Nontrivial zeros of the Riemann zeta function

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ABSTRACT

The Riemann hypothesis, stating that all nontrivial zeros of the Riemann zeta function have real parts equal to $\frac{1}{2}$, is one of the most important conjectures in mathematics. In this paper we prove the Riemann hypothesis by adding an extra unbounded term to the traditional definition, extending its validity to $\operatorname{Re} z > 0$. The Stolz-Cesàro theorem is then used to analyse $\zeta(z)/\zeta(1-z)$ as a ratio of complex sequences. The results are analysed in both halves of the critical strip ($\operatorname{Re} z \in \left(0, \frac{1}{2}\right)$, $\operatorname{Re} z \in \left(\frac{1}{2}, 1\right)$), yielding a contradiction when it is assumed that $\zeta(z) = 0$ in either of these halves.

Keywords: Nontrivial zeros, Riemann functional equation, Riemann zeta function.

1. INTRODUCTION

In 1859 Bernhard Riemann published an article titled, *On the number of primes less than a given quantity* [1]. In that work he speculated that all complex valued nontrivial zeros of the zeta function have a real part equal to $\frac{1}{2}$. This became known as the Riemann hypothesis. Ever since then, mathematicians have endeavoured to prove it. In 1900 David Hilbert included it in his list of 23 most important problems of the twentieth century. And since 2000 it has remained as one of six millennium problems.

The Riemann zeta function with real part, $\operatorname{Re} z > 1$ is traditionally defined by the infinite sum

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \mathbb{C} \cap \{\operatorname{Re} z > 1\}. \tag{1}$$

In this work we start with an integral form of the zeta function, which is analytically continued to the imaginary axis but excludes the only pole at $z = 1$. This form is given by [2]

$$\zeta(z) = \frac{z}{z-1} - z\int_1^{\infty} \{x\} x^{-z-1} dx, \quad z \neq 1, \ \operatorname{Re} z > 0. \tag{2}$$

We further rely on Riemann's functional equation, which implies symmetry on the positions of nontrivial zeros about the critical line ($\operatorname{Re} z = \frac{1}{2}$), and allows the zeta function to be analytically continued to the whole complex plane. This is given by [1]

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z) \tag{3}$$

where

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \quad \operatorname{Re} z > 0$$

is the gamma function extending the factorial function to the entire complex plane. From equation (3) it is straightforward to determine the trivial zeros at $z = -2n, \ n \in \mathbb{N} \setminus \{0\}$. All other zeros are known to lie only within the critical strip defined by $0 < \operatorname{Re} z < 1$ [2].

In this work we prove the Riemann hypothesis in four stages. The first is to write the zeta function in terms of the infinite sum shown in equation (1), which is extended to $\operatorname{Re} z > 0$ by the addition of an extra divergent term. By examining the properties of equation (3) it is shown that $\zeta(z)/\zeta(1-z)$ has no poles or zeros within the critical strip. In the third stage we apply the Stolz-Cesàro theorem [3,4] in the analysis of complex sequences converging to the zeta function. Finally, all of the preceding analysis is drawn together. When we assume $\zeta(z) = 0$ in either half of the critical strip ($\operatorname{Re} z \in (0, \frac{1}{2})$, $\operatorname{Re} z \in (\frac{1}{2}, 1)$) we obtain a contradiction to known properties of the functional equation, leaving only the critical line where zeros may be found.

2. PROOF OF THE RIEMANN HYPOTHESIS

The first task is to prove the following theorem:

Theorem 1: The Riemann zeta function for $\operatorname{Re} z > 0$, in terms of the traditional infinite sum is given by

$$\zeta(z) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z}\right), \quad \operatorname{Re} z > 0. \tag{4}$$

Proof:

We begin with the following known relationship [2, p8]

$$\zeta(z) = \frac{z}{z-1} - z\int_1^\infty \{x\}\, x^{-z-1}dx, \quad \operatorname{Re} z > 0. \tag{2}$$

The first thing to note here is that since $\{x\} = x - \lfloor x \rfloor$ implying that $0 \le \{x\} < 1$, then

$$\int_1^\infty \left|\{x\}\, x^{-z-1}\right| dx < \int_1^\infty \left|x^{-z-1}\right| dx .$$

Because $\operatorname{Re}(z+1) > 1$, the integral on the right of this inequality converges. Therefore, the integral in equation (2) converges, hence defining the zeta function as meromorphic for $\operatorname{Re} z > 0$ with a single pole at $z = 1$. Moreover, the resulting analytic continuation to the imaginary axis is unique [5, p362].

The next step is to rewrite the integral as

$$\int_1^\infty \{x\}\, x^{-z-1}dx = \lim_{N\to\infty}\int_1^N \{x\}\, x^{-z-1}dx$$

while noting that $N \to \infty$. Integrating by parts gives

$$\int_1^N \{x\}\, x^{-z-1}dx = \left[\frac{\{x\}\, x^{-z}}{-z}\right]_1^N - \int_1^N \frac{x^{-z}}{-z}\{x\}'\, dx .$$

And by differentiating the fractional part of *x*, this becomes

$$\int_1^N \{x\}\, x^{-z-1}dx = \left[\frac{\{x\}\, x^{-z}}{-z}\right]_1^N - \int_1^N \frac{x^{-z}}{-z}\left(1 - \sum_{n\in\mathbb{Z}}\delta(x-n)\right)dx .$$

The first term on the right hand side vanishes due to the real part of *z* being positive and $\{1\} = 0$ at the lower limit. By directly evaluating the integral on the right hand side, the integral on the left becomes

$$\int_1^N \{x\} x^{-z-1} dx = -\left[\frac{x^{1-z}}{-z(1-z)} \right]_1^N - \frac{1}{z}\sum_{n=2}^{N} n^{-z}$$
$$= \frac{N^{1-z}-1}{z(1-z)} - \frac{1}{z}\sum_{n=2}^{N} n^{-z}.$$

Note that the lower limit on the sum starts at $n=2$. This is due to the lower limit on the integral being 1, where the discontinuity in $\{x\}$ appearing there is not included. The first discontinuity is therefore seen at $x=2$ corresponding to $n=2$ in the sum.

Substituting back into equation (2) we get

$$\zeta(z) = \frac{z}{z-1} - \frac{N^{1-z}-1}{1-z} + \sum_{n=2}^{N} n^{-z} = \frac{z}{z-1} + \frac{1}{1-z} - \frac{N^{1-z}}{1-z} + \sum_{n=2}^{N} n^{-z}.$$

Here we note that the sum starting at $n=2$ is just the sum starting at $n=1$ minus 1. Therefore, using

$$\sum_{n=2}^{N} n^{-z} = \sum_{n=1}^{N} n^{-z} - \frac{z-1}{z-1}$$

we have

$$\zeta(z) = \sum_{n=1}^{N} n^{-z} - \frac{z-1}{z-1} + \frac{z}{z-1} - \frac{1}{z-1} - \frac{N^{1-z}}{1-z} = \sum_{n=1}^{N} n^{-z} - \frac{z-1}{z-1} + \frac{z-1}{z-1} - \frac{N^{1-z}}{1-z}$$
$$= \sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z}, \quad N \to \infty, \quad \operatorname{Re} z > 0.$$

This shows the validity of equation (4) extending the traditional definition to $\operatorname{Re} z > 0$. Moreover, we see that this reduces to the original definition for $\operatorname{Re} z > 1$ due to the vanishing second term, hence reinforcing the statement that equation (2) is an analytic continuation of equation (1).

□

A key part of the analysis to follow relies on certain properties of the Riemann functional equation. Rearranging equation (3) as

$$F_z \equiv \frac{\zeta(z)}{\zeta(1-z)} = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \tag{5}$$

we quote and prove the following property:

Theorem 2: In equation (5), $|F_z| \in (0,\infty)$ for all points within the critical strip.

Proof:

In equation (5) we need to show that F_z is bounded and nonzero within the critical strip. The first two factors on the right hand side are real multiples of the form, $a^z,\ a \in \mathbb{R}^+$. These can be written as $a^\sigma a^{it}$ where $z = \sigma + it$, $\left|a^{it}\right| = 1$ and $a^\sigma \in (0,\infty),\ \sigma \in \mathbb{R}$. Therefore, $2^z \pi^{z-1}$ has no poles or zeros within the critical strip.

Expanding the sine factor we have

$$\begin{aligned}\sin\left(\frac{\pi z}{2}\right) &= \sin\left(\frac{\pi\sigma}{2} + i\frac{\pi t}{2}\right) \\ &= \sin\left(\frac{\pi\sigma}{2}\right)\cos\left(i\frac{\pi t}{2}\right) + \cos\left(\frac{\pi\sigma}{2}\right)\sin\left(i\frac{\pi t}{2}\right) \\ &= \sin\left(\frac{\pi\sigma}{2}\right)\cosh\left(\frac{\pi t}{2}\right) + i\cos\left(\frac{\pi\sigma}{2}\right)\sinh\left(\frac{\pi t}{2}\right).\end{aligned}$$

The first thing to notice is that the hyperbolic functions are bounded for finite values of their arguments. Also, considering the real part of the right hand side, we always have

$\cosh\left(\frac{\pi t}{2}\right) > 0$, and $\sin\left(\frac{\pi\sigma}{2}\right) \in (0,1)$ for $\sigma \equiv \operatorname{Re} z \in (0,1)$. Therefore, we have

$$\left|\sin\left(\frac{\pi z}{2}\right)\right| \in (0,\infty), \quad \operatorname{Re} z \in (0,1).$$

The final factor in F_z is the gamma function, $\Gamma(1-z)$. It is known, from the infinite product representation of $1/\Gamma(z)$,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}, \quad z \in \mathbb{C}$$

where $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \operatorname{Log} n\right)$

and the Weierstrass factor theorem [5, p384], that $1/\Gamma(z)$ is an entire function with simple zeros at 0 and $-n,\ n \in \mathbb{N}$ [5, p394]. Therefore, $\Gamma(z)$ is analytic on $\mathbb{C}\setminus\{0,-n : n \in \mathbb{N}\}$ and has no zeros. Since $\Gamma(z)$ has no poles or zeros within the critical strip, then the same follows for $\Gamma(1-z)$. Having considered all of the factors of F_z, it follows that it has no poles or zeros within the critical strip. Therefore, F_z is both bounded and nonzero within the critical strip.

□

The next stage is to write F_z in terms of the zeta function in the form of equation (4). This gives us

$$F_z = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z}}{\sum_{n=1}^{N} n^{z-1} - \frac{N^{z}}{z}} \tag{6}$$

which takes the form $F_z = \lim_{N\to\infty} a_N / b_N$ where $a_N, b_N \in \mathbb{C}$. In the case that $\zeta(z) = 0$ then this limit takes the form "$0/0$", which we proceed to analyse using the Stolz-Cesàro theorem. However, in its original form, this theorem is only applicable to real valued sequences. Therefore, the theorem is here extended to complex sequences. This is done for the form, "$0/0$", where the limit converges.

Theorem 3, the Stolz-Cesàro theorem for complex sequences: Let $(a_N)_{N\geq 1}$ and $(b_N)_{N\geq 1}$ be two sequences of complex numbers, where $(a_N) \to 0$ and $(b_N) \to 0$ as $N \to \infty$. Then

$$\lim_{N\to\infty} \frac{a_N - a_{N+1}}{b_N - b_{N+1}} = s \quad \Rightarrow \quad \lim_{N\to\infty} \frac{a_N}{b_N} = s \,.$$

Proof:

For $N \to \infty$ we start with

$$\frac{a_N - a_{N+1}}{b_N - b_{N+1}} \to s$$

which we use as the base case in part of a proof by induction. In the inductive step we need to show that

$$\frac{a_N - a_{N+k}}{b_N - b_{N+k}} \to s \quad \Rightarrow \quad \frac{a_N - a_{N+k+1}}{b_N - b_{N+k+k}} \to s \,.$$

Our starting point for the inductive step is

$\frac{a_N - a_{N+k}}{b_N - b_{N+k}} \to s$ and from the base case we have $\frac{a_{N+k} - a_{N+k+1}}{b_{N+k} - b_{N+k+1}} \to s$.

Using elementary properties of limits [5, p110], we have

$$\frac{a_N - a_{N+k} + a_{N+k} - a_{N+k+1}}{b_N - b_{N+k} + b_{N+k} - b_{N+k+1}} \to s \quad \Rightarrow \quad \frac{a_N - a_{N+k+1}}{b_N - b_{N+k+1}} \to s$$

which completes the inductive step. Therefore, we can say that

$$\lim_{\substack{N\to\infty \\ \nu\to\infty}} \frac{a_N - a_{N+\nu}}{b_N - b_{N+\nu}} = s$$

where $N,\nu \to \infty$ independently. By hypothesis, $a_{N+\nu}, b_{N+\nu} \to 0$ as $\nu \to \infty$. Consequently, this reduces to

$$\lim_{N\to\infty} \frac{a_N}{b_N} = s\,.$$

Therefore, we have

$$\lim_{N\to\infty} \frac{a_N - a_{N+1}}{b_N - b_{N+1}} = s \quad \Rightarrow \quad \lim_{N\to\infty} \frac{a_N}{b_N} = s$$

as required. This concludes the proof of the Stolz-Cesàro theorem for complex sequences in the "$0/0$" case.

□

The analysis so far has established a good position from which we can prove the Riemann hypothesis.

Theorem 4, the Riemann hypothesis: All of the nontrivial zeros of the Riemann zeta function contained within the critical strip, $0 < \operatorname{Re} z < 1$, have real parts equal to $\frac{1}{2}$.

Proof:

We begin by simplifying

$$\frac{a_{N-1} - a_N}{b_{N-1} - b_N} = \frac{a_N - a_{N-1}}{b_N - b_{N-1}} = \frac{\displaystyle\sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z} - \sum_{n=1}^{N-1} n^{-z} + \frac{(N-1)^{1-z}}{1-z}}{\displaystyle\sum_{n=1}^{N} n^{z-1} - \frac{N^{z}}{z} - \sum_{n=1}^{N-1} n^{z-1} + \frac{(N-1)^{z}}{z}}\,. \tag{7}$$

Evaluating the numerator, we have

$$
\begin{aligned}
a_N - a_{N-1} &= \sum_{n=1}^{N} n^{-z} - \sum_{n=1}^{N-1} n^{-z} - \frac{N^{1-z}}{1-z} + \frac{(N-1)^{1-z}}{1-z} \\
&= N^{-z} - \frac{N^{1-z} - (N-1)^{1-z}}{1-z} \\
&= N^{-z} - \frac{N^{1-z} - N^{1-z}\left(1-N^{-1}\right)^{1-z}}{1-z} \\
&= N^{-z} - N^{1-z}\frac{1-\left(1-N^{-1}\right)^{1-z}}{1-z} \\
&= N^{-z} - N^{1-z}\frac{1-\left(1-(1-z)N^{-1} - \tfrac{1}{2}z(1-z)N^{-2} + \cdots\right)}{1-z} \\
&= N^{-z} - N^{1-z}\frac{\left((1-z)N^{-1} + \tfrac{1}{2}z(1-z)N^{-2} + \cdots\right)}{1-z} \\
&= N^{-z} - N^{-z}\frac{\left((1-z) + \tfrac{1}{2}z(1-z)N^{-1} + \cdots\right)}{1-z} \\
&= N^{-z} - N^{-z} - \tfrac{1}{2}zN^{-z-1} + \cdots \\
&= N^{-z}\left(-\tfrac{1}{2}zN^{-1} + \cdots\right).
\end{aligned}
$$

Similarly

$$
b_N - b_{N-1} = N^{z-1}\left(-\tfrac{1}{2}(1-z)N^{-1} + \cdots\right).
$$

By taking the ratio of the dominant terms top and bottom, we have

$$
\frac{a_N - a_{N-1}}{b_N - b_{N-1}} \to \frac{N^{-z}}{N^{z-1}}\left(\frac{z}{1-z}\right) = N^{1-2z}\left(\frac{z}{1-z}\right) \text{ as } N \to \infty
$$

where we note that $0 < \left|z/(1-z)\right| < \infty$.

Hence, by theorem 3, we can state that whenever $\zeta(z) = 0$ then

$$
\left|\frac{\sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z}}{\sum_{n=1}^{N} n^{z-1} - \frac{N^{z}}{z}}\right| = N^{1-2\sigma}\left|\frac{z}{1-z}\right| \to \infty \text{ as } N \to \infty, \quad \sigma \equiv \operatorname{Re} z < \frac{1}{2} \tag{8a}
$$

$$
\frac{\sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z}}{\sum_{n=1}^{N} n^{z-1} - \frac{N^{z}}{z}} = N^{1-2z}\frac{z}{1-z} = 0, \quad \operatorname{Re} z > \frac{1}{2}. \tag{8b}
$$

If we contradict the statement of the Riemann hypothesis and assume that $\zeta(z)=0$ in the lower half of the critical strip, then a divergent equation (8a) implies that

$$\left|2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\right| \to \infty, \quad \operatorname{Re} z < \frac{1}{2}. \tag{9a}$$

Similarly, in the upper half of the critical strip, from equation (8b), we have

$$2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z) = 0, \quad \operatorname{Re} z > \frac{1}{2}. \tag{9b}$$

But because F_z remains bounded everywhere within the critical strip, equation (9a) reveals a contradiction to the assumption that $\zeta(z)=0$ when $\operatorname{Re} z < \frac{1}{2}$. Moreover, since the pattern of zeros is symmetrical about the critical line, there can be no zeros when $\operatorname{Re} z > \frac{1}{2}$ either; which is borne out by equation (9b). Therefore, the only place where zeros can exist within the critical strip, is on the critical line. This concludes the proof of the Riemann hypothesis.

□

Before concluding, we make a few remarks about phase. From the preceding analysis, we note that

$$\frac{a_N - a_{N-1}}{b_N - b_{N-1}} = N^{1-2z}\left(\frac{z}{1-z}\right) \tag{10}$$

which is derived from the assumption that $\zeta(z)=0$. The right hand side can be expanded to read

$$N^{1-2z}\left(\frac{z}{1-z}\right) = N^{1-2\sigma}\exp(-i2t\ln N)\frac{\sqrt{\sigma^2+t^2}\exp(i\theta)}{\sqrt{(1-\sigma)^2+t^2}\exp(i\phi)}$$

where $z=\sigma+it$, $\theta=\tan^{-1}(t/\sigma)$ and $\phi=\tan^{-1}(-t/(1-\sigma))$. Therefore, the overall phase is given by

$$\arg\left[N^{1-2z}\left(\frac{z}{1-z}\right)\right] = \tan^{-1}\left(\frac{t}{\sigma}\right) - \tan^{-1}\left(\frac{-t}{1-\sigma}\right) - 2t\ln N \,.$$

Consequently, because the term, $-2t\ln N$, diverges the phase is infinitely sensitive to small changes in *t*. This means that the phase can be made to fit that of equation (5) whenever $\zeta(z)=0$, irrespective of the value *t* takes. Furthermore, the right hand side of equation (10) matches F_z in equation (5) by modulus everywhere on the critical line as one would expect. However, this does not provide any straightforward way to compute the positions of zeros along the critical line.

3. SUMMARY OF THE PROOF

Given the conditions imposed by equations (4, 5) the proof presented here can be summarised with a single expression, given by

$\zeta(z) = 0 \;\Rightarrow\;$ Equations (8a,b) $\;\Rightarrow\; \operatorname{Re} z = \frac{1}{2}$.

Or equivalently

$\operatorname{Re} z \neq \frac{1}{2} \;\Rightarrow\; \neg$Equations (8a,b) $\;\Rightarrow\; \zeta(z) \neq 0$.

In this work we followed the path

$\zeta(z) = 0 \;\Rightarrow\;$ Equations (8a,b)

followed by

$\operatorname{Re} z \neq \frac{1}{2} \;\Rightarrow\; \neg$Equations (8a,b).

4. CONCLUSION

In this paper we have demonstrated the Riemann hypothesis to be true and that the real parts of all nontrivial zeros are $\frac{1}{2}$. By writing the zeta function in a form that includes the traditional sum extended to $\operatorname{Re} z > 0$, with an appropriate application of the Stolz-Cesàro theorem to complex sequences, it has been shown that, due to a contradiction, no nontrivial zeros can exist when $\operatorname{Re} z \neq \frac{1}{2}$. Otherwise equation (5) would be required to diverge/vanish in the lower/upper half of the critical strip. Since it is established that it does neither, the only occurrences of zeros within the critical strip are at $\operatorname{Re} z = \frac{1}{2}$ as hypothesized.

REFERENCES

[1] Riemann B, 1859, *Ueber die Anzahl der Primzahlen unter einer gegebenen Groesse*, (*On the number of primes less than a given quantity*), Monatsberichte der Berliner Akademie.

[2] Heymann Y, 2020, *An investigation of the non-trivial zeros of the Riemann zeta function*, ArXiv:1804.04700v15.

[3] Stolz O, 1885, *Vorlesungen über allgemeine Arithmetik: nach den Neueren Ansichten*, Leibzig: Teubners, 173-175. https://archive.org/stream/vorlesungenbera01stolgoog#page/n181/mode/1up

[4] Cesàro E, 1888, *Sur la convergence des séries*, Nouvelles annals de mathématiques, Series 3, **7**, 49-59.

[5] Osborne AD, 1999, *Complex Variables and Their Applications*, Addison-Wesley.