Irrationality of $\pi$
Using Just Derivatives

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Abstract

The quest for an irrationality of pi proof that can be incorporated into an analysis (or a calculus) course is still extant. Ideally a proof would be well motivated and use in an interesting way the topics of such a course. In particular $e^{\pi i}$ should be used and the more easily algebraic of derivatives and integrals – i.e. derivatives. A further worthy goal is to use techniques that anticipate those needed for other irrationality and, maybe even, transcendence proofs. We claim to have found a candidate proof.

Introduction

Invariably irrationality proofs use proof by contradiction. The number in question is assumed to be rational and a contradiction is derived. Why does this work? It works because irrational numbers are always changing; their tails change. Assuming that they don’t change, that all zeros or 9s occur, eventually the approximation implicit in an irrational number represented by a rational becomes large enough that it is manifest that the fixed assumption can’t work: there’s a contradiction.

A combination of polynomials with fixed roots and ever changing partial sums of series seem a likely avenue to an irrationality proof. This is especially true as series in the form of a power series or $e^x$ or $e^{ix}$ have partials that double as polynomials. Assuming the polynomial has a certain root and that the series for which the polynomial is a partial is also converging to this number should work to generate the schism mentioned. A natural candidate that embodies these ideas is Euler’s famous formula:

$$e^{\pi i} - 1 = 0.$$
Derivatives of Polynomials

All polynomials are integer polynomials, \( z \) is a complex number, \( n \) and \( j \) are non-negative integers, and \( p \) is a prime number.

**Definition 1.** Given a polynomial \( f(z) \), lowercase, the sum of all its derivatives is designated with \( F(z) \), uppercase.

**Example 1.** If \( f(z) = cz^n \) then

\[
F(z) = \sum_{k=0}^{n} f^{(k)}(z) = cz^n + cnz^{n-1} + cn(n-1)z^{n-2} + \cdots + cn!.
\]

**Lemma 1.** If \( f(z) = cz^n \), then

\[
F(0) = \sum_{k=1}^{\infty} \frac{z^k n!}{(n+k)!}.
\]

**Proof.** As \( F(z) = c(z^n + nz^{n-1} + \cdots + n!) \), \( F(0) = cn! \). Thus,

\[
F(0) = cn!(1 + z/1 + z^2/2! + \cdots + z^n/n! + \cdots)
\]

\[
= cz^n + cnz^{n-1} + \cdots + cn! + cz^{n+1}/(n+1)! + \cdots
\]

\[
= F(z) + cz^n(z/(n+1) + z^2/(n+1)(n+2) + \cdots)
\]

\[
= F(z) + f(z) \sum_{k=1}^{\infty} \frac{z^k n!}{(n+k)!},
\]

giving (1).

**Definition 2.** Let

\[
\delta_n! = \sum_{k=1}^{\infty} \frac{z^k n!}{(n+k)!}.
\]

**Lemma 2.**

\[
\lim_{p \to \infty} \frac{\delta_n!}{(p-1)!} = 0.
\]

**Proof.** We have

\[
\left| \frac{\delta_n!(z)}{(p-1)!} \right| = \left| \frac{z/(n+1) + z^2/(n+1)(n+2) + \cdots e^z}{(p-1)!} \right| < \left| \frac{e^z}{(p-1)!} \right|
\]
and
\[ \lim_{n \to \infty} \left| \frac{e^z}{(p-1)!} \right| = 0. \]
This implies (2). \hfill \Box

**Lemma 3.** If \( F(z) \) is the sum of the derivatives of \( f(z) = c_0 + c_1 z + \cdots + c_n z^n \), then
\[ F(0)e^z = F(z) + \sum_{k=0}^{n} c_k z^k \delta_k(z). \] (3)

**Proof.** Let \( f_j(z) = c_j z^j \), for \( 0 \leq j \leq n \). Using the derivative of the sum is the sum of the derivatives,
\[ F(z) = \sum_{k=0}^{n} (f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n, \]
where \( F_j \) is the sum of the derivatives of \( f_j \). Using Lemma 1,
\[ e^z F_j(0) = F_j(z) + f_j(z) \delta_j(z). \] (4)
and summing (4) from \( j = 0 \) to \( n \), gives
\[ e^z F(0) = F(z) + \sum_{j=0}^{n} f_j(z) \delta_j(z). \]
This is (3). \hfill \Box

**Definition 3.** If \( f_j(z) = c_j z^j \), for \( 0 \leq j \leq n \), then define
\[ \epsilon_n! (f(z)) = \sum_{j=0}^{n} f_j(z) \delta_j(z), \]
where
\[ f(z) = \sum_{j=0}^{n} f_j(z). \]

**Lemma 4.**
\[ \lim_{p \to \infty} \frac{\epsilon_n!(z)}{(p-1)!} = 0. \] (5)
**Proof.** As \( \delta_j(z) < e^z \) for \( j = 0, \ldots, n \),

\[
|\epsilon_n(z)| = \left| \sum_{j=0}^{n} f_j(z) \delta_j(z) \right| \leq e^{|z|} \sum_{j=0}^{n} |f_j(z)| \frac{(p-1)!}{(p-1)!}.
\]

Then, noting

\[
\sum_{j=0}^{n} |f_j(z)| \leq c \sum_{j=0}^{n} |z|^j \leq cn|z|^r,
\]

where \( c = \max\{|c_0|, |c_1|, \ldots, |c_n|\} \) and \( |z|^r = \max\{|z|, |z|^2, \ldots, |z|^n\} \) and

\[
\lim_{p \to \infty} \frac{cn|z|^r}{(p-1)!} = 0,
\]

we arrive at (5). Note: \( r \) will not vary with \( n \).

**Structuring Roots**

There is a relationship between the roots of \( f(z) \) and those of \( F(z) \). This will enable us to structure the roots of polynomials and apply (3) using \( z \) values that are roots of \( f(z) \). A pattern will emerge of the following form

\[
0 = I + \epsilon
\]

where \( I \) is a non-zero integer and \( \epsilon \) is as small as we please: a contradiction.

**Lemma 5.** If polynomial \( f(z) \) has a root \( r \) of multiplicity \( p \), then \( f^{(k)}(r) = 0 \) for \( 0 \leq k \leq p - 1 \) and each term of \( f^{(k)}(r) \), \( p \leq k \leq n \) is a multiple of \( p! \).

**Proof.** Suppose \( r = 0 \) then, for some \( n \) we have \( f(z) = z^p(b_nz^n + \cdots + b_0) \). Now \( f(z) \) has \( b_0z^p \) as its term with minimal exponent. Using the derivative operator, \( D(z^n) = nz^{n-1} \), repeatedly, we see the 0 through \( p - 1 \) derivatives of \( f(z) \) will have a positive exponent of \( z \) in each term. This implies that \( r = 0 \) is a root for these derivatives. Using the product of \( p \) consecutive natural numbers is divisible by \( p! \), terms of subsequent derivatives will be multiples of \( p! \).

If \( r \neq 0 \), then \( f(z) = (z - r)^pQ(z) \), for some polynomial \( Q(z) \). Let \( g(z) = f(z + r) = z^pQ(z + r) \). As \( g^{(k)} = f^{(k)} \) for all \( k \), \( g^{(k)}(0) = f^{(k)}(r) \), and the \( r = 0 \) case applies. \[ \square \]
Lemma 6. If $a$ and $b$ are two non-zero Gaussian integers, then there exist a large enough prime $p$ such that
\[
\frac{|p!a + (p-1)!b|}{(p-1)!} > 1.
\]

Proof. Suppose $a = a_1 + ia_2$ and $b = b_1 + ib_2$.

\[
|p!a + (p-1)!b| = |p!(a_1 + ia_2) + (p-1)!(b_1 + ib_2)|
\]
\[
= (p-1)!|pa_1 + ipa_2 + b_1 + ib_2|
\]
\[
= (p-1)!|(pa_1 + b_1) + i(pa_2 + b_2)|
\]
\[
= (p-1)!\sqrt{(pa_1 + b_1)^2 + (pa_2 + b_2)^2}
\]

The square root contains the sum of two positive or zero integers. Then as both $a$ and $b$ are non-zero Gaussian integers, letting 0 indicate a zero value for a real or complex component and a 1 indicate a non-zero component the possibilities are

\[
a_1b_1| 00 10 01 11 \text{ forcing } a_2b_2| 11 01 10 00.
\]

The only possibility resulting in a zero sum $|pa + b|$ occurs with $b = -pa$ with $b \neq 0$. This is a 11 case. Assuming $a_1$ and $b_1$ are non-zero and $p > \max\{|b_1|\}$, then $p \nmid |b_1|$ and $(pa_1 + b_1)^2$ must be non-zero as, if it is zero then then $pa_1 + b_1 = 0$ and $pa_1 = -b_1$ and $p||b_1|$, a contradiction. So one or both summands are non-zero positive integers. As the square root of a number greater than 1 is greater than 1, the Lemma is established. \qed

Pi is Irrational

Theorem 1. $\pi$ is irrational.

Proof. Suppose not. Then $e^{\pi i} = e^{ri}$ where $r$ is a rational, say $a/b$. Modify the polynomial
\[
z^{p-1}(z - ai/b)^p
\]

\[
to make it an integer polynomial:
\]
\[
f(z) = (bz)^{p-1}(bz - ai)^p.
\]
Then, using Euler’s formula and Lemma 1

\[ 0 = F(0)(e^{ri} + 1) = F(ri) + F(0) + \epsilon_n(f(z)). \]

There is a prime \( p \) large enough that the left hand side of

\[ \left| \frac{\epsilon_n(f(z))}{(p-1)!} \right| = \left| \frac{F(ri) + F(0)}{(p-1)!} \right| \]

is less than one per Lemma 4 and the right hand side is greater than one per Lemma 6, a contradiction. \( \square \)

**Conclusion**

This proof of the irrationality of \( \pi \) uses derivatives and limits at a level of a real analysis course based on Rudin or Apostol [1, 3]. It also anticipates proofs of the transcendence of \( e \) and \( \pi \) [2].

**References**

