Elementary proof of the Collatz conjecture
(also called Syracuse conjecture)
by Ahmed Idrissi Bouyahyaoui

Abstract:
Variables used in the proof are:
x and \((x + V)\) are positive integer variables. \((x + V)\) is the successor of \(x\).
\(V\) is an integer variable of adjustment, at first step \(V_0 = 2\) and \(x_0 = y_0 > 2\).
Representation: \(x = 2^\alpha *(y)\) and \(V = 2^\beta *(z)\), \(\alpha \beta = 0\), \(\alpha\) and \(\beta\) are non negative integer variables, \(y\) and \(z\) are odd integer variables, \(y > 0\).
The Collatz algorithm \((3\times x + 1)\) is applied simultaneously to \(x\) and \((x + V)\), so we have for rule 2 - \((3\times x + 1)\) - of the algorithm and adjustment:
\[(x + V) := (2^\alpha *(3\times y + 1)) + (2^\beta *(3\times z) – (2^\alpha – 1)) = 3(2^\alpha *(y) + 2^\beta *(z)) + 1 > 0\]
That gives:
\[x := 2^\alpha *(3\times y+1) = 2^\alpha’ *(y’),\ V := 2^\beta *(3\times z) - (2^\alpha - 1) = 2^\beta’ *(z’)\] (1)
We deduce the rule:
\[(x := 2^\alpha *(3\times y + 1))^\wedge (V := 2^\beta *(3\times z) – (2^\alpha – 1)) \Rightarrow V < x\]
By recurrence we have:\(x_{i+1} + V_{i+1} := 3(x_i + V_i) + 1 > 0\) \(\Rightarrow x + V > 0\)
As \(x + V > 0\) and \(V < x\), we deduce the rule:
\[(V < x)^\wedge (x + V > 0) \Rightarrow (0 < x + V < 2\times x).\] (0< \(x_i\)+ \(V_i\) <2\(x_i\))
By hypothesis \(S(x_0)\) is a sequence of Syracuse, the rule shows that sequence \(S(x_0+2)\) is bounded because it is upper bounded by sequence of Syracuse \(S(2\times x_0) = 2\times S(x_0)\) and lower bounded by \(0\).

The bounded sequence \(S(x_0+2)\) and the sequence of Syracuse \(S(2\times x_0) = 2\times S(x_0)\) – upper bound - converge to the only trivial cycle: \([4, 2, 1]\).
So by recurrence, every positive integer gives a sequence of Syracuse.

Collatz conjecture (also called Syracuse conjecture)
Algorithm of Collatz:
Let \(x\) a positive integer number.
1 - if \(x\) is even then \(x := x/2\)
2 - if \(x\) is odd then \(x := x \times 3 + 1\)
We repeat 1 - 2 until obtain a cycle (is only cycle ?) or \(x\) tends to infinity.
The cycle \([4, 2, 1]\) is the Collatz conjecture.
The symbol := means: assign value on right to variable on left.
Representation of variables:
x and (x + V) are positive integer variables. (x + V) is the successor of x.
V is a variable of adjustment, at first step V₀ = 2 and x₀ = y₀ > 2.
The variables x and V are written in the form:
x := a*(y) with a := 2ᵃ and V := b*(z) with b := 2ᵇ.
α and β are non negative integer variables, such as α*β = 0.
y and z are odd integer variables, y > 0.
(x + V) := a*(y) + b*(z) = 2ᵃ*(y) + 2ᵇ*(z) and α*β = 0.

Application of the Collatz algorithm:
The Collatz algorithm (3*x + 1) is applied simultaneously to x and (x + V).
The coefficient a is power of 2, the algorithm is applied to the odd part y of
x := a*(y) giving a sequence of Syracuse S(x₀) and the odd part z of V := b*(z) is
multiplied by 3 plus an adjustment.
In operation 3*x + 1, x := a*(3*y + 1) = a’*(y’), x is increased by (a - 1) to subtract from V and we have for V in x + V:  V := b*(3*z) - (a-1) = b’*(z’).
a’ and b’ are power of 2, y’ and z’ are odd integer variables.
So we have the equality
a*(3*y+1) + b*(3*z) - (a-1) = a*(3*y) +1 + b*(3*z) = 3 * (a*(y) + b*(z)) + 1 > 0,
giving 3*(x + V) + 1, with x and V of before the operation 3* + 1, according to
the rule 2 of the algorithm.
The rule 2 and adjustment give:
x := 2ᵃ*(3*y+1), V := 2ᵇ*(3*z) - (2ᵃ -1) (2)
We deduce the rule:
(x := 2ᵃ*(3*y + 1)) ^ (V := 2ᵇ*(3*z) – (2ᵃ – 1)) => V < x. (Vi<xᵢ)

In the line a’*(y’) + b’*(z’), a’ and b’ are divided by gcd(a’,b’) according to the
rule 1 of the algorithm.
If gcd(a’,b’) = 1 then division by 2 is deferred and then we have :
x := 2ᵃ’*(y’), V := 2ᵇ’*(z’) and α’*β’ = 0.

Evaluation of variable of adjustment V:
When x is multiplied by 3 then + 1, V is multiplied by 3.
When x is divided by 2, V is divided by 2.
When x = a(3*y+1), x is increased by (a - 1), V is decreased by (a - 1).
We deduce that V is always less than x.
We deduce the rule:
(V < x) ==> (x + V < 2*x). (xᵢ+Vᵢ < 2*xᵢ)
By hypothesis $S(x_0)$ is a sequence of Syracuse, the rule shows that sequence $S(x_0+2)$ is bounded because it is upper bounded by sequence of Syracuse $S(2\times x_0) = 2\times S(x_0)$.

By recurrence we have : $x_{i+1} + V_{i+1} := 3(x_i + V_i) + 1 > 0$ 
$\implies x + V > 0$
This shows $(x + V)$ is always positive and therefore the sequence $S(x_0+2)$ is lower bounded by $0$ :
$(x + V) > 0$

We deduce the rule :
$(V < x) \land (x + V > 0) \implies (0 < x + V < 2\times x)$. $(0 < x_i+V_i < 2\times x_i)$

**Conclusion :**
The bounded sequence $S(x_0+2)$ and the sequence of Syracuse $S(2\times x_0) = 2\times S(x_0)$ – upper bound - converge to the only trivial cycle : $[4, 2, 1]$.
So by recurrence, every positive integer gives a sequence of Syracuse.
Generation of sequences of Syracuse $S(x_0)$ and $S(x_0 + V_0)$:
Application of Collatz algorithm simultaneously to $x$ and $(x + V)$ to generate sequences of Syracuse $S(x_0)$ and $S(x_0 + V_0)$.

Generation of sequences of Syracuse $S(17)$ and $S(17 + 2) = S(19)$:
$x_0 = 17$ et $x_0 + V_0 = 17 + 2 = 19$

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