Elementary proof of the Syracuse conjecture
by Ahmed Idrissi Bouyahyaoui

Abstract:
Variables used in the proof are:
x and (x + V) are positive integer variables. (x + V) is the successor of x.
V is an integer variable of adjustment, at first step V₀ = 2 and x₀ = y₀ > 2.
Representation: x = 2^α*(y) and V = 2^β*(z), α*β = 0, α and β are non negative integer variables, y and z are odd integer variables, y > 0.

The Collatz algorithm (3* + 1) is applied simultaneously to x and (x + V), so we have for rule 2 - (x3 + 1) - of the algorithm and adjustment:
(x + V) := (2α*(3*y + 1)) + (2β*(3*z) - (2α - 1)) = 3(2α*(y) + 2β*(z)) + 1
That gives:
x := 2α*(3*y+1) = 2α’*(y’), V := 2β*(3*z) - (2α -1) = 2β’*(z’) (1)
We deduce the rule:
(x := 2α*(3*y + 1)) ^ (V := 2β*(3*z) - (2α - 1)) ==⇒ V < x
For x := 1*(y₀) and V := V₀ = 2*(1), rule 2 - (x3 + 1) - of the algorithm gives:
(x + V) := (1*(3*y₀ + 1)) + (2*(3) – (1 – 1)) > 0 ==⇒ x + V > 0
As x + V > 0 and V < x, we deduce the rule:
(V < x) ^ (x + V > 0) ==⇒ (0 < x + V < 2*x)
This rule shows that sequence S(x₀+2) is bounded because it is upper bounded by sequence of Syracuse S(2*x₀) = 2*S(x₀) and lower bounded by 0.
By hypothesis S(x₀) is a sequence of Syracuse.

The bounded sequence S(x₀+2) and the sequence of Syracuse S(2*x₀) = 2*S(x₀) – upper bound - converge to the only trivial cycle : [4, 2, 1].
So by recurrence, every positive integer gives a sequence of Syracuse.

Syracuse conjecture (Collatz conjecture)
Algorithm of Collatz:
Let x a positive integer number.
1 - if x is even then x := x/2
2 - if x is odd then x := x * 3 + 1
We repeat 1 - 2 until obtain a cycle (is only cycle ?) or x tends to infinity.
The symbol := means : assign value on right to variable on left.
Representation of variables:
x and \((x + V)\) are positive integer variables. \((x + V)\) is the successor of \(x\).
\(V\) is a variable of adjustment, at first step \(V_0 = 2\) and \(x_0 = y_0 > 2\).
The variables \(x\) and \(V\) are written in the form:
\[x := a^\alpha(y)\] with \(a := 2^\alpha\) and \(V := b^\beta(z)\) with \(b := 2^\beta\).
\(\alpha\) and \(\beta\) are non negative integer variables, such as \(\alpha \beta = 0\).
\(y\) and \(z\) are odd integer variables, \(y > 0\).
\((x + V) := a^\alpha(y) + b^\beta(z) = 2^\alpha(y) + 2^\beta(z)\) and \(\alpha \beta = 0\).

Application of the Collatz algorithm:
The Collatz algorithm \((3* + 1)\) is applied simultaneously to \(x\) and \((x + V)\).
The coefficient \(a\) is power of 2, the algorithm is applied to the odd part \(y\) of \(x\):
\[x := a^\alpha(y)\] giving a sequence of Syracuse \(S(x_0)\) and the odd part \(z\) of \(V := b^\beta(z)\) is multiplied by 3 plus an adjustment.
In operation \(3* + 1\), \(x := a^\alpha(3^y + 1) = a'^\alpha(y')\), \(x\) is increased by \((a - 1)\) to subtract from \(V\) and we have for \(V\) in \(x + V: V := b^\beta(3^z) - (a - 1) = b'^\beta(z')\).
\(a'\) and \(b'\) are power of 2, \(y'\) and \(z'\) are odd integer variables.
So we have the equality:
\[a^\alpha(3^y + 1) + b^\beta(3^z) - (a - 1) = a^\alpha(3^y) + 1 + b^\beta(3^z) = 3 \times (a^\alpha(y) + b^\beta(z)) + 1,\]
giving \(3^x(x + V) + 1\), with \(x\) and \(V\) of before the operation \(3* + 1\), according to the rule 2 of the algorithm.
The rule 2 and adjustment give:
\[x := 2^\alpha(3^y + 1), V := 2^\beta(3^z) - (2^\alpha - 1) \quad (2)\]
We deduce the rule:
\[(x := 2^\alpha(3^y + 1))^\wedge (V := 2^\beta(3^z) - (2^\alpha - 1)) \implies V < x\]
In the line \(a'^\alpha(y') + b'^\beta(z')\), \(a'\) and \(b'\) are divided by gcd\((a',b')\) according to the rule 1 of the algorithm.
If gcd\((a',b') = 1\) then division by 2 is deferred and then we have:
\[x := 2^\alpha(y'), V := 2^\beta(z')\] and \(\alpha \beta' = 0\).

Evaluation of variable of adjustment \(V\):
When \(x\) is multiplied by 3 then \(+ 1\), \(V\) is multiplied by 3.
When \(x\) is divided by 2, \(V\) is divided by 2.
When \(x = a(3^y + 1)\), \(x\) is increased by \((a - 1)\), \(V\) is decreased by \((a - 1)\).
We deduce that \(V\) is always less than \(x\).
We deduce the rule:
\[(V < x) \implies (x + V < 2^x)\]
This rule shows that sequence \(S(x_0 + 2)\) is bounded because it is upper bounded by sequence of Syracuse \(S(2^x_0) = 2^x S(x_0)\).
By hypothesis $S(x_0)$ is a sequence of Syracuse.
Let $R(x_0)$ the sequence of Syracuse without context of $(x + V)$ \((2)\), division by 2 is done and $R(x_0) \subset S(x_0)$.

Power of 2 is a neutral factor in evolution of sign of $(x + V)$, application of Collatz algorithm \textbf{without division by 2} gives with $x=y_0$ and $V=V_0=2$ (initial data):

$$(x + V) = 1(y_0) + 2(1) > 0$$
$$= 1(3y_0 + 1) + 2(3) - (1 - 1) > 0$$
$$= 1(3^2y_0 + 3 + 1) + 2(3^2) > 0$$

.....

This shows $(x + V)$ is always positive and therefore the sequence $S(x_0+2)$ is lower bounded by 0:

$$(x + V) > 0$$

We deduce the rule:

$$(V < x) \land (x + V > 0) \implies (0 < x + V < 2^x)$$

**Conclusion:**

The \textbf{bounded sequence $S(x_0+2)$} and the sequence of Syracuse $S(2^*x_0) = 2^*S(x_0)$ – upper bound - converge to the only trivial cycle : $[4, 2, 1]$.

So by recurrence, every positive integer gives a sequence of Syracuse.
Generation of sequences of Syracuse $S[x]$ and $S[x + 2]$

Application of Collatz algorithm to generate sequences $S[x_0]$

Generation of sequence of Syracuse $S[17]$
$x_0 = 17$
1(17), 1(52), 4(13), 1(13), 1(40), 8(5), 1(5), 1(16), 16(1), 1(1)

Generation of sequence of Syracuse $S[19]$
$x_0 = 19$
1(19), 1(58), 2(29), 1(29), 1(88), 8(11), 1(11), 1(34), 2(17),
1(17), 1(52), 4(13), 1(13), 1(40), 8(5), 1(5), 1(16), 16(1), 1(1)

Application of Collatz algorithm simultaneously to $x$ and $(x + V)$ to generate sequences of Syracuse $S[x_0]$ and $S[x_0 + V_0]$

$x_0 = 17$ et $x_0 + V_0 = 17 + 2 = 19$


dd

ahmed.idrissi@free.fr
January 2023
INPI – Paris