Using (*3+2^m-1)/2^k Odd Tree to Solve The Collatz Conjecture Problem

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Abstract: Build a special identical equation, use its calculation characters to prove and search for solution of any odd converging to 1 equation through (*3+1)/2^k operation, change the operation to (*3+2^m-1)/2^k, get a solution for this equation. Furthermore, analysis the sequences produced by iteration calculation during the procedure of searching for solution, build a weight function model, prove it monotonically decreases, build a complement weight function model, prove it has many chances to increase to its convergence state. Build a (*3+2^m-1)/2^k odd tree, prove if odd in (*3+2^m-1)/2^k long huge odd sequence can not converge, the sequence must walk out of the boundary of the tree after infinite steps of (*3+2^m-1)/2^k operation.

Key words: Collatz conjecture; $(*3+1)/2^k$ odd sequence; $(*3+2^m-1)/2^k$ odd sequence; weight function; $(*3+2^m-1)/2^k$ odd tree.

1 Introduction About The Collatz Conjecture

The Collatz Conjecture is a famous math conjecture, named after mathematician Lothar Collatz, who introduced the idea in 1937. It is also known as the $3x + 1$ conjecture, the Ulam conjecture[1] etc. Many mathematicians have tried to prove it true or false and have expanded it to more digits scale. But until today, it has not yet been proved.

The Collatz Conjecture concerns sequences of positive integers in which each term is obtained from the previous one as follows: if the previous integer is even, the next integer is the previous integer divided by 2, till to odd. If the previous integer is odd, the next term is the previous integer multiply 3 and plus 1. The conjecture is that these sequences always reach 1, no matter which positive integer is chosen to start the sequence[1].

Here is an example for a typical integer $x = 27$, takes up to 111 steps, increasing or decreasing step by step, climbing as high as 9232 before descending to 1[1].

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

If the conjecture is false, there should exists some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound[1]. No such sequence has been found by human or computer after verified a lot of numbers can reach to 1. It is very difficult to prove these two cases exist or not.

This paper will try to prove the conjecture true from a special view. Because any even can become odd through $\div 2^k$ operation, this paper will research only odd characters in the conjecture sequence. The equivalence conjecture become: with random starting odd x, do $(\times 3+1)\div 2^k$ operation repeatedly, it always converges to 1. The above sequence can be written as following, in which numbers on arrows are k in $\div 2^k$ in each step:

$$27 \xrightarrow{1} 41 \xrightarrow{2} 31 \xrightarrow{1} 47 \xrightarrow{1} 71 \xrightarrow{1} 107 \xrightarrow{1} 161 \xrightarrow{2} 121 \xrightarrow{2} 91 \xrightarrow{1} 137 \xrightarrow{2} 103 \xrightarrow{1}$$
$$155 \xrightarrow{1} 233 \xrightarrow{2} 175 \xrightarrow{1} 263 \xrightarrow{1} 395 \xrightarrow{1} 593 \xrightarrow{2} 445 \xrightarrow{3} 167 \xrightarrow{1} 251 \xrightarrow{1} 377 \xrightarrow{2}$$
$$283 \xrightarrow{1} 425 \xrightarrow{2} 319 \xrightarrow{1} 479 \xrightarrow{1} 719 \xrightarrow{1} 1079 \xrightarrow{1} 1619 \xrightarrow{1} 2429 \xrightarrow{3} 911 \xrightarrow{1} 1367 \xrightarrow{1}$$
$$2051 \xrightarrow{1} 3077 \xrightarrow{4} 577 \xrightarrow{2} 433 \xrightarrow{2} 325 \xrightarrow{4} 61 \xrightarrow{3} 23 \xrightarrow{1} 35 \xrightarrow{1} 53 \xrightarrow{5} 5 \xrightarrow{4} 1$$

2 Build A Equation For The Conjecture

If odd x do n times $(\times 3+1)\div 2^k$ calculation build odd y, can get:

$$y=\frac{3^n x+3^{n-1}+3^{n-2}\times 2^{p1}+3^{n-3}\times 2^{p_1+p_2}\ldots+3\times 2^{p_1+p_2+\ldots+p_{n-2}}+2^{p_1+p_2+\ldots+p_{n-1}}}{2^{p_1+p_2+\ldots+p_n}}$$

In which $p_1 \ldots p_n$ is k in $\div 2^k$ operation in each step.

For example: $(7\times 3+1)\div 2=11$, $(11\times 3+1)\div 2=17$, then $17=\frac{3^2\times 7+3+2}{2^2}$

Suppose odd x can converge to 1 through $(\times 3+1)\div 2^k$ calculation, then y=1, get:

$$3^n x+3^{n-1}+3^{n-2}\times 2^{p_1}+3^{n-3}\times 2^{p_1+p_2}\ldots+3\times 2^{p_1+p_2+\ldots+p_{n-2}}+2^{p_1+p_2+\ldots+p_{n-1}}-2^{p_1+p_2+\ldots+p_n}=0 \quad \text{Formula (1)}$$

We know $(1\times 3+1)\div 2^2=1$, and can do any times this kind of operation. That is to say, 1 do random n steps $(\times 3+1)\div 2^2$ operation can converge to 1, have:

$$3^n+3^{n-1}+3^{n-2}\times 2^2+3^{n-3}\times 2^4\ldots+3\times 2^{2n-4}+2^{2n-2}-2^{2n}=0$$

Below use this model to prove and search for a solution of Formula (1) for any odd x converging to 1.

3 Solution to The Any Odd Converging to 1 Equation

First with odd x do reform:

$x=a_m\times 3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0$, $a_m \ldots a_0$=0, 1 or 2. Then:

$3^n x=3^n\times(a_m\times 3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0)$

If a_m>1 or a_m=1 but

$(a_{m-1}\times 3^{n+m-1}+\ldots+a_1\times 3^{n+1}+a_0\times 3^n)>(3^{n+m-1}+3^{n+m-2}\times 2^2\ldots+3^n\times 2^{2(m-1)})$, make

$x=3^{m+1}-3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0$ or

$x=3^{m+1}-2\times 3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0$

Build identical equation:

$$3^{n+m}+3^{n+m-1}+3^{n+m-2}\times 2^2+3^{n+m-3}\times 2^4\ldots+3^{n-1}\times 2^{2m}\ldots+3\times 2^{2(n+m)-4}+2^{2(n+m)-2}-2^{2(n+m)}=0 \quad \text{Formula (2)}$$

If x can converge to 1, Formula (1) and Formula (2) should be equivalent. Below try to reform Formula (2) to form of Formula (1), if success it proves that equation for Formula (1) has solution.

First let:

$(3^{n+m-1}+3^{n+m-2}\times 2^2\ldots+3^n\times 2^{2(m-1)})-(a_{m-1}\times 3^{n+m-1}+\ldots+a_1\times 3^{n+1}+a_0\times 3^n)=t_n\times 3^n$,
because x is odd, this is odd minus even, t_n should be odd.

Because the max value of x-3^m is $2\times 3^{m-1}+2\times 3^{m-2}+\ldots+2\times 3+2$, min value is $-3^{m-1}+1$, then t_n has a range:

From $(3^{m-1}+3^{m-2}\times 2^2\ldots+2^{2(m-1)})-(2\times 3^{m-1}+2\times 3^{m-2}+\ldots+2\times 3+2)$ to
$(3^{m-1}+3^{m-2}\times 2^2\ldots+2^{2(m-1)})-(-3^{m-1}+1)$.

Change t_n to binary form and let:

$t_n \times (2+1) \times 3^{n-1} + 3^{n-1} \times 2^{2m} - 3^{n-1} = t_{n-1} \times 3^{n-1}$, this is just with 3^n part multiply (2+1) become 3^{n-1} part, and plus corresponding part in Formula (2), minus corresponding part in Formula (1). From now on, t_{n-1} become even, write t_{n-1} in $odd \times 2^p$ form. Continue:

$t_{n-1} \times (2+1) \times 3^{n-2} + 3^{n-2} \times 2^{2m+2} - 3^{n-2} \times 2^{p_1} = t_{n-2} \times 3^{n-2}$, and let 2^{p1} be the lowest bit of odd part of t_{n-1}).

Watch Formula (1) and Formula (2), in general, if do not consider $2^{p1+\cdots}$ part (because consider $2^{p1+\cdots}$ as the lowest bit of odd part of t_{i-2}) in Formula (1), part in Formula (2) is bigger than corresponding part in Formula (1). Hence after a few times of $t_{i-1} \times (2+1)$, value of t_{i-2} is mainly determined by corresponding part in Formula (2). And, after $t_{i-1} \times (2+1)$, odd part should add 1 or 2 bits, if add 1 bit, $+2^{2m+2}$ should operate in MSB bit; if add 2 bits, $+2^{2m+2}$ should operate in MSB-1 bit. Both cases odd part adds 2 bits after $+2^{2m+2}$ operation, if MSB bit of t_{i-2} is 2^k, k should be odd.

For example:

$$3 + 2^2 = 7, 7 \times (2+1) + 2^4 - 1 = 9 \times 2^2, 9 \times 2^2 \times (2+1) + 2^6 - 2^2 = 21 \times 2^3$$

Continue:

$t_{n-2} \times (2+1) \times 3^{n-3} + 3^{n-3} \times 2^{2m+4} - 3^{n-3} \times 2^{p_1+p_2} = t_{n-3} \times 3^{n-3}$, let 2^{p1+p2} be the lowest bit of odd part of t_{n-2}. Because LSB bit sequence number of odd part of t_i increases continuously, this can be finished easily.

Watch t_i($i<n$ and decreases step by step), during iteration, the count of succession 1 in the highest part should be unchanged or increased. Why? This is because of characters of odd multiply 3 and $+2^{2m}$ operation. If t_{i-1} is with binary form 10…, obviously, count of succession 1 in highest part of t_{i-2} is unchanged or increased. Similar to binary form 110…. Suppose t_{i-1} is with binary form 1…1(k>2 bits of 1)0…, do $\times (2+1)$, head part should become 101…1(k-2 bits of 1)01…, do $+2^{2m}$, become 1…1(k bits of 1)01…, if tail part carry 1 to head part after doing $\times (2+1)$, head part become 1…1(k+1 bits of 1)0….

Do this iteration continuously, count of succession 1 in the highest part of odd part of t_i is unchanged or increased, LSB bit sequence number is also increased. Hence, finally, t_i is much possible to become form of 11…10...0, just $2^k \times (2^j - 1)$ form (k+j=even). Stop here, do not do $\times (2+1)$ again, odd x already converge to 1. Do $-2^{2(n+m)}$ operation, it should operate in MSB+1 bit, because MSB bit sequence number of $+2^{2k}$ is forever equal to MSB+1 bit sequence number of the previous item. Hence subtractive result can be equal to $-2^{p_1+p_2+\ldots+p_n}$, thus get a solution of Formula (1).

Below give a specific example, x=7.

We know, with 7 do $(\times 3 + 1) \div 2^k$, have:

$$7 \xrightarrow{1} 11 \xrightarrow{1} 17 \xrightarrow{2} 13 \xrightarrow{3} 5 \xrightarrow{4} 1$$

Suppose:

$$3^n \times 7 + 3^{n-1} + 3^{n-2} \times 2^{p_1} + 3^{n-3} \times 2^{p_1+p_2} \ldots + 3 \times 2^{p_1+p_2+\ldots+p_{n-2}} + 2^{p_1+p_2+\ldots+p_{n-1}} - 2^{p_1+p_2+\ldots+p_n} = 0$$

$$3^n \times 7 = 3^n \times (2 \times 3 + 1) = 3^n \times (3^2 - 3 + 1) = 3^{n+2} - 3^{n+1} + 3^n$$

Build:

$$3^{n+2} + 3^{n+1} + 3^n \times 2^2 + 3^{n-1} \times 2^4 \ldots + 3 \times 2^{2n} + 2^{2n+2} - 2^{2n+4} = 0$$

$$3^{n+1} + 3^n \times 2^2 + 3^{n+1} - 3^n = (2^3 + 1) \times 3^n$$

*(2+1) and $+2^4$: $(2^3+1) \times (2+1) \times 3^{n-1} + 2^4 \times 3^{n-1} = (2^5 + 2^3 + 2 + 1) \times 3^{n-1}$

-3^{n-1}: $(2^5 + 2^3 + 2 + 1) \times 3^{n-1} - 3^{n-1} = (2^5 + 2^3 + 2) \times 3^{n-1}$

*(2+1) and $+2^6$: $(2^5 + 2^3 + 2) \times (2+1) \times 3^{n-2} + 2^6 \times 3^{n-2} = (2^7 + 2^5 + 2^4 + 2^3 + 2^2 + 2) \times 3^{n-2}$,

Let p_1=1, and delete item 2:

$$(2^7+2^5+2^4+2^3+2^2+2-2)\times 3^{n-2}=(2^7+2^5+2^4+2^3+2^2)\times 3^{n-2}$$

*(2+1) and +2^8: $(2^7+2^5+2^4+2^3+2^2)\times(2+1)\times 3^{n-3}+2^8\times 3^{n-3}=(2^9+2^8+2^5+2^4+2^2)\times 3^{n-3}$

Let p_1+p_2=2, and delete item 2^2:

$$(2^9+2^8+2^5+2^4+2^2-2^2)\times 3^{n-3}=(2^9+2^8+2^5+2^4)\times 3^{n-3}$$

*(2+1) and +2^{10}: $(2^9+2^8+2^5+2^4)\times(2+1)\times 3^{n-4}+2^{10}\times 3^{n-4}=(2^{11}+2^{10}+2^8+2^7+2^4)\times 3^{n-4}$

Let $p_1+p_2+p_3$=4, and delete item 2^4:

$$(2^{11}+2^{10}+2^8+2^7+2^4-2^4)\times 3^{n-4}=(2^{11}+2^{10}+2^8+2^7)\times 3^{n-4}$$

*(2+1) and +2^{12}: $(2^{11}+2^{10}+2^8+2^7)\times(2+1)\times 3^{n-5}+2^{12}\times 3^{n-5}=(2^{13}+2^{12}+2^{11}+2^7)\times 3^{n-5}$

Let $p_1+p_2+p_3+p_4$=7, and delete item 2^7:

$$(2^{13}+2^{12}+2^{11}+2^7-2^7)\times 3^{n-5}=(2^{13}+2^{12}+2^{11})\times 3^{n-5}$$

Now become 111…, the highest bit is 2^{13}, iteration finished, steps n=5. And

$2^{13}+2^{12}+2^{11}-2^{(2\times 5+4)}=-2^{11}=-2^{p_1+\ldots+p_5}$.

This way, get a solution of Formula (1), in which the value of n and p_i is exactly same with the result got from calculating directly.

4 Convergence Regularity Of Collatz Conjecture Sequence

4.1 Equivalence of (*3+1)/2^k and (*3+2^m-1)/2^k operation

If calculate directly with odd through $(\times 3+1)\div 2^k$ operation, the odd sequence built (called Sequence (1)) has no obvious converge regularity, elements in the sequence vary sometimes big, sometimes small. But if do operation as introduced in above section, we would find convergence regularity of the odd sequence built (called Sequence (2)) is more obvious.

If add two corresponding elements in each step in these two odd sequences, should be exactly 2^k(k is different with different elements). Such as

$7+9=16, 11+21=32, 17+47=64$ … in above example.

In general, first element in Sequence (1) can be written as(a_{m-2}…a_0=0, 1 or 2, a_{m-1}=-2, -1, 0, 1 or 2):

$$x=3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0$$

First element in Sequence (2) is:

$$a=(3^{m-1}+3^{m-2}\times 2^2\ldots+2^{2(m-1)})-(a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0)$$

and, then

$x+a=3^m+3^{m-1}+3^{m-2}\times 2^2\ldots+2^{2(m-1)}=2^{2m}$, is just the same form with Formula (2), and 2m should be the MSB+1 bit sequence number of x or a(along with the increase of a in Sequence (2), 2m should be the MSB+1 bit sequence number of a, because each corresponding part in Formula (2) is bigger than which in Formula (1). In fact we can select first a>x manually as long as sum of these two odds is 2^k).

Below prove next elements also satisfy above regularity.

Suppose a in Sequence (2) and x in Sequence (1) satisfy above regularity, and:

$a=2^m+a_{m-1}\times 2^{m-1}+\ldots+a_1\times 2+1$,

$x=2^{m+1}-a$, then

$3a+2^{m+1}-1=3\times 2^m+3\times a_{m-1}\times 2^{m-1}+\ldots+3\times a_1\times 2+3+2^{m+1}-1$,

$3x+1=3\times 2^{m+1}-3\times 2^m-3\times a_{m-1}\times 2^{m-1}-\ldots-3\times a_1\times 2-3+1$,

$(3x+1)+(3a+2^{m+1}-1)=4\times 2^{m+1}=2^k$

This states that the lowest bit of odd part of (3x+1) and ($3a+2^{m+1}-1$) is equal, add these two odd parts should be 2^i(i<k).

Through above introduction we know, with odd do $(\times 3+1)\div 2^k$ operation in the Collatz Conjecture odd sequence, equivalently, with the corresponding odd do $(\times 3+2^m-1)\div 2^k$ using above iteration method. We can easily prove that odd 1…10a(a is in binary base) is equivalent to odd 10a in second method, count of succession 1 bits in the head part only represent the iteration steps roughly.

4.2 Weight Function and Its Monotonically Decrease Character

Build a simple weight model:

$$w_i=\frac{\text{value of all 0 bits in odd part in } t_i}{2^{2k}} \qquad \text{Definition (1)}$$

Which 2^{2k} is corresponding addition part in t_i in its step(we can also use the sum of t_i and its corresponding part in original sequence 2^{2k} or 2^{2k+1} as denominator). Simply we can use w_i to represent the weight of value of all 0 bits in odd part in t_i. Specially, with any odd a, which highest bit is 2^m, define w_i for this odd:

$$w_{[a]}=\frac{\text{value of all 0 bits in odd a}}{2^{m+1}} \qquad \text{Definition (2)}$$

At this time, use MSB+1 bit of each odd as 2^m in $(\times 3+2^m-1)\div 2^k$ operation in each step. Although the denominator may be bigger than which in Definition (1), the regularity is same.

t_i sequence in above example is: 9,42,188,816,3456,14336

odd part sequence is: 9,21,47,51,27,7

w_i sequence is(according to Definition (1)):

(2+4)/4=1.5,(4+16)/16=1.25,64/64=1,(64+128)/256=0.75,512/1024=0.5,0/4096=0

Below prove w_i monotonically decreases except some special cases.

In fact, only one non-convergence case 0 bits in t_i do not shift right or bit-count reduce when t_i has not converged. This is:

101->1011.

This case w_i do not change, both are 1/4, according to Definition (2). But next step 1011->11 , t_i converges, hence this case is not worth worrying about.

Suppose with odd a do $(\times 3+1)\div 2^k$ operation, and use x represent iteration steps. Reform w_i as following(according to Definition (1)), the numerator part is exactly equal to 0 bits in t_i:

$$w(x)=\frac{3^x a+3^{x-1}+3^{x-2}\times 2^{p_1}+3^{x-3}\times 2^{p_1+p_2}\ldots+3\times 2^{p_1+p_2+\ldots+p_{x-2}}+2^{p_1+p_2+\ldots+p_{x-1}}-2^{p_1+p_2+\ldots+p_x}}{2^{2k_x}}$$

Obviously w(x) is continuous derivable when a is in odd domain definition and x is in positive integer domain definition, and is bounded(>=0).

Now try to take the derivative of w(x).

Here the derivation definition of the numerator and denominator is: (y(x+1)-y(x))/(x+1-x).

Then the derivation of the numerator is:

$$2\times(3^x a+3^{x-1}+3^{x-2}\times 2^{p_1}+\ldots+3\times 2^{p_1+p_2+\ldots+p_{x-2}}+2^{p_1+p_2+\ldots+p_{x-1}})+2^{p_1+p_2+\ldots+p_x}+2^{p_1+p_2+\ldots+p_x}-2^{p_1+p_2+\ldots+p_{x+1}}$$

The derivation of the denominator is: $2^{2k_x+2}-2^{2k_x}=3\times 2^{2k_x}$

Then

$$w'(x)=\frac{2\times 2^{2k_x}\times 2^{p_1+p_2+...+p_x}+3\times 2^{2k_x}\times 2^{p_1+p_2+...+p_x}-2^{2k_x}\times 2^{p_1+p_2+...+p_{x+1}}-(3^x a+3^{x-1}+...+2^{p_1+p_2+...+p_{x-1}})\times 2^{2k_x}}{2^{4k_x}}$$

$$=\frac{(5-2^{p_{x+1}})\times 2^{2k_x}\times 2^{p_1+p_2+...+p_x}-b\times 2^{p_1+p_2+...+p_x}\times 2^{2k_x}}{2^{4k_x}}=\frac{(5-2^{p_{x+1}}-b)\times 2^{p_1+p_2+...+p_x}}{2^{2k_x}}$$

Which b is the odd after odd a doing x steps $(\times 3+1)\div 2^k$ operation. that is:

$$3^x a+3^{x-1}+3^{x-2}\times 2^{p_1}+...+3\times 2^{p_1+p_2+...+p_{x-2}}+2^{p_1+p_2+...+p_{x-1}}=b\times 2^{p_1+p_2+...+p_x}$$

Observe $w'(x)$, when b>3, $w'(x)<0$, w(x) monotonically decreases. Only when b=1(this case $2^{p_{x+1}}$ should be equal to 4), or when b=3, $2^{p_{x+1}}=2$, $w'(x)=0$. Second case of b=3 is the exception case introduced above, the corresponding odd part of t_i is with form '101', is not worth worrying about. First case is convergence case.

Totally, this kind of iteration calculation has these cases after doing $(\times 3+2^m-1)\div 2^k$ as following:

Case 1: odd tail part decreases one bit, head part does not increase one bit, this case tail part should insert one bit of 1 and with zero or more 0 changing to 1, totally 1 bits weight should increase in tail part.

Case 2: odd tail part decreases one bit, head part increases one bit, if corresponding odd in $(\times 3+1)\div 2^k$ sequence changes bigger, is just because tail part carries one bit of 1 to head part.

Case 3: odd tail part decreases two bits, head part does not increase one bit, tail part 0 bits should shift right.

Case 4: odd tail part decreases two bits, head part increases one bit.

Case 5: odd tail part decreases three or more bits, head part increases zero or one bit.

5 The Complement Weight Function Of $W_{[a]}$

5.1 Convergence Regularity of Complement Weight Function

To avoid proving weight function $W_{[a]}$ converging to 0(it is not easily to prove strictly the numerator part must be equal to 0 finally), build its complement weight function. Build:

$w_{c[a]}=\frac{a}{2^{m+1}}$, the highest bit of a is 2^m.

Through the proof and introduction above, we know $W_{c[a]}$ monotonically increases except when corresponding odd b_i in $(\times 3+1)\div 2^k$ sequence of a_i is 1 or 3, and these exception cases are not worth worrying about. And we also know the convergence state of $W_{c[a]}$ is $\frac{2^k-1}{2^k}$.

Suppose odd a_0, a_1, a_2 are three elements in order in $(\times 3+2^m-1)\div 2^k$ sequence, a_0 is equal to a, then

$w_{c[a_0]}=\frac{a}{2^{m+1}}$, $w_{c[a_1]}=\frac{3a+2^{m+1}-1}{2^{m+3}}$, $w_{c[a_2]}=\frac{3^2 a+3\times 2^{m+1}-3+2^{m+3}-2^p}{2^{m+5}}$, where 2^p is 2^k in first step $(\times 3+2^m-1)\div 2^k$ operation.

$$w_{c[a_1]}-w_{c[a_0]}=\frac{3a+2^{m+1}-1-4a}{2^{m+3}}=\frac{2^{m+1}-a-1}{2^{m+3}},$$

$$w_{c[a_2]}-w_{c[a_1]}=\frac{3^2 a+3\times 2^{m+1}-3+2^{m+3}-2^p-12a-4\times 2^{m+1}+4}{2^{m+5}}=\frac{3\times 2^{m+1}-3a-2^p+1}{2^{m+5}},$$

$$\frac{w_{c[a_2]} - w_{c[a_1]}}{w_{c[a_1]} - w_{c[a_0]}} = \frac{3\times 2^{m+1} - 3a - 2^p + 1}{2^{m+5}} \times \frac{2^{m+3}}{2^{m+1} - a - 1} = \frac{3}{4} + \frac{4 - 2^p}{4\times(2^{m+1} - a - 1)}$$

Observe this formula, when 2^p is equal to 2 or 4, $\frac{w_{c[a_2]} - w_{c[a_1]}}{w_{c[a_1]} - w_{c[a_0]}}$ is $\geq \frac{3}{4}$, suppose this ratio is $\frac{3}{4}$, then

$$w_{c[a_n]} = \frac{a}{2^{m+1}} + \frac{2^{m+1} - a - 1}{2^{m+3}} \times (1 + \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3 + ...(\frac{3}{4})^{n-1}),$$

When n->∞, $w_{c[a_n]} = \frac{a}{2^{m+1}} + \frac{2^{m+1} - a - 1}{2^{m+3}} \times 4 = \frac{2^{m+1} - 1}{2^{m+1}}$, this is a convergence state, in actual case, it needs a limit number n steps to reach to(or bigger than) $\frac{2^{m+1} - 1}{2^{m+1}}$, because the ratio is $\geq \frac{3}{4}$.

when 2^p is bigger than 4, $\frac{w_{c[a_2]} - w_{c[a_1]}}{w_{c[a_1]} - w_{c[a_0]}}$ is $< \frac{3}{4}$, but still $> \frac{1}{2}$, $W_{c[a]}$ also increases, $W_{c[a]}$ can converge in $\frac{2^k - 1}{2^k}$ (k is any positive integer), not only $\frac{2^{m+1} - 1}{2^{m+1}}$. This increases the convergence chance of $W_{c[a]}$.

Observe the varying of fraction in lowest terms of $W_{c[a]}$, the denominator part is equal, smaller, or 2 times of previous(because the numerator part at least can be divided by 2 in each step) in each step, when is equal, the numerator part should increase, it is possible to converge, when is 2 times of previous, the total value also increase, when is smaller, the total value should not only be bigger than the value of front $W_{c[a]}$ with same denominator part(if exists), but also be bigger than all $W_{c[a]}$ follow it. And in long sequence, usually appear smaller case, it has many chances to appear $\frac{2^k - 1}{2^k}$, especially when the front element is already close to its convergence state. For example, suppose 177/256 is in sequence, if some following element with same denominator part 256 appear after many steps, its value should be bigger than all the elements between 177/256 and itself, it is much possible to be equal to 255/256.

Continuously observe $W_{c[a]}$, even in the 2 times case, elements are closer to convergence state by themselves. Suppose the denominator part of fraction in lowest terms of $w_{c[a_1]} = \frac{3a + 2^{m+1} - 1}{2^{m+3}}$ is 2^{m+2},

$$\frac{2^{m+2} - 1}{2^{m+2}} - \frac{(3a + 2^{m+1} - 1) \div 2}{2^{m+2}} = \frac{3\times 2^m - \frac{3}{2}a - \frac{1}{2}}{2^{m+2}}$$

$$\frac{2^{m+1} - 1}{2^{m+1}} - \frac{a}{2^{m+1}} = \frac{2^{m+1} - a - 1}{2^{m+1}}$$

$$\frac{3\times 2^m - \frac{3}{2}a - \frac{1}{2}}{2^{m+2}} - \frac{2^{m+1} - a - 1}{2^{m+1}} = \frac{3\times 2^m - \frac{3}{2}a - \frac{1}{2} - 2^{m+2} + 2a + 2}{2^{m+2}} = \frac{\frac{1}{2}a - 2^m + \frac{3}{2}}{2^{m+2}} = \frac{a - 2^{m+1} + 3}{2^{m+3}}$$

$2^m<a<2^{m+1}-1$, if a is not equal to 11…101, which is very close to its convergence state 11…1, the above formula is <0. Thus prove the above conclusion.

Below give an example of start number 27 in $(\times 3 + 1) \div 2^k$ odd sequence to verify, some decimals are written in the form which is easily to be judged equal to, bigger or smaller than 0.75.

Odds in $(\times 3 + 2^m - 1) \div 2^k$ sequence are:

37,87,97,209,441,917,1887,1927,1957,3959,3993,8037,16151,16209,32505,65141,130479,130627,65369,130821,261767,261861,523863,523969,1048097,2096433,4193225,8386989,16774787,8387697,1

6775849,33552381,67105787,16776639,16776783,16776891,4194243,2097129,4194269,8388555,1048571,262143

$W_{c[a]}$ sequence:

37/64,87/128,97/128,209/256,441/512,917/1024,1887/2048,1927/2048,1957/2048,3959/4096,3993/4096,8037/8192,16151/16384,16209/16384,32505/32768,65141/65536,130479/(65536*2),130627/(65536*2),65369/65536,130821/(65536*2),261767/(65536*4),261861/(65536*4),523863/(65536*8),523969/(65536*8),1048097/(65536*16),2096433/(65536*32),4193225/(65536*64),8386989/(65536*128),16774787/(65536*256),8387697/(65536*128),16775849/(65536*256),33552381/(65536*512),67105787/(65536*1024),16776639/(65536*256),16776783/(65536*256),16776891/(65536*256),4194243/(65536*64),2097129/(65536*32),4194269/(65536*64),8388555/(65536*128),1048571/(65536*16),262143/262144

$w_{c[a_{i+1}]} - w_{c[a_i]}$ sequence:

13/128,10/128,15/256,23/512,35/1024,53/2048,40/2048,30/2048,45/4096,34/4096,51/8192,77/16384,58/16384,87/32768,131/65536,197/(65536*2),148/(65536*2),111/(65536*2),83/(65536*2),125/(65536*4),94/(65536*4),141/(65536*8),106/(65536*8),159/(65536*16),239/(65536*32),359/(65536*64),539/(65536*128),809/(65536*256),607/(65536*256),455/(65536*256),683/(65536*512),1025/(65536*1024),769/(65536*1024),144/(65536*256),108/(65536*256),81/(65536*256),15/(65536*64),11/(65536*64),17/(65536*128),13/(65536*128),1/(65536*16)

$\frac{w_{c[a_{i+2}]} - w_{c[a_{i+1}]}}{w_{c[a_{i+1}]} - w_{c[a_i]}}$ sequence:

10/13≈0.77,0.75,0.77,0.76,0.76,0.755,0.75,0.75,0.76,0.75,0.755,0.753,0.75,0.753,0.752,0.751,0.75,0.748,0.753,0.752,0.75,0.752,0.75,0.752,0.751,0.751,0.750,0.750,0.749,0.751,0.750,0.750,0.749,0.75,0.75,0.741,0.73,0.77,0.76,0.62

5.2 An Equivalent Description of Collatz Conjecture

Through above we know $w_{c[a_1]} = \frac{3a + 2^{m+1} - 1}{2^{m+3}}$, it can be written in following forms:

$$w_{c[a_1]} = \frac{3a + 2^{m+1} - 1}{2^{m+3}} = \frac{4a + 2^{m+1} - a - 1}{2^{m+3}} = \frac{4a + b - 1}{2^{m+3}}$$

$w_{c[a_1]} = \frac{2a + [\frac{b-1}{2}]}{2^{m+2}}$, b-1<>0 mod 4, or

$w_{c[a_1]} = \frac{a + [\frac{b-1}{4}]}{2^{m+1}}$, b-1≡ 0 mod 4, in which b is the corresponding odd of a in $(\times 3 + 1) \div 2^k$ sequence, b-1 reflects the 0-bits in the tail part of a.

Then Collatz Conjecture can be described as: With any odd a in range of 2^k to 2^{k+1}-1, set its initial goal set is 2^{j+1}-1(j<=k), its tail part is b, do operation: try to do (b-1) divided by 4, if can not, shift left one bit of a, plus the result of shifting right one bit of b(the 0-bits in the tail part of a), and add 2^{k+2}-1 to goals set of a, this operation makes the 0-bits in the tail part of a shift right or count reduce; if can, a plus the result of (b-1) divided by 4, this operation not only makes the 0-bits in the tail part of a shift right or count reduce, but also reduces the odds count about 1/4 to its goal 2^{k+1}-1, furthermore, if the last result is even, it can reduce a fraction of using 2^{k+1} as denominator, this makes it can reach its previous goal 2^{j+1}-1(j<=k) possibly. Do these operations repeatedly, it has unlimited chances to reach to one of its goal set.

Through above we know, if $(\times 3+1)\div 2^k$ sequence have only /2 and(or) /4 cases, the sequence can never converge, /2 case makes goal of a in $(\times 3+2^m-1)\div 2^k$ sequence larger, /4 case needs ∞ steps. But it is not possible in long sequence, this is determined by the regularity of tail binary bits of odd doing $(\times 3+1)\div 2^k$ operation. Odds of form *10…01(many 0), result can do /4, Odds of form *11…11(many 1), result can do /2, these two cases must become other forms after several steps. Odds with other forms, themselves and their following steps must appear alternately /2, /4, /2^k(k>2) cases.

6 (*3+2^m-1)/2^k Odd Tree and Its Convergence Regularity

6.1 (*3+2^m-1)/2^k Odd Tree and Its Characters

We call 2^k are the properties of odds after doing $(\times 3+2^m-1)\div 2^k$ operation. See following tree:

…

L6: 129(321.1) 131(81.3) 133(327.1) 135(165.2) 137(333.1) 139(21.5) 141(339.1) 143(171.2) 145(345.1) 147(87.3) 149(351.1) 151(177.2) 153(357.1) 155(45.4) 157(363.1) 159(183.2) 161(369.1) 163(93.3) 165(375.1) 167(189.2) 169(381.1) 171(3.8) 173(387.1) 175(195.2) 177(393.1) 179(99.3) 181(399.1) 183(201.2) 185(405.1) 187(51.4) 189(411.1) 191(207.2) 193(417.1) 195(105.3) 197(423.1) 199(213.2) 201(429.1) 203(27.5) 205(435.1) 207(219.2) 209(441.1) 211(111.3) 213(447.1) 215(225.2) 217(453.1) 219(57.4) 221(459.1) 223(231.2) 225(465.1) 227(117.3) 229(471.1) 231(237.2) 233(477.1) 235(15.6) 237(483.1) 239(243.2) 241(489.1) 243(123.3) 245(495.1) 247(249.2) 249(501.1) 251(63.4) 253(507.1) 255

L5: 65(161.1) 67(41.3) 69(167.1) 71(85.2) 73(173.1) 75(11.5) 77(179.1) 79(91.2) 81(185.1) 83(47.3) 85(191.1) 87(97.2) 89(197.1) 91(25.4) 93(203.1) 95(103.2) 97(209.1) 99(53.3) 101(215.1) 103(109.2) 105(221.1) 107(7.6) 109(227.1) 111(115.2) 113(233.1) 115(59.3) 117(239.1) 119(121.2) 121(245.1) 123(31.4) 125(251.1) 127

L4: 33(81.1) 35(21.3) 37(87.1) 39(45.2) 41(93.1) 43(3.6) 45(99.1) 47(51.2) 49(105.1) 51(27.3) 53(111.1) 55(57.2) 57(117.1) 59(15.4) 61(123.1) 63

L3: 17(41.1) 19(11.3) 21(47.1) 23(25.2) 25(53.1) 27(7.4) 29(59.1) 31

L2: 9(21.1) 11(3.4) 13(27.1) 15

L1: 5(11.1) 7

L0: 3

In above tree, a.b in () means result is $a*2^b$ after front odd doing $(\times 3+2^m-1)\div 2^k$ operation, m_th layer has 2^m elements, the last element is the convergence state. Characters of 2^k are very regular, for example, upward from a specific layer, positions of 2 are 1+2i(i>=0), upward from another specific layer, positions of 2^2 are 4+4i, positions of 2^3 are 2+8i, positions of 2^4 are 14+16i…, this can be easily proved strictly. For example, odds of position 2+8i in m layer are 2^{m+1}-1+(2+8i)*2, (0=<i<=[(2^{m-1}-1)/4]).

$$3\times(2^{m+1}-1+(2+8i)\times 2)+2^{m+2}-1=2^{m+3}+2^{m+1}+48i+8$$

Can be divided by 2^3, result is odd if m+1>3. And because the highest bit of the result odd is 2^m, it must be in m-1 layer, downward one layer from m layer.

Through above, we can easily prove that if the property of an odd is 2^1, it moves upward one layer(and also moves forward some location), if the property of an odd is 2^2, it moves forward in the same layer, if the property of an odd is 2^k(k>2), it moves downward k-2 layers(and also moves forward some location).

In this tree, because element count of each layer is 2 times of which of the downward layer, we can transform all positions to one specific layer, m-2 layer transform to m-1 layer do $\times 2$, m layer transform to m-1 layer do $\div 2$, etc. Then all transform positions(to m-1 layer) can not exceed 2^{m-1}!

Next try to prove odds in any layer can converge. We know loop odd sequence and divergence odd sequence both are long non-convergence sequence(expand loop sequence in $(\times 3+2^m-1)\div 2^k$ sequence). Below suppose the research sequence is long huge(odds in $(\times 3+1)\div 2^k$ sequence are huge, for example $>2^{10}+1$) sequence.

6.2 Transform Position Sequence and Its Convergence

Suppose a is an odd in m-1 layer, its highest bit is 2^m.

Position of a in m-1 layer is: $\frac{a-2^m+1}{2}$,

$3\times a+2^{m+1}-1=b\times 2^{p_1}$, b is in layer $m-p_1+1$

Position of b in $m-p_1+1$ layer is: $\frac{b-2^{m-p_1+2}+1}{2}$,

Position of b in m-1 layer is: $\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}$

$3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}=c\times 2^{p_1+p_2}$, c is in layer $m+3-p_1-p_2$

Position of c in $m+3-p_1-p_2$ layer is: $\frac{c-2^{m+4-p_1-p_2}+1}{2}$

Position of c in m-1 layer is: $\frac{c-2^{m+4-p_1-p_2}+1}{2^{5-p_1-p_2}}$

$$\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}-\frac{a-2^m+1}{2}=\frac{b+1-2^{2-p_1}\times a-2^{2-p_1}}{2^{3-p_1}}$$

$$\frac{c-2^{m+4-p_1-p_2}+1}{2^{5-p_1-p_2}}-\frac{b-2^{m-p_1+2}+1}{2^{3-p_1}}=\frac{c+1-2^{2-p_2}\times b-2^{2-p_2}}{2^{5-p_1-p_2}}$$, ratio p is:

$$p=\frac{c+1-2^{2-p_2}\times b-2^{2-p_2}}{2^{5-p_1-p_2}}\times\frac{2^{3-p_1}}{b+1-2^{2-p_1}\times a-2^{2-p_1}}=\frac{c\times 2^{p_2}+2^{p_2}-2^2\times b-2^2}{2^2}\times\frac{2^{p_1}}{b\times 2^{p_1}+2^{p_1}-2^2\times a-2^2}$$

$$=\frac{c\times 2^{p_1+p_2}+2^{p_1+p_2}-2^{2+p_1}\times b-2^{2+p_1}}{2^2}\times\frac{1}{3\times a+2^{m+1}-1+2^{p_1}-2^2\times a-2^2}$$

$$=\frac{3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}+2^{p_1+p_2}-2^{2+p_1}\times b-2^{2+p_1}}{2^2}\times\frac{1}{2^{m+1}+2^{p_1}-a-5}$$

$$=\frac{3^2\times a+3\times 2^{m+1}-3+2^{m+3}-2^{p_1}+2^{p_1+p_2}-2^2\times(3\times a+2^{m+1}-1)-2^{2+p_1}}{2^2}\times\frac{1}{2^{m+1}+2^{p_1}-a-5}$$

$$=\frac{3\times 2^{m+1}-3\times a-5\times 2^{p_1}+2^{p_1+p_2}+1}{2^2}\times\frac{1}{2^{m+1}+2^{p_1}-a-5}$$

$$=\frac{3\times(2^{m+1}+2^{p_1}-a-5)+2^{p_1+p_2}-8\times 2^{p_1}+16}{2^2\times(2^{m+1}+2^{p_1}-a-5)}$$

$$=\frac{3}{4}+\frac{2^{p_1+p_2}-8\times 2^{p_1}+16}{2^2\times(2^{m+1}+2^{p_1}-a-5)}=\frac{3}{4}+\frac{2^{p_1}\times(2^{p_2}-8)+16}{2^2\times(2^{m+1}+2^{p_1}-a-5)}$$

Because in long huge sequence, $2^{m+1}-a$ is very big, the ratio p is very close to 3/4. Only these cases ratio p<3/4: $p_2=1$, $p_1>=2$; $p_2=2$, $p_1>=3$. This is to say: cases of (forward, upward), (downward, upward), (downward, forward) ratio <3/4; cases of (upward, upward), (downward, downward), (upward, downward), (upward, forward), (forward, downward) ratio >3/4; case of (forward, forward) ratio =3/4. Obviously, the ratio regularity is also suitable for all following steps.

Generally, Transform position increment from odd a to b is:

$$\Delta = \frac{b - 2^{m-p_1+2} + 1}{2^{3-p_1}} - \frac{a - 2^m + 1}{2} = \frac{b + 1 - 2^{2-p_1} \times a - 2^{2-p_1}}{2^{3-p_1}}$$

$$= \frac{b \times 2^{p_1} + 2^{p_1} - 2^2 \times a - 2^2}{2^3} = \frac{3 \times a + 2^{m+1} - 1 + 2^{p_1} - 2^2 \times a - 2^2}{2^3}$$

$$= \frac{2^{m+1} - a + 2^{p_1} - 5}{2^3}$$

Only when $2^{m+1} - a = 1$, $p_1 = 2$ or $2^{m+1} - a = 3$, $p_1 = 1$, the transform position increment is equal to 0, these two cases are convergence state or quasi convergence state. Other cases transform position increment is bigger than 0(even in expand loop sequence). The bigger p_1 is, the bigger position increment will get.

We can deduce the common transform position formula:

$$s_0 = \frac{a - 2^m + 1}{2} = 2^{m-1} - \frac{2^{m+1} - a - 1}{2} = 2^{m-1} + \frac{1 - b_0}{2}$$

$$s_1 = \frac{3 \times a + 2^{m+1} - 1 - 2^{m+2} + 2^{p_1}}{2^3} = 2^{m-1} - \frac{3^1 \times (2^{m+1} - a) + 1 - 2^{p_1}}{2^{2\times1+1}} = 2^{m-1} + \frac{(1 - b_1) \times 2^{p_1}}{2^{2\times1+1}}$$

$$s_2 = \frac{3^2 a + 3 \times 2^{m+1} - 3 + 2^{m+3} - 2^{p_1} - 2^{m+4} + 2^{p_1+p_2}}{2^5} = 2^{m-1} - \frac{3^2 \times (2^{m+1} - a) + 3 + 2^{p_1} - 2^{p_1+p_2}}{2^{2\times2+1}} = 2^{m-1} + \frac{(1 - b_2) \times 2^{p_1+p_2}}{2^{2\times2+1}}$$

...

$$s_i = 2^{m-1} + \frac{2^{p_1+p_2+\dots p_i} - 3^i \times (2^{m+1} - a) - 3^{i-1} - 3^{i-2} \times 2^{p_1} - \dots 2^{p_1+p_2+\dots p_{i-1}}}{2^{2i+1}},$$

$$= 2^{m-1} + \frac{(1 - b_i) \times 2^{p_1+p_2+\dots p_i}}{2^{2i+1}}$$

in which b_i is the corresponding odd in $(\times 3 + 1) \div 2^k$ sequence.

Continue to calculate:

$$s_{(1,1)} - s_{(2)} = \frac{2^2 - 3^2 \times (2^{m+1} - a) - 3 - 2}{2^5} - \frac{2^2 - 3 \times (2^{m+1} - a) - 1}{2^3} = \frac{3 \times (2^{m+1} - a) - 13}{2^5} > 0,$$

When in long huge sequence, where $s_{(2)}$ is virtual transform position.

$$s_{(1,2,2)} - s_{(2,2)} = \frac{2^5 - 3^3 \times (2^{m+1} - a) - 3^2 - 3^1 \times 2 - 2^3}{2^7} - \frac{2^4 - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5}$$

$$= \frac{2^5 - (3^3 \times (2^{m+1} - a) + 3^2 + 3^1 \times 2 + 2^3) - 4 \times 2^4 + 4 \times (3^2 \times (2^{m+1} - a) + 3 + 2^2)}{2^7}$$

$$= \frac{3^2 \times (2^{m+1} - a - 3)}{2^7} > 0$$

$$s_{(1,2,2,2)} - s_{(2,2,2)} = \frac{2^{5+2} - 3 \times (3^3 \times (2^{m+1} - a) + 3^2 + 3^1 \times 2 + 2^3) - 2^5}{2^{7+2}} - \frac{2^{4+2} - 3 \times (3^2 \times (2^{m+1} - a) + 3 + 2^2) - 2^4}{2^{5+2}}$$

$$= \frac{3 \times 2^5 - 3 \times (3^3 \times (2^{m+1} - a) + 3^2 + 3^1 \times 2 + 2^3)}{2^{7+2}} - \frac{3 \times 2^4 - 3 \times (3^2 \times (2^{m+1} - a) + 3 + 2^2)}{2^{5+2}}$$

$$= \frac{3}{4} \times \frac{3^2 \times (2^{m+1} - a - 3)}{2^7} > 0$$

$$s_{(1,2,2,2,2)} - s_{(2,2,2,2)} = \frac{3}{4} \times \frac{3}{4} \times \frac{3^2 \times (2^{m+1} - a - 3)}{2^7} > 0$$

...

When in long huge sequence.

Similar:

$$s_{(1,1,2,2)} - s_{(2,2)} = \frac{2^6 - 3^4 \times (2^{m+1} - a) - 3^3 - 3^2 \times 2 - 3 \times 2^2 - 2^4}{2^9} - \frac{2^4 - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5}$$

$$= \frac{3^2 \times (7 \times (2^{m+1} - a) - 17)}{2^9} > 0$$

$$s_{(1,1,2,2,2)} - s_{(2,2,2)} = \frac{3}{4} \times \frac{3^2 \times (7 \times (2^{m+1} - a) - 17)}{2^9} > 0$$

$$s_{(1,1,2,2,2,2)} - s_{(2,2,2,2)} = (\frac{3}{4})^2 \times \frac{3^2 \times (7 \times (2^{m+1} - a) - 17)}{2^9} > 0$$

...

When in long huge sequence.

This means $s_{(1,1)} > s_{(2)}$ and $s_{(1,1,2,2)} - s_{(2,2)}$ for same start odd a, then can get conclusion $s_{(p_1 \dots p_i,1,1)} > s_{(p_1 \dots p_i,2)}$ and $s_{(p_1 \dots p_i,1,1,2,2)} > s_{(p_1 \dots p_i,2,2)}$ as long as $2^{m+1} - a$ do i steps $(\times 3 + 1) \div 2^k$ is still huge. Furthermore, can get following conclusions:

$$s_{(p_1 \dots p_i,1\dots,1,1,1)} > s_{(p_1 \dots p_i,1\dots,1,2)} > s_{(p_1 \dots p_i,1\dots,1,1)} > s_{(p_1 \dots p_i,1\dots,2)} > \dots > s_{(p_1 \dots p_i,2)}$$

$s_{(p_1 \dots p_i,1\dots,1,2^+)} > s_{(p_1 \dots p_i,1\dots,1,1)} > s_{(p_1 \dots p_i,1\dots,2)} > \dots > s_{(p_1 \dots p_i,2)}$, in which $2^+ >= 2$

$s_{(p_1 \dots p_i,1\dots,1,2^+,1\dots,1,2^+)} > s_{(p_1 \dots p_i,1\dots,1,2^+,2)} \geq s_{(p_1 \dots p_i,1\dots,1,2,2)} > s_{(p_1 \dots p_i,2,2)}$,

and more $s_{(p_1 \dots p_i,1\dots,1,2^+,1\dots,1,2^+,1\dots,1,2^+)} \dots$

Obviously, $s_{(2^+,2)} - s_{(2,2)} = \frac{2^{2^+ + 2} - 3^2 \times (2^{m+1} - a) - 3 - 2^{2^+}}{2^5} - \frac{2^{2+2} - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5} = \frac{3 \times 2^{2^+} - 3 \times 2^2}{2^5} \geq 0$

This is to say, if we delete all upward steps((1) steps) between two (2^+) steps in long huge sequence and change all (2^+) steps to (2) steps, final virtual transform position is smaller than original.

Suppose $p_1 = 2^+$, last step is (2^+), exists many (1) and (2^+) steps in the middle, rebuild a new z steps sequence from original using above method.

Not like 2^{m+1}-a is always huge odd in each step in original sequence as we supposed. In new virtual sequence it can become small or decimal, but its value must always >1 before ∞ steps. Because if 2^{m+1}-a=1+x(x>0), look:

$$\frac{(1 + x) \times 3 + 1}{4} = 1 + \frac{3x}{4} > 1 .$$

Then in any step, $\Delta_{new} = \frac{2^{m+1} - a + 2^2 - 5}{2^3} > 0$, ratio

$$p_{new} = \frac{3}{4} + \frac{2^2 \times (2^2 - 8) + 16}{2^2 \times (2^{m+1} + 2^2 - a - 5)} = \frac{3}{4} .$$

Hence the transform position increment ratio of new sequence is always 3/4. Use new sequence to estimate the transform position of original sequence. Final transform position of original sequence is(can also be gotten from the common transform position formula):

$$s_{final} > s_{new} = \frac{a - 2^m + 1}{2} + (\frac{2^{m+1} - a + 2^2 - 5}{2^3}) \times (1 + \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3 + \dots + (\frac{3}{4})^{z-1})$$

If long huge sequence is non-convergence, it must appear (2^+) steps continuously after some upward steps each time, the count of (2^+) steps must be infinite.

When z->∞,

$$s_{final} > s_{new} = \frac{a - 2^m + 1}{2} + (\frac{2^{m+1} - a - 1}{2^3}) \times 4 = 2^{m-1}$$

Walk out of boundary of the tree. This means, the transform position of original sequence must reach to or become bigger than 2^{m-1} before a limit steps. Long huge sequence must become a small sequence(once one element becomes a small odd in our range, the sequence becomes), or converge before a limit steps, otherwise overstep the boundary of the tree(it is not possible in real world).

Still has one puzzle, the equivalent elements of elements in m-1 layer(by adding binary 1 before head) are all in right half part in m layer, it seems to exist many loops. It is of course not correct, this is because, although they are equivalent, their functions are different. Other odds can change to them, they can also converge. If some long sequence exist loops, the transform position(to m-1 layer) must become bigger than 2^{m-1}, it is contradictory.

Maybe it is possible to use proportional sequence of ratio 3/4 to estimate the convergence steps for long huge sequence. Use number property $2^{p_1} = 8$ or $2^{p_1} = 16$ of an odd of m-1 layer as start odd, final transform position must be bigger than 2^{m-1} in limit steps n using ratio 3/4, indicates that the convergence step count should be smaller than n multiply a number(because we delete some upward steps, the suitable value of the number is difficult to get, but should not be very large).

7 Conclusion

This way, we have proved that the Collatz Conjecture is true.

References

[1]Wikipedia, TheFreeDictionary.com mirror. Collatz Conjecture. la.thefreedictionary.com