapping continues till \( r \) is even.

For value of \( r \) being even, we define said set as \( r_b \).

Example: Say, \( n_x = 13, r_0 = 13 \) (by definition)
- For \( r_y = r_1 \): because \( y \) is odd, \( r_y = \frac{r_{y-1} + 1}{2} \) implies \( r_1 = \frac{r_0 + 1}{2} = 7 \), so \( r_1 = 7_1 \)
  - Since value of \( r \) in \( r_1 \) is odd, we extent the set further;
- For \( r_y = r_2 \): because \( y \) is even, \( r_y = \frac{r_{y-1} - 1}{2} \) implies \( r_2 = \frac{r_1 - 1}{2} = 3 \), so \( r_2 = 3_2 \)
  - Since value of \( r \) in \( r_2 \) is odd, we extend the set further.
- For \( r_y = r_3 \): because \( y \) is odd, \( r_y = \frac{r_{y-1} + 1}{2} \) implies \( r_3 = \frac{r_2 + 1}{2} = 2 \), so \( r_2 = 2_3 \)
  - Since value of \( r \) in \( r_3 \) is even, we cannot extend the set further. Thus, \( b=3 \) and \( r_b = 2_3 \)

**Definition 2:** \( \{r_b\} \)

\( r_b = r_y | r \text{ in } r_y = 2k, k \in \mathbb{Z}^+ \)

Since, \( r_b \) is same as \( r_y \) with the only condition is that value of \( r \) in \( r_y \) is even. So, \( r_b \) carries the same definition as \( r_y \)

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</tr>
<tr>
<td>( b \equiv 0 \text{ mod } 2 ) (b=even)</td>
<td>( r_b = \frac{r_{b-1} - 1}{2} )</td>
</tr>
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</table>

If one applies relevant map on \( r_b \) where value of \( r \) is even, result is a rational solution which is not a positive integer or zero, thus is invalid.
Remark: For condition \( r = 0 \), we use the classification of zero being even described by Penner 1999, p. 34: Lemma B.2.2

Define: \( \{ R_b \} = \{ \{ r_1 \} \cup \{ r_2 \} \cup \{ r_3 \} \cup \{ r_4 \} \cup \{ r_5 \} \cup \{ r_6 \} \cup \ldots \} \)

**Lemma 1.0:** There does not exist \( n_x \) that is a subset of \( \{ U' \} \), and does not have an associated representation in \( \{ r_b \} \). In other words, all elements of \( \{ U' \} \) are a subset of \( r_b \) such that \( b = 1 \rightarrow \infty \).

\[
\forall \; \, n_x \in \{ U' \} \exists \; \{ r_b(n_x) \} \in R_b \; \text{for} \; b = 1 \rightarrow \infty | 3n_x + 1 = n_s, 2^b \; \& \; n_x, n_s = 2k - 1 \; \& \; k, b \in \mathbb{Z}^+
\]

\[
\Rightarrow q\{ U' \} = \sum_{b=1}^{\infty} q\{ r_b \} \mid \text{for } b \equiv 1 \mod 2, r_b = \frac{r_{b-1} + 1}{2} \; \& \; b \equiv 0 \mod 2, r_b = \frac{r_{b-1} - 1}{2} \; \& \; b \in \mathbb{Z}^+
\]

**Proof:** let number of elements in any given set be represented by \( q(x) \)

| \( L1 \) | \( \forall \; \, n_x \in \{ U' \} \exists \; \{ r_b(n_x) \} \in R_b \; \text{for} \; b = 1 \rightarrow \infty \)
| --- | --- |
| \( L1.1 \) | \( q\{ U' \} = \) total number of elements in universal set \( \{ U' \} \) \( q\{ r_y \} = \) total number of elements in set \( \{ r_y \} \)
| \( \text{Proof:} \) | By definition |
| \( L1.2 \) | **Base Case** \( q\{ r_1 \} = \frac{1}{2^1} q\{ U' \} \)
| \( \text{Proof:} \) | By definition 2 \( \{ \text{even}_{r_y} \} = \{ r_b \} \) \& \( \{ \text{odd}_{r_y} \} = \{ r_{y+1} \} \) \( q\{ r_{y+1} \} = q\{ \text{odd}_{r_y} \} = q\{ \text{odd}_{r_{y+1}} \} + q\{ \text{even}_{r_{y+1}} \} \)
| | Quantity of odd numbers are equal to quantity of even numbers \( q\{ \text{odd}_{r_y} \} = q\{ \text{even}_{r_y} \} \) \( q\{ \text{even}_{r_y} \} = \frac{1}{2} q\{ r_{y+1} \} \) \( q\{ r_{b=1} \} = \frac{1}{2} q\{ r_{b=0} \} = \frac{1}{2^1} q\{ U' \} \)
| \( L1.3 \) | \( q\{ r_{b=2} \} = \frac{1}{2^2} q\{ U' \} \)
| \( \text{Proof:} \) | \( q\{ r_{b=1} \} = q\{ r_{y=2} \} + q\{ r_{b=2} \} = 2q\{ r_{b=2} \} \) \( q\{ r_{b=2} \} = \frac{1}{2} q\{ r_{b=1} \} = \frac{1}{2^2} q\{ U' \} \)
| \( L1.4 \) | **Mathematical Induction Assumed case** \( q\{ r_{b=x} \} = \frac{1}{2^{b=x}} q\{ U' \} \mid x \in \mathbb{Z}^+ \)
| \( \text{Proof:} \) | Assumed for induction \( q\{ r_{b=(x+1)} \} = \frac{1}{2^{b=(x+1)}} q\{ U' \} \mid x \in \mathbb{Z}^+ \)
Proof: \[ q(r_{b=x}) = q\{r_{y=(x+1)}\} + q\{r_{b=(x+1)}\} = 2q\{r_{b=(x+1)}\} \]
\[ q\{r_{b=(x+1)}\} = \frac{1}{2} q\{r_{b=x}\} = \frac{1}{2^{b=(x+1)}} q(U') \]

| \( b \to \infty \) | \[ q(U') = \sum_{b=1}^{\infty} q(r_b) \]
|---|---|

Proof:
\[ \sum_{b=1}^{\infty} q(r_b) = q(r_{b=1}) + q(r_{b=2}) + q(r_{b=3}) + q(r_{b=4}) + q(r_{b=5}) ... \]

Using L1.6
\[ \sum_{b=1}^{\infty} q(r_b) = q(U') \left( \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} \ldots \right) = q(U')(1) = q(U') \]

L1.0
\[ \forall n_x \in \{U'\} \exists \{r_b(n_x)\} \in R_b \text{ for } b = 1 \to \infty \]

Proof: By L1.6

| **Theorem 1.0:** for all values of \( n_x \), the \( r_b \) has well defined values that depend upon the parity of \( b \) |

\[ \iff b = \text{even}, r_b = \frac{3n_x - 2^b + 1}{3.2^b} \land \iff b = \text{odd}, r_b = \frac{3n_x + 2^b + 1}{3.2^b} | 3n_x + 1 = n_s, 2^b \text{ & } n_x, n_s = 2k - 1 \& a, k, b \in \mathbb{Z}^+ \]

Proof:

<table>
<thead>
<tr>
<th><strong>T1.0</strong></th>
<th><strong>Condition</strong></th>
</tr>
</thead>
</table>
| \[ \iff b = \text{even}, r_b = \frac{3n_x - 2^b + 1}{3.2^b} \land \iff b = \text{odd}, r_b = \frac{3n_x + 2^b + 1}{3.2^b} | 3n_x + 1 = n_s, 2^b \text{ & } n_x, n_s = 2k - 1 \& a, k, b \in \mathbb{Z}^+ \]
| **T1.1** | **IF** |

\[ r_b = \frac{r_{b-1} \pm 1}{2} \]

Proof: By definition D2

<table>
<thead>
<tr>
<th><strong>T1.2.1</strong></th>
<th><strong>If b=even b=2</strong></th>
</tr>
</thead>
</table>

\[ r_2 = \frac{3n_x - 2^2 + 1}{3.2^2} \]

Proof:
\[ r_2 = \frac{r_{2-1} - 1}{2^1} = \frac{\frac{n_x+1}{2^1} - 1}{2^1} = \frac{n_x - 3}{3.2^2} = \frac{3n_x - 3}{3.2^2} = \frac{3n_x - 2^2 + 1}{3.2^2} \]

| **T1.2.2** | **b = 2a | \in \mathbb{Z}^+** |
|---|---|

\[ r_{2a} = \frac{3n_x - 2^{2a} + 1}{3.2^{2a}} \]
\begin{table}
\renewcommand{\arraystretch}{1.3}
\begin{tabular}{|c|c|}
\hline
Proof: & Assumed for induction \\
\hline
T1.2.3 & \( b = 2a + 2 | \) \\
& \( a \in \mathbb{Z}^+ \) \\
\hline
& \( r_{2a+2} = \frac{3n_x - 2^{2a+2} + 1}{3 \cdot 2^{2a+2}} \) \\
\hline
\end{tabular}
\end{table}

\begin{proof}
Using \textbf{T1.2.2} \\
\[ r_{2a+2} = \frac{r_{2a+2-1} - 1}{2^1} \implies r_{2a+2} = \frac{3n_x - 2^{2a} + 1}{3 \cdot 2^1} - 1 \\
\]
\[ r_{2a+2} = \frac{3n_x - 2^{2a+2} + 1}{3 \cdot 2^{2a+2}} \text{ (by algebra)} \]
\end{proof}

\begin{tabular}{|c|c|}
\hline
T1.2.4 & Then, \\
\hline
& \( r_b = \frac{3n_x - 2^b + 1}{3 \cdot 2^b} \) \\
\hline
\end{tabular}

\begin{proof}
Using mathematical induction in \textbf{T1.2.2} \& \textbf{T1.2.3} and substituting \( 2a \) with \( b \) \\
\end{proof}

\begin{tabular}{|c|c|c|}
\hline
T1.3.1 & If, \( b=\text{odd} \) \\
& Base case \( b=1 \) \\
\hline
& \( r_1 = \frac{3n_x + 2^1 + 1}{3 \cdot 2^1} \) \\
\hline
\end{tabular}

\begin{proof}
Using definition \textbf{D2.1} \\
\end{proof}

\begin{tabular}{|c|c|c|}
\hline
T1.3.2 & \( b = 2a + 1 | \) \\
& \( a \in \mathbb{Z}^+ \) \\
\hline
& \( r_{2a+1} = \frac{r_{2a} + 1}{2^1} \) \\
\hline
\end{tabular}

\begin{proof}
Using \textbf{T1.2.2} \\
\[ r_{2a+1} = \frac{r_{2a} + 1}{2^1} \implies r_{2a+1} = \frac{3n_x - 2^{2a} + 1}{3 \cdot 2^1} + 1 \\
\]
\[ r_{2a+1} = \frac{3n_x + 2^{2a+1} + 1}{3 \cdot 2^{2a+1}} \text{ (by algebra)} \]
\end{proof}

\begin{tabular}{|c|c|c|}
\hline
T1.0 & THEN \\
\hline
& \( if \ b = \text{even}, r_b = \frac{3n_x - 2^b + 1}{3 \cdot 2^b} \) \& \( \text{if} \ b = \text{odd}, r_b = \frac{3n_x + 2^b + 1}{3 \cdot 2^b} \) \\
\hline
\end{tabular}

\begin{proof}
By \textbf{T1.2.4} \& \textbf{T1.3.3} \\
\end{proof}

Upon calculating based on Theorem 1, for values in \( r_b \), we get;
\[ r_1 = \frac{n_x + 1}{2^1}, r_2 = \frac{n_x - 1}{2^2}, r_3 = \frac{n_x + 3}{2^3}, r_4 = \frac{n_x - 5}{2^4}, r_5 = \frac{n_x + 11}{2^5}, r_6 = \frac{n_x - 21}{2^6} \ldots \]

\textbf{Theorem 2.0:}
\[ \forall (r_b + r_{b-1}) = n_s \mid r_b \in n_x \& 3n_x + 1 = n_x, 2^b \& n_x, n_x = 2k - 1 \& k, b \in \mathbb{Z}^+ \]

We establish the operation \"\( r_b + r_{b-1} \)\" is identical to application of \( 3n+1 \) (on odd) followed by \( n/2 \) (on even) till we get odd.

**Proof:**

<table>
<thead>
<tr>
<th>T2.0</th>
<th>[ \forall (r_b + r_{b-1}) = n_s \mid r_b \in n_x &amp; 3n_x + 1 = n_x, 2^b &amp; n_x, n_x = 2k - 1 &amp; k, b \in \mathbb{Z}^+ ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>T2.1</td>
<td>IF [ \forall (r_b + r_{b-1}) = n_s \rightarrow \forall (r_{\text{even}} + r_{b-1}) = n_s \rightarrow \forall (r_{\text{odd}} + r_{b-1}) = n_s ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Since, parity of ( b ) seems to play a role, we put in the effort to study each case separately.</td>
</tr>
<tr>
<td>T2.2.1</td>
<td>If, Case 1: ( b = \text{even} = 2^j \mid j \in \mathbb{Z}^+ ) [ r_{2j} = \frac{3n_x - 2^j + 1}{3.2^j} &amp; r_{2k-1} = \frac{3n_x + 2^{j-1} + 1}{3.2^{j-1}} ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Using <strong>Theorem 1</strong></td>
</tr>
<tr>
<td>T2.2.2</td>
<td>[ r_{2j} + r_{2j-1} = \frac{(3n_x + 1)}{2^j} ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Using Algebra</td>
</tr>
<tr>
<td>T2.2.3</td>
<td>Then [ \forall (r_{\text{even}} + r_{b-1}) = n_s ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Substitute ( 2j ) with ( \text{even} ) &amp; ( 2j-1 ) with ( \text{b-1} ) in <strong>T2.2.2</strong> and equate with <strong>D0.2</strong></td>
</tr>
<tr>
<td>T2.3.1</td>
<td>If, Case 2: ( b = \text{odd} = 2^j + 1 \mid j \in \mathbb{Z}^+ ) [ r_b = \frac{3n_x + 2^{j+1} + 1}{3.2^{j+1}} &amp; r_{2j+1} = \frac{3n_x - 2^{j+1-1} + 1}{3.2^{j+1-1}} ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Using <strong>Theorem 1</strong></td>
</tr>
<tr>
<td>T2.3.2</td>
<td>[ r_{2j+1} + r_{2j+1-1} = \frac{(3n_x + 1)}{2^{j+1}} ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Using Algebra</td>
</tr>
<tr>
<td>T2.3.3</td>
<td>Then, [ \forall (r_{\text{odd}} + r_{b-1}) = n_s ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Substitute ( 2j+1 ) with ( \text{odd} ) &amp; ( 2j+1-1 ) with ( \text{b-1} ) in <strong>T2.3.2</strong> and equate with <strong>D0.2</strong></td>
</tr>
<tr>
<td>T2.0</td>
<td>THEN, [ \forall (r_b + r_{b-1}) = n_s ]</td>
</tr>
<tr>
<td>Proof:</td>
<td>Using <strong>T2.2.3</strong> &amp; <strong>T2.3.3</strong></td>
</tr>
</tbody>
</table>
Let \( g(n_x) = r_b(n_x) + r_{b-1}(n_x) \)

Then, \((r_b + r_{b-1}) = n_s \Rightarrow g(n_x) = f(n_x)\)

Thus, we create an identical function to the collatz transformations.

**Theorem 2** can also be re-written in an interesting form: sum of two continued fractions (using definition \( r_b \) and \( r_{b-1} \)) of for all the possible positive integer values of \( b \):

\[
(r_b + r_{b-1}) = n_s \Rightarrow \lim_{b\to\infty} \left( \frac{n_x + 1}{2} - 1 + \frac{1}{2} \right) = \frac{3n_x + 1}{2b}
\]

The continued fraction expression is pretty simple to prove. One may reach the same conclusion without going through **Theorem 1**

Now, we explore if there exists some element \( n_x \), which under defined collatz transformations becomes infinity.

\[ n_x \equiv n_x \mid n_x = \infty \]

**Corollary 1.0**: We identify the condition when any given element after undergoing transformation will definitely increase.

if \( b = 1, \forall n_s > \forall n_x \) & if \( b > 1, \forall n_s < \forall n_x \) \( 3n_x + 1 = n_s, 2^b \) & \( n_x, n_s = 2k - 1 \) & \( b \in \mathbb{Z}^+ \)

increase/decrease: condition for any transformation = \( \begin{cases} \text{for } b = 1, & \forall n_s > \forall n_x \mid n_s > 1 \\ \text{for } b > 1, & \forall n_s < \forall n_x \mid n_s > 1 \end{cases} \)

**Proof:**

| C1.0 | Condition | \[
\begin{align*}
(r_b + r_{b-1}) &= n_s \\
\lim_{b\to\infty} \left( \frac{n_x + 1}{2} - 1 + \frac{1}{2} \right) &= \frac{3n_x + 1}{2b}
\end{align*}
\]
<table>
<thead>
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<tbody>
<tr>
<td>C1.1</td>
<td>If</td>
<td>( r_b + r_{b-1} = n_s )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By <strong>Theorem 2</strong></td>
<td></td>
</tr>
<tr>
<td>C1.2.1</td>
<td>If Case 1: ( b = 1 )</td>
<td>( r_1 + r_0 = n_s )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By definition <strong>D1</strong>: ( r_0 = n_x )</td>
<td></td>
</tr>
<tr>
<td>C1.2.2</td>
<td>Then</td>
<td>( n_s &gt; n_x )</td>
</tr>
<tr>
<td>Proof:</td>
<td>( r_1 + r_0 = \frac{n_x + 1}{2} + n_x &gt; n_x \Rightarrow n_s &gt; n_x )</td>
<td></td>
</tr>
<tr>
<td>C1.3.1</td>
<td>If Case 2: ( b = 2 )</td>
<td>( n_s = r_2 + r_1 )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By <strong>Theorem 2</strong></td>
<td></td>
</tr>
</tbody>
</table>
\[
C1.3.2 \quad \begin{aligned}
\text{Proof:} \\
ns &= \frac{n_x - 1}{2^2} + \frac{n_x + 1}{2} = \frac{3n_x + 1}{4}
\end{aligned}
\]

\[
C1.3.2.1 \quad \text{If } n_x = 1 \quad \text{Then} \\
n_s = n_x
\]

\[
\text{Proof:} \\
3n_x + 1 = n_x \cdot 2^2 \text{ if } n_x = 1 \text{ then } n_s = 1
\]

\[
C1.3.2.2 \quad \text{If } n_x > 1 \quad \text{Then} \\
n_x > n_s
\]

\[
\text{Proof:} \\
3n_x + 1 = n_x \cdot 2^2 \text{ and } n_x = 1 + n' \Rightarrow n_s = \frac{3 + 3n'}{4} = 1 + \frac{3n'}{4} \\
n' = 2k' \text{ and } k' \in \mathbb{Z}^+
\]

\[
C1.4.1 \quad \text{If Case 3: } b \geq 3 \\
3n_x + 1 = n_s \cdot 2^{2^3}
\]

\[
\text{Proof:} \quad \text{By definition } D0.2: \text{ because } b \geq 3
\]

\[
C1.4.2 \quad \text{Then} \\
n_x > n_s
\]

\[
\text{Proof:} \quad \text{if } n_s > n_x, \text{ then } n_s = n_x + j \mid j \in \mathbb{Z}^+ \\
3n_x + 1 = n_x \cdot 2^{2^3} \Rightarrow 3n_x + 1 = (n_x + j) \cdot 2^{2^3} \\
1 - j \cdot 2^{2^3} = n_x \cdot (2^{2^3} - 3) \\
\text{for } j \geq 1, \text{ left hand side is negative, implying } n_x \text{ is negative, implying } n_x \notin \mathbb{Z}^+. \text{ This is false.}
\]

\[
C1.5 \quad n_s < n_x \text{ with } b > 2
\]

\[
\text{Proof:} \quad \text{By } C1.3.2.2 \text{ and } C1.4.2
\]

\[
C1.0 \quad \text{THEN} \\
\text{if } b = 1, \forall n_s > \forall n_x \wedge \text{if } b > 1, \forall n_s < \forall n_x
\]

\[
\text{Proof:} \quad \text{By } C1.2.2 \text{ and } C1.5
\]

We consider applying transformation on some number multiple times such that it will definitely increase in all the applied transformations. Thus, the sub condition as per Corollary 1; \(n_s\) is always greater than \(n_x\) during all of these multiple transformations needs to be probed.

**Corollary 2.0:**

\[
r_1(s) = \frac{3}{2} r_1(x) \mid r_1(x) \text{ is } r_b \text{ for } n_x, r_1(s) \text{ is } r_b \text{ for } n_s \text{ and } b = 1, 3n_x + 1 = n_s \cdot 2^b \text{ and } n_x, n_s \\
= 2k - 1 \& k \in \mathbb{Z}^+
\]
$r_1(s)$ is the value of $r_b$ for $n_s$. Similarly, $r_1(x)$ is the value of $r_b$ for $n_x$. When we repeatedly apply transformation: we always label the element that we apply transformation upon as $n_x$, the transformed element is always labelled as $n_s$.

Example: Say, $n_x = 9$ then $n_s = 7$, now apply transformation on 7, so 7 becomes $n_x = 11$, $r_b(11) = 6_1$ then $n_s = 17$, $r_b(11) = 4_2$... and so on.

Proof:

| C2.0 | Condition | \( r_1(s) = \frac{3}{2} r_1(x) | \) $r_1(x)$ is $r_b$ for $n_x$, $r_1(s)$ is $r_b$ for $n_s$ & $b = 1, 3 n_x + 1 = n_s. 2^b$ & $n_x, n_s = 2k - 1$ & $k \in \mathbb{Z}^+$
| C2.1 | IF | $r_b$ for $n_x = r_b(x)$ & $r_b$ for $n_s = r_b(s)$ | $3 n_x + 1 = n_s. 2^b$
| Proof: | By definition |
| C2.2 | | $n_x = 2 r_1(x) - 1$ & $n_s = 2 r_1(s) - 1$
| Proof: | By algebra on definition of $r_1$
| \( r_1(x) = \frac{n_x + 1}{2} \) & \( r_1(s) = \frac{n_s + 1}{2} \)
| C2.3 | | $r_1(x) = n_s - n_x$
| Proof: | $r_1 + r_0 = n_s \Rightarrow r_1(x) + n_x = n_s$
| C2.4 | | $r_1(x) = (2 r_1(s) - 1) - (2 r_1(x) - 1)$
| Proof: | Using substitution of $n_s$ & $n_x$ from C2.2 in C2.3
| C2.0 | THEN | $r_1(s) = \frac{3}{2} r_1(x)$
| Proof: | Using algebra on C2.4

Corollary 1 implies for $n$ greater than 1; $b$ greater than 1 is the only condition for increase during transformations. Corollary 2 implies for $n$ greater than 1, an element can grow finite number of times, as any number $(3 r_1(x))$ that is divided by 2 will eventually result; an odd number. Thus, after some finite number of transformations, the element $n$ will definitely decrease because $b$ happens to be greater than 1. We do not conclude that $n$ reaches a value less than itself, we only conclude that for all there does not exist $n$ that can grow continuously infinite number of times.

Thus, the transformational process, $n$ continuously grows and transforms to infinity; that is described by the following equation
\[ n_{u_1} \equiv n_{u_2} \equiv n_{u_3} \equiv n_{u_4} \equiv n_{u_5} \ldots \equiv n_{u_\infty} \mid n_{u_1} < n_{u_2} < n_{u_3} < n_{u_4} < n_{u_5} < \ldots < n_{u_\infty} \text{ where } n_{u_\infty} = \infty \land r_b = r_1 \forall n_{u_1}, n_{u_2}, n_{u_3}, n_{u_4}, n_{u_5} \ldots \]

is false and invalid. One concludes that continuous increase to infinity is not possible.

**Notation:**

\(<\neq>\) is used to describe relationship between 2 elements; one element may be greater than or smaller than the other element, but both the elements are not equal.

Note: It would seem improper to use "\(<\neq>\)" notation describing any series. However, It is okay to use such notation in the context of our analysis; we don't know if when they are larger or smaller to adjacent element, all we know is none of the elements in the series can be equal to any other element. We consider every element during the transformational process to be not equal to any other element, as that would imply, the elements loops, thus \(n\) cannot transform to infinity.

Consider the transformational process described as:

\[ n_{u_1} \equiv n_{u_2} \equiv n_{u_3} \equiv n_{u_4} \equiv n_{u_5} \ldots \equiv n_{u_\infty} \mid n_{u_1} <\neq n_{u_2} <\neq n_{u_3} <\neq n_{u_4} <\neq n_{u_5} \ldots \]

The transformation from \(n_{u_1}\) to \(n_{u_\infty}\) with discontinuous growth may be described by the above equation. So, it is still possible for some number to grow to infinity at a relatively slower rate.

Hence, the question of discontinuous growth to infinity remains valid and thus open.

**Proposition 1.0:**

\[ n_x \neq \infty \mid 3n_x + 1 = n_s, 2^b \land n_x, n_s = 2k - 1 \& k, b \in \mathbb{Z}^+ \]

We prove proposition by contradiction.

**Proof:**

<table>
<thead>
<tr>
<th>P1.0</th>
<th>Condition</th>
<th>( n \neq \infty \mid 3n_x + 1 = n_s, 2^b \land n_x, n_s = 2k - 1 &amp; k, b \in \mathbb{Z}^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1.1</td>
<td>IF</td>
<td>( n_s \equiv \infty \mid n_x, n_s = 2k + 1 &amp; k, b \in \mathbb{Z}^+ )</td>
</tr>
<tr>
<td>Proof:</td>
<td>Assumed to establish contradiction</td>
<td></td>
</tr>
<tr>
<td>P1.2</td>
<td></td>
<td>( 3r_b \pm 1 = n_s )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By applying definition 2 on Theorem 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( r_b + r_{b-1} = 3r_b \pm 1 )</td>
</tr>
<tr>
<td>P1.3</td>
<td></td>
<td>( r_b \notin {U} )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By P1.1 &amp; P1.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 3r_b \pm 1 = \infty \Rightarrow r_b = \frac{\infty \pm 1}{3} )</td>
</tr>
<tr>
<td>P1.4</td>
<td></td>
<td>( \forall \ r_b \in {U} )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By Definition 2</td>
<td></td>
</tr>
</tbody>
</table>
∀ r, b ∈ ℤ⁺ & 𝑍⁺ ∈ {𝑈} ⇒ r_b ∈ {𝑈}

P1.0 THEN 

Proof: By contradiction in P1.3 & P1.4

Thus, no number can transform to infinity.

Corollary 3.0

\( n_s \not\equiv 0 \mod 3 \mid 3n_x + 1 = n_s2^b, n_x, n_s = 2k - 1 \& k, b \in ℤ^+ \)

Despite the argument being trivial in nature, we will still prove it as it is instrumental in our study of
the conjecture.

Proof:

<table>
<thead>
<tr>
<th>C3.0</th>
<th>( n_s \not\equiv 0 \mod 3 \mid 3n_x + 1 = n_s2^b, n_x, n_s = 2k - 1 &amp; k, b \in ℤ^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C3.1</td>
<td>IF ( 3n_x + 1 = n_s2^b, n_x, n_s = 2k - 1 &amp; k, b \in ℤ^+ )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By definition</td>
</tr>
<tr>
<td>C3.2</td>
<td>( n_s \equiv 0 \mod 3 \Rightarrow n_s = 3j \mid j \in ℤ^+ )</td>
</tr>
<tr>
<td>Proof:</td>
<td>Assumed to establish contradiction</td>
</tr>
<tr>
<td>C3.3</td>
<td>( n_x \not\in ℤ^+ )</td>
</tr>
<tr>
<td>Proof:</td>
<td>( n_x = \frac{3j2^b - 1}{3} = j2^b - \frac{1}{3} \Rightarrow n_x \not\in ℤ^+ )</td>
</tr>
<tr>
<td>C3.0</td>
<td>Then ( n_s \not\equiv 0 \mod 3 )</td>
</tr>
<tr>
<td>Proof:</td>
<td>By contradiction in C3.1 &amp; C3.3</td>
</tr>
</tbody>
</table>

Corollary 4.0

if \( n_1 \equiv 1 \mod 3 \& n_2 \equiv 1 \mod 3 \), then \( n_3 \equiv 1 \mod 3 \)

if \( n_1 \equiv 2 \mod 3 \& n_2 \equiv 2 \mod 3 \), then \( n_3 \equiv 1 \mod 3 \)

if \( n_1 \equiv 1 \mod 3 \& n_2 \equiv 2 \mod 3 \), then \( n_3 \equiv 2 \mod 3 \) \mid n_1, n_2, n_3, k_1, k_2, k_3 \in ℤ

The above arguments may be written in multiplicative format;

\( 2 \mod 3 \ast 2 \mod 3 \equiv 1 \mod 3 \land 1 \mod 3 \ast 1 \mod 3 \equiv 1 \mod 3 \land 1 \mod 3 \ast 2 \mod 3 \equiv 2 \mod 3 \)

Proof:

| C4 | \( \Leftarrow n_1 \equiv 1 \mod 3 \& n_2 \equiv 1 \mod 3, n_3 \equiv 1 \mod 3 \) \land   |
|    | \( \Leftarrow n_1 \equiv 2 \mod 3 \& n_2 \equiv 2 \mod 3, n_3 \equiv 1 \mod 3 \) \land |
|    | \( \Leftarrow n_1 \equiv 1 \mod 3 \& n_2 \equiv 2 \mod 3, n_3 \equiv 2 \mod 3 \) |
**Proof:**

- **C4.0**
  - **THEN**
  - \( \iff n_1 \equiv 1 \mod 3 \land n_2 \equiv 1 \mod 3, n_3 \equiv 1 \mod 3 \)
  - \( \iff n_1 \equiv 2 \mod 3 \land n_2 \equiv 2 \mod 3, n_3 \equiv 2 \mod 3 \)
  - \( \iff n_1 \equiv 1 \mod 3 \land n_2 \equiv 2 \mod 3, n_3 \equiv 2 \mod 3 \)
  - **Proof:**
    - By definition

**Theorem 3.0:** There is a well-defined relationship between \( n_s \) *modulo* 3 & parity of \( b \) in \( 2^b \). One can determine \( n_s \) *modulo* 3 by the parity of \( b \) and vice versa. The relationship is independent of \( n_x \).

- if \( n_s \equiv 2 \mod 3 \), then \( b = \text{odd} \) & if \( n_s \equiv 1 \mod 3 \), then \( b = \text{even} \), \( |3n_x + 1 = n_s \cdot 2^b \land n_s \in \mathbb{Z}^+ \)

**Proof:**

- **T3.0**
  - \( \iff n_s \equiv 2 \mod 3, b = \text{odd} \land n_s \equiv 1 \mod 3, b = \text{even} \land 3n_x + 1 = n_s \cdot 2^b \land n_s \in \mathbb{Z}^+ \)

**T3.1**

- **IF**
  - **T3.0**
    - \( \neg b = \text{odd} \Rightarrow b = \text{even} \)
    - \( n_s \equiv 2 \mod 3 \) is false
  - \( \neg b = \text{even} \Rightarrow b = \text{odd} \)
    - \( n_s \equiv 1 \mod 3 \) is false
  - **Proof:**
    - *Modus Tollens*

**T3.2.1**

- If, Case 1: \( b = \text{even} \& 3n_x + 1 = n_s \cdot 2^{\text{even}} \)
| 
|  
| $n_s = 3j + 2$ | $n_s \equiv 2 \text{mod}3 = 3j + 2$ |

**Proof:**

By definition of $n_s$ in terms of $3n+1$

| 
|  
| T3.2.2 | $3n_x + 1 = (3j + 2).2^{\text{beven}}$ |

**Proof:**

By definition of $n_s$ in terms of $n_s \mod 3$

| 
|  
| T3.2.3 | $3n_x + 1 = 3j.2^{\text{beven}} + 2^{\text{bodd}}$ |

**Proof:**

Since $2.2^{\text{bodd}} = 2^{\text{beven}}$

| 
|  
| T3.2.4 | Then 

$\neg b = \text{odd} \Rightarrow b = \text{even}$, $\neg n_s \equiv 2 \text{mod}3$

Thus,

$b = \text{odd}, n_s \equiv 2 \text{mod}3$ |

**Proof:**

Using Corollary 4.0 & T3.2.3

$3n_x + 1 = 3j.2^{\text{beven}} + 2^{\text{bodd}} \Rightarrow$

$1 \text{mod}3 \equiv 0 \text{mod}3 * 1 \text{mod}3 + 2 \text{mod}3$  

$1 \text{mod}3 \equiv 2 \text{mod}3$ is $\Leftrightarrow$, thus false

| 
|  
| T3.3.1 | If. Case 2: 

$b = \text{odd} \& n_s = 3j + 1$

$3n_x + 1 = n_s * 2^{\text{bodd}}$  

$n_s \equiv 1 \text{mod}3 = 3j + 1$ |

**Proof:**

By definition of $n_s$ in terms of $3n+1$

| 
|  
| T3.3.2 | $3n_x + 1 = (3j + 1).2^{\text{bodd}}$ |

**Proof:**

By definition of $n_s$ in terms of $n_s \mod 3$

| 
|  
| T3.3.3 | $3n_x + 1 = 3j.2^{\text{bodd}} + 2^{\text{bodd}}$ |

**Proof:**

Since $1.2^{\text{bodd}} = 2^{\text{bodd}}$

| 
|  
| T3.3.4 | Then 

$\neg b = \text{odd} \Rightarrow b = \text{even}$, $\neg n_s \equiv 1 \text{mod}3$

Thus,

$b = \text{even}, n_s \equiv 1 \text{mod}3$ |

**Proof:**

Using Corollary 4.0 & T3.3.3

$3n_x + 1 = 3j.2^{\text{bodd}} + 2^{\text{bodd}} \Rightarrow$

$1 \text{mod}3 \equiv 0 \text{mod}3 * 2 \text{mod}3 + 2 \text{mod}3$  

$1 \text{mod}3 \equiv 2 \text{mod}3$ is $\Leftrightarrow$, thus false

| 
|  
| T3.0 | THEN 

$\neg b = \text{odd}$, $\neg n_s \equiv 2 \text{mod}3 \land \neg b = \text{even}$, $\neg n_s \equiv 1 \text{mod}3$

$b = \text{even}, n_s \equiv 1 \text{mod}3 \land b = \text{odd}, n_s \equiv 2 \text{mod}3$ |

**Proof**

By T3.2.4 & T3.3.4

**∎**
Definition 3.0 \( \{n_x\} \): The set \( n_x \), is a set that contains all the possible values of \( n_x \) that would satisfy definition 1.0;

\[
n_x \equiv n_x' \quad 3n_x + 1 = n_x \cdot 2^b \implies n_x = \frac{n_x' \cdot 2^b - 1}{3} \quad \text{for all valid values of } b
\]

The set \( \{n_x\} \) is infinitely large with values represented by;

\[
\left\{ \frac{n_x' \cdot 2^b - 1}{3}, \frac{n_x' \cdot 2^{b+2} - 1}{3}, \frac{n_x' \cdot 2^{b+4} - 1}{3}, \frac{n_x' \cdot 2^{b+6} - 1}{3}, \ldots \right\}
\]

if \( n_s \cong 2 \mod 3 \), \( \beta = 1 \) & if \( n_s \cong 1 \mod 3 \), \( \beta = 2 \) & \( z \in \mathbb{Z}^+ \)

All the above give the same result \( n_s \) upon application of \( n/2 \).

Note: We have only even numbers and not odd numbers being added to the exponent of \( 2^\beta \) in the above set representation because Theorem 3 dictates; the parity of \( b \) has to be the same, if we happen to add odd number to the exponent of \( 2^\beta \), then the parity of \( b \) changes, thus we would not get any valid solution for \( n_x \).

Theorem 4.0: All elements of \( \{n_x\} \) can be expressed in the form of its adjacent element.

\[
n_{x1+(z+1)} = 4n_{x1+z} + 1 | \\
\{n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3}, n_{x1+4}, n_{x1+5}, n_{x614} \ldots n_{x1+z}, n_{x1+(z+1)}\} \in \{n_x\} & \\
3n_x + 1 = n_s' \cdot 2^b & n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3} \ldots, n_x, n_s = 2k - 1 \& k, b \in \mathbb{Z}^+
\]

Note: The notation \( n_{x1+1} \) instead of \( n_{x2} \), would seem a bit strange.

There is a method to the madness;

\( n_x \equiv n_s \implies n_x = \frac{n_s \cdot 2^{b-1}}{3} \) with \( b=1 \) for \( n_s \equiv 2 \mod 3 \) or \( b=2 \) for \( n_s \equiv 1 \mod 3 \)

\( n_{x1} \): We refer this as base case. It is the first/smallest solution such that

\[
n_{x1} = \frac{n_s' \cdot 2^1 - 1}{3} | n_s \equiv 2 \mod 3 \lor n_{x1} = \frac{n_s' \cdot 2^2 - 1}{3} | n_s \equiv 1 \mod 3
\]

Since, we write exponential of 2 in the form \( 2^{b+2z} (z \in \mathbb{Z}^+) \)

\( n_{x1+1}, n_{x1+2}, n_{x1+3} \ldots \) represent:

\[
n_{x1+1} = \frac{n_s' \cdot 2^{1+2} - 1}{3} | n_s \equiv 2 \mod 3 \lor n_{x1+1} = \frac{n_s' \cdot 2^{b+2} - 1}{3} | n_s \equiv 1 \mod 3
\]

\[
n_{x1+2} = \frac{n_s' \cdot 2^{1+4} - 1}{3} | n_s \equiv 2 \mod 3 \lor n_{x1+2} = \frac{n_s' \cdot 2^{b+4} - 1}{3} | n_s \equiv 1 \mod 3
\]

\[
n_{x1+3} = \frac{n_s' \cdot 2^{1+6} - 1}{3} | n_s \equiv 2 \mod 3 \lor n_{x1+3} = \frac{n_s' \cdot 2^{b+6} - 1}{3} | n_s \equiv 1 \mod 3
\]

The notation makes sense as the \( z \) in the expression \( 2^{b+2z} \) is referred to as \( n_{x1+z} \). It creates a simple direct link to the additional component of exponent (2z) in expression; \( 2^{b+2z} \). The notation also helps identifying parity of \( n_s \) modulo3 which will be evident as we study further.
**Proof:**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>T4.0</td>
<td>$n_{x1+(z+1)} = 4n_{x1+z} + 1$</td>
</tr>
<tr>
<td>T4.1</td>
<td>IF ${n_s, 2^\beta, n_s, 2^{\beta+2}, n_s, 2^{\beta+4}, n_s, 2^{\beta+6}, n_s, 2^{\beta+8}, n_s, 2^{\beta+10}, ... } \equiv n_s$</td>
</tr>
<tr>
<td>Proof:</td>
<td>The said set transforms to $n_s$ upon application of $n/2$. Parity of $b$ has been maintained same, complying with theorem 3.</td>
</tr>
<tr>
<td>T4.2.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>${n_x} = \left[ \frac{n_s, 2^\beta - 1}{3}, \frac{n_s, 2^{\beta+2} - 1}{3}, \frac{n_s, 2^{\beta+4} - 1}{3}, \frac{n_s, 2^{\beta+6} - 1}{3}, \frac{n_s, 2^{\beta+8} - 1}{3}, \frac{n_s, 2^{\beta+10} - 1}{3}, ... \right]$</td>
</tr>
<tr>
<td>Proof:</td>
<td>By definition</td>
</tr>
<tr>
<td>T4.2.2</td>
<td>Let $[n_x] = [n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3}, n_{x1+4}, n_{x1+5}, n_{x1+6}, n_{x1+7} ... n_{x1+8} ...]$</td>
</tr>
<tr>
<td>Proof:</td>
<td>By definition</td>
</tr>
<tr>
<td>T4.3</td>
<td>Base case $n_{x1+1} = 4n_{x1} + 1$</td>
</tr>
<tr>
<td>Proof:</td>
<td>By substitution of $n_s, 2^{\beta+2}$ $n_{x1} = \frac{n_s, 2^\beta - 1}{3} \Rightarrow n_s, 2^{\beta+2} = 2^2(3n_{x1} + 1)$ $n_{x1+1} = \frac{n_s, 2^{\beta+2} - 1}{3} = \frac{2^2(3n_{x1} + 1) - 1}{3}$ By algebra, we get; $n_{x1+1} = 4n_{x1} + 1$</td>
</tr>
<tr>
<td>T4.4</td>
<td>Mathematical Induction $n_{x1+z} = 4n_{x1+(z-1)} + 1$ $\frac{n_s, 2^{\beta+2(z-1)} - 1}{3}, n_{x1+(z+1)} = \frac{n_s, 2^{\beta+2z} - 1}{3}$</td>
</tr>
<tr>
<td>Proof:</td>
<td>Assumed for induction</td>
</tr>
<tr>
<td>T4.0</td>
<td>THEN, $n_{x1+(z+1)} = 4n_{x1+z} + 1$</td>
</tr>
<tr>
<td>Proof:</td>
<td>$n_{x1+(z+1)} = \frac{n_s, 2^{\beta+2(z+1)} - 1}{3}$ $n_{x1+(z+1)} = \frac{4n_s, 2^{\beta+2(z)} - 1}{3}$ Using $3n_{x1+z} + 1 = n_s, 2^{\beta+2z}$ $n_{x1+(z+1)} = \frac{4(3n_{x1+z} + 1) - 1}{3}$</td>
</tr>
</tbody>
</table>

**Corollary 5.0:** Congruence modulo 3 is well ordered irrespective of the first solution, $0 \mod 3$ is followed by $1 \mod 3$ is followed by $2 \mod 3$ is followed by $0 \mod 3$ is followed by $1 \mod 3$ and so on...
(if \( n_1 \equiv 0 \mod 3 \) then \( n_1 \equiv 1 \mod 3, n_1 \equiv 2 \mod 3, n_1 \equiv 0 \mod 3, \ldots \) \( \land \) (if \( n_1 \equiv 1 \mod 3 \), then \( n_1 \equiv 2 \mod 3, n_1 \equiv 0 \mod 3, \ldots \) \( \land \) (if \( n_1 \equiv 2 \mod 3, n_1 \equiv 0 \mod 3, n_1 \equiv 1 \mod 3, \ldots \)) | \( \{n_1, n_1+1, n_1+2, n_1+3, n_1+4, \ldots \} \in \{n_x\} \) & \( 3n_1 + 1 = n_x, 2^b \land n_x, n_5 = 2k - 1 \) \& \( b, b \in \mathbb{Z}^+\)

**Proof:**

<table>
<thead>
<tr>
<th>C5.0</th>
<th>(if ( n_1 \equiv 0 \mod 3 ), ( n_1+1 \equiv 1 \mod 3, n_1+2 \equiv 2 \mod 3, n_1+3 \equiv 0 \mod 3, \ldots ) ( \land ) (if ( n_1 \equiv 1 \mod 3 ), then ( n_1 \equiv 2 \mod 3, n_1 \equiv 0 \mod 3, \ldots ) ( \land ) (if ( n_1 \equiv 2 \mod 3, n_1 \equiv 0 \mod 3, n_1 \equiv 1 \mod 3, \ldots ))</th>
<th>( {n_1, n_1+1, n_1+2, n_1+3, n_1+4, \ldots } \in {n_x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C5.1</td>
<td>IF</td>
<td>( n_1 \equiv 0 \mod 3 \Rightarrow n_1 = 3</td>
</tr>
<tr>
<td>C5.2.1</td>
<td>Case1: If</td>
<td>( n_1 \equiv 0 \mod 3 \Rightarrow n_1 = 3</td>
</tr>
<tr>
<td>C5.2.2</td>
<td></td>
<td>( n_1+1 \equiv 1 \mod 3 )</td>
</tr>
<tr>
<td>C5.2.3</td>
<td></td>
<td>( n_1+2 \equiv 2 \mod 3 )</td>
</tr>
<tr>
<td>C5.2.4</td>
<td></td>
<td>( n_1+3 \equiv 0 \mod 3 )</td>
</tr>
<tr>
<td>C5.2.5</td>
<td></td>
<td>( n_1+4 \equiv 1 \mod 3 )</td>
</tr>
<tr>
<td>C5.2.6</td>
<td>Then</td>
<td>( n_1+5 \equiv 2 \mod 3, n_1+6 \equiv 1 \mod 3, n_1+7 \equiv 0 \mod 3, n_1+8 \equiv 1 \mod 3, n_1+9 \equiv 2 \mod 3, n_1+10 \equiv 0 \mod 3, \ldots )</td>
</tr>
<tr>
<td>C5.2.7</td>
<td></td>
<td>( n_1+1 \equiv 1 \mod 3, n_1+2 \equiv 2 \mod 3, n_1+3 \equiv 0 \mod 3, n_1+4 \equiv 1 \mod 3, n_1+5 \equiv 2 \mod 3, n_1+6 \equiv 1 \mod 3, n_1+7 \equiv 0 \mod 3, \ldots )</td>
</tr>
<tr>
<td>C5.3.1</td>
<td>Case 2: If</td>
<td>Let ( n_1 \equiv 1 \mod 3 \Rightarrow n_1 = 3</td>
</tr>
</tbody>
</table>
Proof: By definition

\[ n_{x1+1} \equiv 2 \text{ mod } 3, n_{x1+2} \equiv 0 \text{ mod } 3, n_{x1+3} \equiv 1 \text{ mod } 3, n_{x1+4} \equiv 2 \text{ mod } 3, \ldots \]

Proof: By \textbf{C5.2.3}, \textbf{C5.2.4}, \textbf{C5.2.2}

\[ (\text{if } n_{x1} \equiv 0 \text{ mod } 3, n_{x1+1} \equiv 1 \text{ mod } 3, n_{x1+2} \equiv 2 \text{ mod } 3 \ldots ) \land 
(\text{if } n_{x1} \equiv 1 \text{ mod } 3, n_{x1+1} \equiv 2 \text{ mod } 3, n_{x1+2} \equiv 0 \text{ mod } 3 \ldots ) \land 
(\text{if } n_{x1} \equiv 2 \text{ mod } 3, n_{x1+1} \equiv 0 \text{ mod } 3, n_{x1+2} \equiv 1 \text{ mod } 3, \ldots ) \]

Proof: By \textbf{C5.2.7}, \textbf{C5.3.2}, \textbf{C5.4.2}

\[ n_{x1} \equiv 2 \text{ mod } 3 \Rightarrow n_{x1} = 3m + 2 | m \in \mathbb{Z}^+ \]

Classify elements of \{U'\} by using sets \( n_s \mod 9 \) and \( n_s \mod 27 \) definition.
<table>
<thead>
<tr>
<th>Dist1</th>
<th>Dist2</th>
<th>Dist3</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 mod 9</td>
<td>16 mod 27</td>
<td>0 mod 3</td>
</tr>
<tr>
<td>7 mod 9</td>
<td>20 mod 27</td>
<td>2 mod 9</td>
</tr>
<tr>
<td>8 mod 9</td>
<td>8 mod 27</td>
<td>5 mod 9</td>
</tr>
<tr>
<td>8 mod 9</td>
<td>26 mod 27</td>
<td>8 mod 9</td>
</tr>
</tbody>
</table>

Table 2.0: $n_x \mod 9$ for $n_s \mod 9$ & $n_s \mod 27$

Theorem 5.0

All elements in $n_x$ are well ordered and for all $n_s$ and $n_x$ modulo 9 is well distributed.

$$values \ of \ n_x \mod 9 \ \forall \ n_x \mod 9 \ \text{is \ well \ distributed} \ |
3n_x + 1 = n_s, 2b \ \& \ n_s \neq n_x \ \& \ n_x = 2k - 1 \ \& \ j, k, b \in \mathbb{Z}^+,$$

$$q(n_s \equiv 1 \mod 9) = q(n_s \equiv 2 \mod 9) = q(n_s \equiv 4 \mod 9) = q(n_s \equiv 5 \mod 9) = q(n_s \equiv 7 \mod 9) = q(n_s \equiv 8 \mod 9) \ \text{for} \ \{U\}, n_s \neq 0 \mod 3$$

Proof:

| T5. | All elements in $n_x$ are well ordered and for all $n_s$, $n_x$ modulo 9 is well distributed. |
| T5.1 | IF $q(n_s \equiv 1 \mod 9) = q(n_s \equiv 2 \mod 9) = q(n_s \equiv 4 \mod 9)$ |
| | $= q(n_s \equiv 5 \mod 9) = q(n_s \equiv 7 \mod 9)$ |
| | $= q(n_s \equiv 8 \mod 9) \ \text{for} \ \{U\}$ |

Proof:

Based upon the fact that there are always and exactly 3 sets of odd elements modulo 9, between 9(m) and 9(m+1) depending if m is odd or even.

If m is odd, then odd elements that lie in between 9m & 9(m+1) are congruent to 2 mod 9, 4 mod 9 and 8 mod 9

If m is even, then odd elements that lie in between 9m & 9(m+1) are congruent to 1 mod 9, 5 mod 9 and 7 mod 9

T5.2.1

Table 2.0

Proof:

Substitute 1 mod 9 with 1+9j, 2 mod 9 with 2+9j, 4 mod 9 with 4+9j, 5 mod 9 with 5+9j, 7 mod 9 with 7+9j, 8 mod 9 with 8+9j; Find $n_{x1}$ with $\beta = 1$ or 2 as per $n_s$ modulo 3 following theorem 3 | $1 + 9k, 2 + 9j, 4 + 9j, 5 + 9j, 7 + 9j, 8 + 9j = 2k - 1, \ k, j \in \mathbb{Z}^+$
Proof: by \textbf{table 2.0}

T5.2.3

values of \( n \mod 9 \) \& \( s \mod 9 \) is well distributed

Proof: By \textbf{T5.2.2}

T5.0 THEN, All elements in \( n \) are well ordered and for all \( n \) and \( n \) modulo 9 is well distributed.

Proof: By \textbf{C5} and \textbf{T5.2.3}

\textbf{Theorem 5} is based upon modular analysis implying the cyclic nature of transformation from \{\( \tau_t \)} from \{\( \tau_{t+1} \)}. Thus, one may extend the understanding to whole universal set \{\( U' \)} and all the reverse transformations elements can go through.

\textbf{Notation:}

\( \equiv \equiv \) double equivalence implies more than 1 transformation.\( n_s \equiv \equiv n_s \) implies that some number \( n_s \) transforms to \( n_s \) with more than 1 transformation; \( n_x \neq n_s \)

\textbf{Proposition 2:} some number loops to itself

Proposition 2.a: the loop happens with single transformation such that \( n_s \equiv n_s \)

Proposition 2.b: the loop happens with more than one transformation such that \( n_s \equiv \equiv n_s \) such that \( n_x \neq n_s \)

\textbf{Proposition 2.a} \( n_s \equiv n_s \): Case for single transformation loop has trivial solution \( 1 \equiv 1 \) with no other possible solution. \( n_x = n_s \)

\[ 3n_x + 1 = n_x^2 b^b \Rightarrow 3n_s + 1 = n_s^2 b^b \Rightarrow n_s (2^b - 3) = 1 \Rightarrow (2^b - 3) = \frac{1}{n_s} \]

For any value of \( n_s > 1 \), right hand side gives a rational solution and on the right-hand side, no value of \( b \) could dish out rational solution. Thus, no other value of \( n_s \) satisfies the condition \( n_s \equiv n_s \)

\[ n_s \equiv n_s | n_x = n_s = 1 \]

\textbf{Definition 4.0:}

\{\( \tau_0 \)} is an arbitrarily defined ordered set that is similar to \{\( U' \)} such that \{\( U' \)\} = \{1\} \( \cup \) \{\( \tau_0 \)\}
All elements of \( \{ \tau_0 \} \) are zero reverse transformations away from \( \{ U' \} \). The element “1” is excluded as \( n_x \equiv n_s \mid n_x = n_s = 1 \)

\( \tau_t \) is a set that contains all elements that are \( t \) reverse transformations away from its associated element in \( \{ U' \} \)

\( \{ \tau_t \} = \{ \tau_t^{1m3} \} \cup \{ \tau_t^{2m3} \} \cup \{ \tau_t^{0m3} \} \)

\( \{ \tau_t^{1m3} \} \) is a set that contains all the elements that congruent to 1 mod3 and are \( t \) reverse transformations away from its associated element in \( \{ U' \} \)

\( \{ \tau_t^{2m3} \} \) is a set that contains all the elements that congruent to 2 mod3 and are \( t \) reverse transformations away from its associated element in \( \{ U' \} \)

\( \{ \tau_t^{0m3} \} \) is a set that contains all the elements that congruent to 0 mod3 and are \( t \) reverse transformations away from its associated element in \( \{ U' \} \)

Let \( q \) be an element counting function such that \( q(\tau_t^{1m3}) \) represents total number of elements that are 1 mod3 and are \( t \) reverse transformations away from \( \{ U' \} \). Similarly, \( q(\tau_t^{2m3}) \) represents total number of elements that are 2 mod3 that are \( t \) reverse transformations away from \( \{ U' \} \) and \( q(\tau_t^{0m3}) \) represents total number of elements that are 0 mod3 that are \( t \) reverse transformations away from \( \{ U' \} \).

\( q(\tau_t^{0m3}) \) refers to inverse of \( q(\tau_t^{0m3}) \) that is elements that are not 0 mod3.

\[ q(\tau_t^{0m3}) = q(\tau_t^{1m3}) \cup q(\tau_t^{2m3}) \]

All elements that are no congruent to 0 mod3, will have a representation in \( \{ \tau_{t+1} \} \)

\[ q(\tau_{t+1}) = q(\tau_t^{0m3}) = q(\tau_t^{1m3}) \cup q(\tau_t^{2m3}) \]

\[ q(\tau_0) = q(\{ U' \} - [1]) \]

The notation \( q(\{ U' \} - [1]) \) implies; all elements of \( \{ U' \} \) except the element “1”.

**Proposition 2.b** \( n_s \equiv n_s \): We explore the possibility for any number to loop with more than one transformation such that \( n_x \neq n_s \).

**Methodology for checking validity of proposition 2.b**

Loop \( n_s \equiv n_s \) implies that when the number of transformations \( t \to \infty \) one should have a valid value for \( n_s \) and all the possible interim values of \( n_x \) such that no value of \( n_x \) for any given \( n_s \) can be congruent 0 mod3, such that \( n_x \neq n_s \)

\[ n_x \not\equiv 0 \text{ mod3} \]

\[ \Rightarrow \forall n_s: (n_x \equiv 1 \text{ mod3} \lor n_x \equiv 2 \text{ mod3}) \]

**Corollary 3:** \( n_s \) cannot be congruent to 0 mod3 even as \( t \to \infty \). Using elimination of \( n_s \equiv 0 \text{ mod3} \) as the set \( \tau_0 \) is expanded to \( \tau_1 \) expanded to \( \tau_2 \) expanded to \( \tau_3 \ldots \) expanded to \( \tau_{t \to \infty} \), one can test if loop is possible. If there is some element that loops with more than one transformation then said process
of elimination should leave us with some definite value with \( n_s \) and \( n_x \) not being congruent to 0 mod3.

For the set \( \{\tau_0\} \), \( t = 0 \)

\[
\{\tau_0\} = \{\tau_0^{1m3}\} \cup \{\tau_0^{2m3}\} \cup \{\tau_0^{0m3}\}
\]

Using Corollary 5; One third of all elements would be eliminated as they are congruent to 0 mod3.

For deriving \( \{\tau_1\} \) from \( \{\tau_0\} \), continue with reverse transformation for rest non-eliminated elements;

\[
\{\tau_1\} = \{\tau_0^{-0m3}\} = \{\tau_0^{1m3}\} \cup \{\tau_0^{2m3}\}
\]

For the set \( \{\tau_1\} \), \( t = 1 \)

\[
\{\tau_1\} = \{\tau_0^{-0m3}\} = \{\tau_0^{1m3}\} \cup \{\tau_0^{2m3}\} = \{\tau_1^{1m3}\} \cup \{\tau_1^{2m3}\} \cup \{\tau_1^{0m3}\}
\]

Using Corollary 5; One third of all elements would be eliminated as they are congruent to 0 mod3.

For deriving \( \{\tau_2\} \) from \( \{\tau_1\} \), continue with reverse transformation for rest non-eliminated elements; And so on...

Note: For deriving \( \{\tau_t\} \) from \( \{\tau_{t-1}\} \), we do not consider the infinite values of \( n_x \) for any given \( n_s \), we only consider the base solution of \( n_x \) that is \( n_{x1} \) as other solutions like \( n_{x1+1}, n_{x1+2}, \ldots \) would automatically be considered as it already exists in \( \{U'\} \).

Including all the possible values of \( \{n_x\} \), gives us infinite solutions for every element in \( \{U'\} \) and going back just one more step would break our analysis because of the infinites popping up everywhere.

We keep the relationship between \( \{\tau_t\} \) and \( \{\tau_{t-1}\} \) as bijective and invertible to able to keep track of number of elements in every set by avoiding the abyss of infinites. Also, no element is kept out of our study as all the possible solutions of \( \{n_x\} \) are already a part of \( \{U'\} \).

Example:

\[
\{\tau_0\} = \{U'\} = \{1\} \cup \{3, 5, 7, 9, 11, 13, 15, \ldots \}
\]

<table>
<thead>
<tr>
<th>{U'}</th>
<th>{\tau_0}</th>
<th>{\tau_1}</th>
<th>{\tau_2}</th>
<th>{\tau_3}</th>
<th>{\tau_4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Does not exist ( n_x = n_s )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>Does not exist ( n_s \equiv 0 \text{ mod3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>Does not exist ( n_s \equiv 0 \text{ mod3} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>9</td>
<td>Does not exist ( n_s \equiv 0 \text{ mod3} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>Does not exist ( n_s \equiv 0 \text{ mod3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>7</td>
<td>9</td>
<td>Does not exist ( n_s \equiv 0 \text{ mod3} )</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>13</td>
<td>17</td>
<td>11</td>
<td>7</td>
<td>9</td>
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<tr>
<td>15</td>
<td>15</td>
<td>Does not exist ( n_s \equiv 0 \text{ mod3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>17</td>
<td>11</td>
<td>7</td>
<td>9</td>
<td>Does not exist</td>
</tr>
</tbody>
</table>
Table 1.0: deriving \{τ_t\} from \{τ_{t−1}\}

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>19</td>
<td>19</td>
<td>25</td>
<td>33</td>
<td>Does not exist (n_s \equiv 0 \mod 3)</td>
</tr>
<tr>
<td>21</td>
<td>21</td>
<td>15</td>
<td>Does not exist (n_s \equiv 0 \mod 3)</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>25</td>
<td>33</td>
<td>Does not exist (n_s \equiv 0 \mod 3)</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>27</td>
<td>19</td>
<td>25</td>
<td>33</td>
</tr>
</tbody>
</table>

Consider; \(n_s = 11\), \(\{n_x\} = \{7, 29, 117, 469, 1877\ldots\}\)

For \(n_s = 11\), \(\{n_{x1} = 7, n_{x1+1} = 29, n_{x1+2} = 117, n_{x1+3} = 469, n_{x1+4} = 1877\ldots\}\)

At \(T=1\), only consider \(n_s = 7\), other values like 29, 117, 469 etc may be ignored as they are going to be evaluated in their respective rows row.

\[
\forall n_s \not\equiv n_s | \exists n_x \mid 3n_x + 1 = n_s, 2^b \wedge n_s \not\equiv n_x \wedge n_x = 2k - 1 \wedge j, j', k, b \in \mathbb{Z}^+ &
\]

\[
\ldots n_s \equiv n_{(t-1)x} \equiv \ldots \equiv n_{2x} \equiv n_{1x} \equiv n_s \equiv n_{n(t-1)x} \equiv \ldots \equiv n_{2x} \equiv n_{1x} \equiv n_x \equiv n_s \equiv n_{(t-1)x} \ldots
\]

Proof:

\[
P2.b.0 \quad \begin{array}{l}
\text{if } n_s \geq 3, \forall n_s \not\equiv n_s
\end{array}
\]

\[
P2.b.1 \quad \begin{array}{l}
\text{if } \ldots n_s \equiv n_{(t-1)x} \equiv \ldots \equiv n_{2x} \equiv n_{1x} \equiv n_s \equiv n_{n(t-1)x} \equiv \ldots \equiv n_{2x} \equiv n_{1x} \equiv n_x \equiv n_s \equiv n_{(t-1)x} \ldots
\end{array}
\]

Proof:

By definition: if a loop exists then one may continue transformation (forward or backward) infinite times

\[
P2.b.2.1 \quad \begin{array}{l}
\not\exists n_{1x}, \text{if } n_x \equiv 0 \mod 3
\end{array}
\]

Proof:

\[
n_x \equiv 0 \mod 3 = 3j
\]

\[
n_{1x} = \frac{n_x 2^b - 1}{3} = \frac{3j 2^b - 1}{3} = \frac{3j' - 1}{3} \mid j' = j^2
\]

\[
n_{1x} \equiv \frac{2 \mod 3}{3} \not\in \{U'\}
\]

\[
P2.b.2.2 \quad \begin{array}{l}
\text{if } n_x \equiv 0 \mod 3, \text{then } n_s \not\equiv n_s
\end{array}
\]

Proof:

\[
n_s \equiv n_{n(t-1)x}
\]

if \(n_x \equiv 0 \mod 3 \Rightarrow \not\exists n_{1x} \Rightarrow \not\exists n_{2x} \Rightarrow \not\exists n_{3x} \ldots \Rightarrow \not\exists n_{n(t-1)x}
\]

Due to contradiction the condition is false.
### P2.b.3.1

\[ q(\tau_{t+1}) = q(\tau_t^{om3}) \]

**Proof:** Using P2.b.2.2

### P2.b.3.2

\[ q(\tau_t) = q(\tau_t^{0m3}) + q(\tau_t^{1m3}) + q(\tau_t^{2m3}) \]

**Proof:** According to Theorem 5 for all numbers upon applying reverse transformation there is ordered distribution of elements modulo 3. Addition being commutative, the order of elements modulo 3 does not matter.

### P2.b.3.3

\[ q(\tau_{t+1}) = \frac{2^1}{3^1} q(\tau_t^{0m3}) \]

**Proof:** \( q(\tau_{t+1}) \) is the set of elements that are one more reverse transformation from \( q(\tau_t) \). According to P2.2.2, \( q(\tau_t^{0m3}) \) will not have any representation in \( q(\tau_{t+1}) \) as relevant \( n_{1x} \) does not exist. Eliminating such elements, we have:

\[ q(\tau_{t+1}) = q(\tau_t^{0m3}) = q(\tau_t^{1m3}) + q(\tau_t^{2m3}) = q(\tau_{t}) - q(\tau_t^{0m3}) \]

Using Theorem 5 describing one third of elements being 0 mod3

\[ q(\tau_{t+1}) = q(\tau_t^{0m3}) = q(\tau_t) - \frac{1}{3} q(\tau_t) = \frac{2}{3} q(\tau_t) \]

### P2.b.4.1 At t=0

\[ q(\tau_0^{om3}) = \frac{2^0}{3^1} q((U') - [1]) \equiv 0 \mod 3 \]

\[ q(\tau_0^{om3}) = \frac{2^1}{3^1} q((U') - [1]) \not\equiv 0 \mod 3 \]

**Proof:** According to Theorem 5: one third of elements are 0 mod3

\[ q(\tau_0^{om3}) = \frac{1}{3} q((U') - [1]) = \frac{2^0}{3^1} q((U') - [1]) \equiv 0 \mod 3 \]

Elements that are not 0 mod3 have their respective \( n_{1x} \) represented in the set \( q(\tau_0^{om3}) \)

\[ q(\tau_0^{om3}) = q((U') - [1]) - q(\tau_0^{om3}) \]

\[ = q((U') - [1]) - \frac{1}{3} q((U') - [1]) \]

\[ = \frac{2^1}{3^1} q((U') - [1]) \not\equiv 0 \mod 3 \]

### P2.b.4.2 At t=1

\[ q(\tau_1^{om3}) = \frac{2^1}{3^2} q((U') - [1]) \& \ q(\tau_1^{om3}) = \frac{2^2}{3^2} q((U') - [1]) \]

**Proof** According to Theorem 5: one third of elements are 0 mod3

\[ q(\tau_1^{om3}) = \frac{1}{3} q(\tau_0^{om3}) = \frac{1}{3} \cdot \frac{2^1}{3^1} q((U') - [1]) = \frac{2^1}{3^2} q((U') - [1]) \]
Elements that are not 0 mod 3 have their respective $n_{1x}$ represented in the set $q(\tau_i^{0m3})$

$$q(\tau_i^{0m3}) = q(\tau_i) - q(\tau_i^{0m3}) = \frac{2^2}{3^2} q((U') - [1])$$

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<th>Mathematical Induction</th>
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<td>Assumed case for mathematical induction</td>
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<td></td>
<td>Using P2.b.4.4</td>
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<tr>
<td></td>
<td>$q(\tau_i^{0m3}) = \frac{2^{t+1}}{3^{t+2}} q((U') - [1])$</td>
</tr>
<tr>
<td></td>
<td>&amp; $q(\tau_i^{0m3}) = \frac{2^{t+2}}{3^{t+2}} q((U') - [1])$</td>
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<th>P2.b.4.4</th>
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<tbody>
<tr>
<td><strong>Proof:</strong></td>
</tr>
<tr>
<td>Using P2.b.4.4</td>
</tr>
<tr>
<td>$q(\tau_i^{0m3}) = \frac{2^{t+1}}{3^{t+1}} q((U') - [1])$</td>
</tr>
<tr>
<td>$q(\tau_i^{0m3}) = \frac{2^{t+2}}{3^{t+2}} q((U') - [1])$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>P2.b.5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Proof:</strong></td>
</tr>
<tr>
<td>$q((U') - [1]) = \sum_{t=0}^{t=\infty} q(\tau_i^{0m3})$</td>
</tr>
</tbody>
</table>

| | |
| | \[
| \sum_{t=0}^{t=\infty} q(\tau_i^{0m3}) = q(\tau_0^{0m3}) + q(\tau_1^{0m3}) + q(\tau_2^{0m3}) + q(\tau_3^{0m3}) \\
+ q(\tau_4^{0m3}) + q(\tau_5^{0m3}) + q(\tau_6^{0m3}) + q(\tau_7^{0m3})... \\
\sum_{t=0}^{t=\infty} q(\tau_i^{0m3}) = q((U') - [1]) (\frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7} \\
+ \frac{2^7}{3^8} ...)
| |
| |
| |
| let $\frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7} + \frac{2^7}{3^8} + \frac{2^8}{3^9} + \cdots = s$ |
| \[
\left(\frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7} + \frac{2^7}{3^8} + \frac{2^8}{3^9} + \cdots\right) = s - \frac{1}{3}
| |
\[ s = 1 \]
\[ \sum_{t=0}^{\infty} q(t t^{0 \text{m}3}) = q([U'] - [1]) \]

<table>
<thead>
<tr>
<th>P2.b.0</th>
<th>THEN</th>
<th>( n_s \not\equiv n_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof:</td>
<td>Using ( \text{P2.b.5.1} ) upon applying reverse transformation, the total number of elements that are 0 mod3 is equal to total number of elements in ( {U'} - [1] ) implying all the elements of ( {U'} - [1] ) reach 0 mod3.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Using ( \text{P2.b.2.2} ): none of the elements that are 0 mod3 loop.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>None of the elements can loop under given transformational conditions.</td>
<td></td>
</tr>
</tbody>
</table>

Alternatively, one could prove that \( n_s \equiv n_s | n_s \not\equiv n_x \) for any and all arbitrary element/s by just using Corollary 5 encountering similar expression mentioned in proof of \( \text{P2.b.5.1} \).

However, in the negative domain loop exists, example: \(-7 \equiv -5 \equiv -7\), but we don’t care as it is out of domain of the conjecture.

Possible solutions at \( t \to \infty \) may be represented as; \( n_s \equiv \infty \lor n_s \equiv n_s | n_s \not\equiv n_s \)

\[ s = p \lor q \lor r | p = (n_s \equiv \infty), q = (n_s \equiv n_s | n_s \not\equiv 1), r = (n_s \equiv n_s = 1) \]

Using \( \text{P1.0, P2.b.0 \& P2.a.0} \), we know

\[ \forall n_s \not\equiv \infty \lor n_x \not\equiv n_x \lor n_s \equiv n_s = 1 \]

\[ -p \lor q \lor s = r \]

\[ \forall \lim_{t \to \infty} n_s \equiv 1 \]

Thus, the conjecture is true.

\[ \blacksquare \]

**References:** Also known as 3n + 1 problem, the 3n + 1 conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem. The sequence of numbers involved is sometimes referred to as the hailstone sequence, hailstone numbers or hailstone numerals


4. Terrence Tao: Almost all orbits of Collatz map attain almost bounded values https://doi.org/10.48550/arXiv.1909.03562


