MULTINOMIAL DEVELOPMENT

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Abstract:
In this paper we obtain the multinomial theorem following the numbers $A_p^n$ and $C_p^n$ (Vandermonde’s identity generalization). Using this notion we obtain generalization of products of numbers in arithmetic progression, arithmetic regression and their sum. From the generalisation we propose (define) the arithmetics sequences product.

1 INTRODUCTION

In combinatorial analysis we have numbers that allow us to calculate the cardinal of a set according to a well determined model among which we can quote: the numbers $n^p$, $A_p^n$ and $C_p^n$.

Let $n$ and $p$ be two natural numbers. The drawing of $p$ balls in an urn containing $n$ balls, models many counting problems [2]

\[ C_p^n = \binom{n}{p} = \frac{n!}{p!(n-p)!} \]

\[ A_p^n = p!C_p^n \]

The number $C_p^n$ plays an important role in enumerative combination and other discipline fields, see Chen and Kho [1] and Comtet [2] for more details on binomial coefficients. The term binomial coefficient comes from the fact that numbers $C_p^n$ appear as coefficients in the development of $(x+y)^n$.

\[ (x+y)^n = \sum_{i=0}^{n} C_n^i x^{n-i} y^i \]
The multinomial coefficients see [8] which are a generalization of the binomial coefficients allow to extend the development to more numbers. Let \( n \) and \( m \) be non-zero natural numbers, \( x_1, x_2, \ldots, x_m \) real numbers:

\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n}^n \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}
\]

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}
\]

In the case of binomials, the sum of the powers of numbers in arithmetic progression using Bernoulli polynomials gives (see [5] and [7] for more details)

\[
\sum_{k=0}^{n-1} (m + kr)^p = \frac{n^p}{p+1} (B_{p+1}(n + \frac{d}{r}) - B_{p+1}(\frac{d}{r}))
\]

The sum of the first \( n \) products of \( p \) consecutive integers is given by the formula of M. LAISANT [6]. Let \( n \) and \( p \) be natural numbers:

\[
S_p(n) = \sum_{k=1}^{n} k(k+1)(k+2)(k+p) = \frac{1}{p+1} \prod_{i=0}^{p} (n+i)
\]

Suppose \( m, r \) are non-zero positive reals and \( n \) is a non-zero natural number, the Euler gamma function [4] and [9] allows to generalize the product of \( n \) consecutive non-zero integers.

\[
\prod_{k=0}^{n-1} (m + kr) = r^n \frac{\Gamma(n + 1 + \frac{m-r}{r})}{\Gamma(1 + \frac{m-r}{r})}
\]

For every non-negative integer \( n \) we have:

\[
\Gamma(n + 1) = n\Gamma(n) = n(n - 1)\Gamma(n - 1) = n!
\]

The results of this paper are organized as follows: in section 2 the formula of the multinomial following the numbers \( A_p^n \) and \( C_p^n \). In section 3 the formulas of product of \( p \) numbers in arithmetic progression or regression (divided by \( p! \)) are presented. In section 4 a generalization of the formula of M. LAISANT will be presented by proposing afterwards the arithmetic sequences produced.
2 MULTINOMIAL DEVELOPMENT

2.1 Multinomial development according to the number $A_n^p$

2.1.1 Theorem 1

Let $m$ and $n$ be two non-zero natural numbers and $x_1, x_2, \ldots, x_m$ natural numbers $n \leq x_i$. Then,

$$A_{(x_1+x_2+\ldots+x_m)}^n = \sum_{k_1+k_2+\ldots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} A_{x_1}^{k_1} A_{x_2}^{k_2} \ldots A_{x_m}^{k_m}$$

Proof: we prove by recurrence on $m$

$$i) m = 1; A_{x_1}^n = \sum_{k_1=n} \binom{n}{k_1} A_{x_1}^{k_1} = \binom{n}{n} A_{x_1}^n = A_{x_1}^n$$

ii) let’s assume the equality is true for $m > 1$ we have:

$$A_{(x_1+x_2+\ldots+x_m+x_{m+1})}^n = \sum_{k_1+k_2+\ldots+k_{m+1}=n} \binom{n}{k_1, k_2, \ldots, k_m, k_{m+1}} A_{x_1}^{k_1} A_{x_2}^{k_2} \ldots A_{x_m}^{k_m} A_{x_{m+1}}^{k_{m+1}}$$

$$= \binom{n}{i} A_{(x_1+x_2+\ldots+x_m)}^{n-i} A_{x_{m+1}}^i$$ (Binomial formula)

$$= \sum_{i=0}^n \binom{n}{i} \left[ \sum_{k_1+k_2+\ldots+k_m=n-i} \binom{n-i}{k_1, k_2, \ldots, k_m} A_{x_1}^{k_1} A_{x_2}^{k_2} \ldots A_{x_m}^{k_m} A_{x_{m+1}}^i \right]$$

$$= \sum_{k_{m+1}=0}^n \binom{n}{k_{m+1}} \left[ \sum_{k_1+k_2+\ldots+k_m=n-k_{m+1}} \binom{n-k_{m+1}}{k_1, k_2, \ldots, k_m} A_{x_1}^{k_1} A_{x_2}^{k_2} \ldots A_{x_m}^{k_m} A_{x_{m+1}}^{k_{m+1}} \right]$$

$$= \sum_{k_{m+1}=0}^n \binom{n}{k_{m+1}} \left[ \sum_{k_1+k_2+\ldots+k_m+k_{m+1}=n} \binom{n}{k_1, k_2, \ldots, k_m, k_{m+1}} A_{x_1}^{k_1} A_{x_2}^{k_2} \ldots A_{x_m}^{k_m} A_{x_{m+1}}^{k_{m+1}} \right]$$

$$A_{(x_1+x_2+\ldots+x_m+x_{m+1})}^n = \sum_{k_1+k_2+\ldots+k_m+k_{m+1}=n} \binom{n}{k_1, k_2, \ldots, k_m, k_{m+1}} A_{x_1}^{k_1} A_{x_2}^{k_2} \ldots A_{x_m}^{k_m} A_{x_{m+1}}^{k_{m+1}}$$
Example

\[ A^3_{(7+3+11)} = \sum_{i+j+k=3} \binom{3}{i,j,k} A_i^3 A_j^3 A_k^3 \]

\[ = A_3^3 A_3^0 A_1^0 + 3 A_2^3 A_3^1 A_1^0 + 3 A_1^3 A_3^2 A_1^0 + 3 A_1^3 A_1^3 A_1^1 + 3 A_3^1 A_1^3 A_1^1 + 6 A_1^3 A_3^3 A_1^1 \]

\[ + 3 A_2^3 A_3^2 A_1^1 + 3 A_2^3 A_1^3 A_1^1 + A_0^3 A_3^3 A_1^1 + A_0^3 A_1^3 A_3^1 + A_1^3 A_3^3 A_1^1 \]

\[ A^3_{(7+3+11)} = 7980 \]

2.1.2 Formula of the binomial according to the number \( A^p_n \)

Let \( m, n \) be natural numbers and \( p \) a non-zero natural number, \( p \leq m, p \leq n \) then:

\[ A^p_{(m+n)} = \sum_{i=0}^{p} C^p_i A^p_m A^i_n \]

Proof (see 3.2)

2.2 Multinomial development according to the number \( C^p_n \) (multinomial vandermonde’s identity)

2.2.1 Theorem 2

Let \( m \) and \( n \) be two non-zero natural numbers and \( x_1, x_2, ..., x_m \) natural numbers \( n \leq x_i \). Then,

\[ C^m_{(x_1+x_2+...+x_m)} = \sum_{k_1+k_2+...+k_m=n} C^{k_1}_{x_1} C^{k_2}_{x_2} C^{k_m}_{x_m} \]

Proof: From theorem 1 we have:
\[ A^n_{x_1+x_2+\ldots+x_m} = \sum_{k_1+k_2+\ldots+k_m=n} \binom{n}{k_1,k_2,\ldots,k_m} A^{k_1}_{x_1} A^{k_2}_{x_2} \ldots A^{k_m}_{x_m} \]

\[ n!C^n_{x_1+x_2+\ldots+x_m} = \sum_{k_1+k_2+\ldots+k_m=n} \frac{n!}{k_1!k_2! \ldots k_m!} C^{k_1}_{x_1} C^{k_2}_{x_2} \ldots C^{k_m}_{x_m} \]

\[ n!C^n_{x_1+x_2+\ldots+x_m} = \sum_{k_1+k_2+\ldots+k_m=n} \frac{n!}{k_1!k_2! \ldots k_m!} C^{k_1}_{x_1} C^{k_2}_{x_2} \ldots C^{k_m}_{x_m} \]

\[ C^n_{x_1+x_2+\ldots+x_m} = \sum_{k_1+k_2+\ldots+k_m=n} C^{k_1}_{x_1} C^{k_2}_{x_2} \ldots C^{k_m}_{x_m} \]

Example

\[ C^3_{7+3+11} = \sum_{i+j+k=3} C^i_7 C^j_3 C^k_{11} \]

\[ = C^0_7 C^0_3 C^3_{11} + C^1_7 C^0_3 C^2_{11} + C^2_7 C^0_3 C^1_{11} + C^3_7 C^0_3 C^0_{11} + C^0_7 C^1_3 C^2_{11} + C^1_7 C^1_3 C^1_{11} + C^2_7 C^1_3 C^0_{11} + C^3_7 C^1_3 C^0_{11} + C^3_7 C^2_3 C^0_{11} \]

\[ C^3_{7+3+11} = 1330 \]

2.2.2 Formula of the binomial according to the number \( C^p_n \)

Let \( m, n \) be natural numbers and \( p \) a non-zero natural number, \( p \leq m, p \leq n \) then :

\[ C^p_{m+n} = \sum_{i=0}^{p} C^p_i C^m_n (Vandermond's identity) \]

Proof (see 3.3)

3 FORMULA OF DECOMPOSITION OF PRODUCT INTO SUM

3.1 Formula for decomposition of \( p \) product of numbers in arithmetic progression into sum

Let the product be defined by :

\[ \prod_{k=n}^{n+p-1} (m+kr) = (m+nr)(m+(n+1)r)\ldots(m+(n+p-1)r) \]
Let’s set: \( D_{m,r}^{k,p} = \prod_{i=1}^{k}(m + (p - i)r) \) with \( D_{m,r}^{0,p} = 1 \)

Theorem 3:
Let \( m \) be a non-zero real number, \( n \) be a natural number and \( p \) a non-zero natural number, then:

\[
\prod_{k=n}^{n+p-1} (m + kr) = \sum_{i=0}^{p} C_{i}^{n} D_{m,r}^{p-i,p \cdot r \cdot i} A_{n}
\]

Proof:

i) \( p = 1 \),
\[
\prod_{k=n}^{n+1-1} (m + kr) = \sum_{i=0}^{1} C_{i}^{1} D_{m,r}^{1-i,1 \cdot r \cdot i} A_{n}
\]
\[
m + nr = C_{0}^{1} D_{m,r}^{0,0} A_{n}^{0} + C_{1}^{1} D_{m,r}^{0,1} A_{n}^{1}
\]
\[
m + nr = m + nr
\]

ii) let’s assume the equality is true for \( p > 1 \) we have:

\[
\prod_{k=n}^{n+p-1} (m + kr) = \sum_{i=0}^{p} C_{i}^{p} D_{m,r}^{p-i,p \cdot r \cdot i} A_{n}
\]
\[
\prod_{k=n}^{n+p} (m + kr) = (m + nr)(m + (n + 1)r) \ldots (m + (n + p - 1)r)(m + (n + p)r)
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} A_{n}^{i} (m + (n + p)r)
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} A_{n}^{i} ((m + pr) + nr)
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} (m + pr) D_{m,r}^{p-1,i} A_{n}^{i} + \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} n A_{n}^{i}
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} (m + pr) D_{m,r}^{p-1,i} A_{n}^{i} + \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} n A_{n}^{i}
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} (m + pr) D_{m,r}^{p-1,i} A_{n}^{i} + \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} n A_{n}^{i}
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} (m + pr) D_{m,r}^{p-1,i} A_{n}^{i} + \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} n A_{n}^{i}
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} (m + pr) D_{m,r}^{p-1,i} A_{n}^{i} + \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} n A_{n}^{i}
\]
\[
= \sum_{i=0}^{p} C_{i}^{0} (m + pr) D_{m,r}^{p-1,i} A_{n}^{i} + \sum_{i=0}^{p} C_{i}^{0} D_{m,r}^{p-1,i} n A_{n}^{i}
\]
\[
\prod_{k=n}^{n+p} \left( m + kr \right) = \sum_{i=0}^{p} C_p^i D_{m,r}^{p+1-i} p^{1+i} A_i^1 + \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i} p^{1+i+1} A_i^{i+1}
\]

\[
+ \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} i A_i^i = \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} n A_i^i
\]

\[
= \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-1-i} p^{1+i+1} A_i^{i+1}
\]

\[
+ \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} i A_i^i + C_p^i D_{m,r}^{p-1-i} p^{i+1} p A_i^i - \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} n A_i^i
\]

\[
= \sum_{i=0}^{p-1} C_p^i (m + ir) D_{m,r}^{p-1-i} p^{i+2} i A_i^i + \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} i A_i^i + C_p^i D_{m,r}^{p-1-i} p^{i+1} p A_i^i
\]

\[
+ \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} (n - i) A_i^i - \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} n A_i^i
\]

\[
= \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p+1-i} p^{1+i+1} A_i^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} A_i^{i+1}
\]

\[
+ \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} i A_i^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-1-i} p^{i+1} p A_i^i + \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} i A_i^i
\]

\[
+ \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} A_i^{i+1} - \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} n A_i^i
\]

\[
= \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} A_i^i + \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} A_i^{i+1}
\]

\[
+ \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} A_i^{i+1} - \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} i A_i^i
\]

\[
= \sum_{i=0}^{p-1} C_p^i D_{m,r}^{p-i} p^{1+i+1} A_i^i + \sum_{i=0}^{p-1} C_p^i (p - i) D_{m,r}^{p-1-i} p^{i+2} A_i^{i+1}
\]
Let the product be defined by:

\[ \prod_{k=n}^{n+p} (m + kr) = C^0_p D^{p+1, p+1}_m, r A^0_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^i_n + \sum_{i=0}^{p-1} C^i_p D^{p-i, p+1}_m, r A^{i+1}_n + C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n \\
+ C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n = C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n \\
+ C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n = C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n \\
+ C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n = C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n \\
+ C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n = C^0_p D^{p+1, p+1}_m, r A^{p+1}_n + \sum_{i=1}^{p} C^i_p D^{p+1-i, p+1}_m, r A^{i+1}_n \]

\[ \prod_{k=n}^{n+p} (m + kr) = \sum_{i=0}^{p+1} C^i_{p+1} D^{p+1-i, p+1}_m, r A^{i}_n \]

**Example**

\[ \prod_{k=7}^{7+3-1} (5 + 2k) = \sum_{i=0}^{3} C^i_{3} D^{3-i, 3}_5, 2 A^{i}_7 \]

\[ = C^3_{3} D^{3, 3}_5, 2 A^{3}_7 + C^2_{3} D^{2, 3}_5, 2 A^{2}_7 + C^1_{3} D^{1, 3}_5, 2 A^{1}_7 + C^0_{3} D^{0, 3}_5, 2 A^{0}_7 \]

\[ = C^3_{3} (9 \times 7 \times 5)2^3 A^{3}_7 + C^2_{3} (9 \times 7)2^2 A^{2}_7 + C^1_{3} 9 \times 2^1 A^{1}_7 + C^0_{3} 2^0 A^{0}_7 \]

\[ \prod_{k=7}^{7+3-1} (5 + 2k) = 9177 \]

### 3.2 Formula for decomposition of p product of numbers
in arithmetic regression into sum

Let the product be defined by:

\[ \prod_{k=0}^{p-1} (m + (n - k)r) = (m + nr)(m + (n - 1)r)...(m + (n - p + 1)r) \]
Let’s set: $$D^k_{m,r} = \prod_{i=1}^{k}(m - (i - 1)r)$$ with $$D^0_{m,r} = 1$$

Theorem 4:

Let $$m$$ be a non-zero real number, $$n$$ be a natural number and $$p$$ a non-zero natural number, then:

$$
\prod_{k=0}^{p-1}(m + (n - k)r) = \sum_{i=0}^{p} C^p_i D^p_{m,r} r^i A_n^i
$$

Proof:

i) $$p = 1$$,

$$\prod_{k=0}^{1-1}(m + (n - k)r) = \sum_{i=0}^{1} C^1_i D^1_{m,r} r^i A_n^i$$

$$m + nr = C^0_1 D^1_{m,r} r^0 A_n^0 + C^1_1 D^1_{m,r} r^1 A_n^1$$

$$m + nr = m + nr$$

ii) let’s assume the equality is true for $$p > 1$$ we have:

$$\prod_{k=0}^{p-1}(m + (n - k)r) = \sum_{i=0}^{p} C^p_i D^p_{m,r} r^i A_n^i$$
Theorem 5: (This theorem is a deduction from Theorem 3)

Let the product be defined by:

\[ \prod_{k=0}^{P} (m + (n - k)r) = (m + nr)(m + (n - 1)r)...(m + (n - p + 1)r)(m + (n - p)r) \]

\[ = \sum_{i=0}^{P} C_{i}^{p} D_{m,r}^{p-i} r^{i} A_{n}^{i} (m + (n - p)r) \]

\[ = \sum_{i=0}^{P} C_{i}^{p} D_{m,r}^{p-i} r^{i} A_{n}^{i} ((m - (p - i)r) + (n - i)r) \]

\[ = \sum_{i=0}^{P} C_{i}^{p} D_{m,r}^{p-i} r^{i} A_{n}^{i} (m - (p - i)r) + \sum_{i=0}^{P} C_{i}^{p} D_{m,r}^{p-i} r^{i+1} A_{n}^{i} (n - i) \]

\[ = \sum_{i=0}^{P} C_{i}^{p} D_{m,r}^{p+1-i} r^{i} A_{n}^{i} + \sum_{i=0}^{P} C_{i}^{p} D_{m,r}^{p-i} r^{i+1} A_{n}^{i+1} \]

\[ = C_{0}^{p} D_{m,r}^{p+1} A_{0}^{0} + \sum_{i=1}^{P} C_{i}^{p} D_{m,r}^{p+1-i} r^{i} A_{n}^{i} + \sum_{i=0}^{P} C_{p}^{i} D_{m,r}^{p-i} r^{i+1} A_{n}^{i+1} + C_{p}^{0} D_{m,r}^{0} r^{P+1} A_{n}^{P+1} \]

\[ = C_{0}^{p} D_{m,r}^{p+1} A_{0}^{0} + \sum_{i=1}^{P} C_{i}^{p} D_{m,r}^{p+1-i} r^{i} A_{n}^{i} + \sum_{i=1}^{P} C_{i}^{p} D_{m,r}^{p+1-i} r^{i} A_{n}^{i} + C_{p}^{0} D_{m,r}^{0} r^{P+1} A_{n}^{P+1} \]

\[ = C_{p+1}^{0} D_{m,r}^{p+1} A_{0}^{0} + \sum_{i=1}^{P} C_{i}^{p+1} D_{m,r}^{p+1-i} r^{i} A_{n}^{i} + C_{p+1}^{0} D_{m,r}^{0} r^{P+1} A_{n}^{P+1} \]

\[ \prod_{k=0}^{P} (m + (n - k)r) = \sum_{i=0}^{P+1} C_{i}^{p} D_{m,r}^{p+1-i} r^{i} A_{n}^{i} \]

Remark: For \( r = 1 \), \( D_{m,1}^{p-i} = A_{m}^{p-i} \) we find the binomial formula according to the number \( A_{n}^{i} \)

3.3 Formula for decomposition of p product of numbers in arithmetic progression divided by p! into sum

Let the product be defined by: \[ \prod_{k=n}^{n+p-1} \frac{(m + kr)}{p!} \]

Let’s set: \( D_{m,r}^{k,p} = \prod_{i=1}^{k} (m + (p - i)r) \) with \( D_{m,r}^{0,p} = 1 \)

Theorem 5: (This theorem is a deduction from Theorem 3)

Let \( m \) be a non-zero real number, \( n \) be a natural number and \( p \) a non-zero
natural number, then:

\[
\prod_{k=n}^{n+p-1} (m + k r) = \frac{p!}{p!} \sum_{i=0}^{p} D_{m,r}^{p-i,p} A_i
\]

Proof:

We have:

\[
\prod_{k=n}^{n+p-1} (m + k r) = \sum_{i=0}^{p} C_i D_{m,r}^{p-i,p} A_i
\]

3.4 Formula for decomposition of product of numbers in arithmetic regression divided by p! into sum

Let the product be defined by:

\[
\prod_{k=0}^{n} (m + (n - k) r) = \frac{p!}{p!}
\]

Let’s set: \(D_{m,r}^{k} = \prod_{i=1}^{k} (m - (i - 1) r)\) with \(D_{m,r}^{0} = 1\)

Theorem 6: (This theorem is a deduction from Theorem 4)

Let \(m\) be a non-zero real number, \(n\) be a natural number and \(p\) a non-zero natural number, then:

\[
\prod_{k=0}^{n} (m + (n - k) r) = \sum_{i=0}^{p} D_{m,r}^{p-i,m,r} (p - i)! C_i
\]

Proof:
We have:

\[ \prod_{k=0}^{p-1} (m + (n - k)r) = \sum_{i=0}^{p} C_p^i D_{m,r}^{-i} A_n^i \]

\[ \frac{\prod_{k=0}^{p-1} (m + (n - k)r)}{p!} = \frac{1}{p!} \sum_{i=0}^{p} C_p^i D_{m,r}^{-i} A_n^i \]

\[ = \sum_{i=0}^{p} C_p^i \frac{D_{m,r}^{-i}}{p!} r^i A_n^i \]

\[ = \sum_{i=0}^{p} \frac{p!}{i!(p-i)!} \frac{D_{m,r}^{-i}}{p!} r^i C_n^i \]

Remark: For \( r = 1 \), \( D_{m,1}^{p-1} = A_m^{p-1} \) we find the binomial formula according to the number \( C_n^p \).

4 SUMS COMPUTATIONS

4.1 Sum of \( p \) products of numbers in arithmetic progression

Consider the sum defined by:

\[ S_p = m(m + r)...(m + (p-1)r) + (m + r)(m + 2r)...(m + pr) + (m + (n-1)r)...(m + (n + p - 2)r) \]

Let’s set:

\[ D_{m,r}^{k,p} = \prod_{i=1}^{k} (m + (p - i)r) \] with \( D_{m,r}^{0,p} = 1 \)

Theorem 7: (This theorem is a deduction from Theorem 3)

Let \( m \) be a non-zero real number, \( n \) be a natural number and \( p \) a non-zero natural number, then:

\[ \sum_{j=0}^{n-1} \prod_{k=j}^{p-1} (m + kr) = \sum_{i=0}^{p} C_p^i D_{m,r}^{-i} A_n^{i+1}_{i+1} \]

Proof

\[ m(m + r)...(m + (p - 1)r) = C_p^0 (m + (p - 1)r)...(m + r) mr^0 A_0^0 \]

\[ (m + r)(m + 2r)...(m + pr) = C_p^0 (m + (p - 1)r)...(m + r) mr^0 A_1^1 + C_p^1 (m + (p - 1)r)...(m + r)r^1 A_1^1 \]

\[ (m + 2r)(m + 3r)...(m + (p + 1)r) = C_p^0 (m + (p - 1)r)...(m + r) mr^0 A_2^2 + C_p^1 (m + (p - 1)r)...(m + r)r^1 A_2^2 \]

\[ \vdots \]
From [6] we have:

Consider the sum defined by:

\[ S = \sum \text{products of numbers in arithmetic progression divided by } p! \]

Let's set:

\[ D_{m,r}^{p-1}(m + (p - 1)r) = C_p^0(m + (p - 1)r) + C_p^1(m + (p - 1)r)^2 + ... + C_p^{p-1}(m + (p - 1)r)^p A_p^p \]

\[ \vdots \]

\[ (m + (n - 1)r)(m + nr) + (m + (n + p - 2)r) = C_p^{m}(m + (p - 1)r) + C_p^1(m + (p - 1)r) + ... + C_p^{p-1}(m + (p - 1)r)^p A_p^p \]

Summing member to member we get:

\[ S_p = C_p^0(m + (p - 1)r)(m + nr) + C_p^1(m + (p - 1)r) + ... + C_p^{p-1}(m + (p - 1)r)^p A_p^p \]

From [6] we have:

\[ S_p = \sum_{i=0}^{p} C_p^i D_{m,r}^{p-i,p} A_{n+1}^{i+1} \]

### 4.2 Sum of p products of numbers in arithmetic progression divided by p!

Consider the sum defined by:

\[ S_p' = C_p^0 \frac{m}{1} + C_p^1 \frac{m + (p - 1)r}{2} + C_p^2 \frac{m + 2r}{3} + ... + C_p^{p-1} \frac{m + (p - 1)r}{p} A_m^p \]

Let's set:

\[ D_{m,r}^{p-1}(m + (p - 1)r) = \prod_{i=1}^{k} (m + (p - i)r) \]

Theorem 8: (This theorem is a deduction from Theorem 5)

Let \( m \) be a non-zero real number, \( n \) be a natural number and \( p \) a non-zero natural number, then:

\[ \sum_{j=0}^{n-1} \frac{\prod_{k=j}^{p-1} (m + kr)}{p!} = \sum_{i=0}^{p} D_{m,r}^{p-i,p} (p-i)! r^i C_n^{i+1} \]
Proof:

\[ S'_p = \frac{m}{2} \sum_{i=0}^{p-1} \frac{m + (i-1)r}{p} + \frac{m + (n-1)r}{2} \sum_{i=0}^{p-1} \frac{m + (n + p - 2)r}{p} + \cdots + \]

\[ S'_p = \frac{1}{p!} S_p. \]

\[
S'_p = \frac{1}{p!} \sum_{i=0}^{p} C^i_p D^{p-i} p_i r^i A^{i+1}_n \frac{A^{i+1}_n}{i+1}
\]

\[ = \sum_{i=0}^{p} C^i_p D^{p-i} p_i r^i A^{i+1}_n \frac{A^{i+1}_n}{i+1}
\]

\[ = \sum_{i=0}^{p} \frac{p!}{i!(p-i)!} \frac{D^{p-i} p_i r^i n i}{p^i} r^i (i+1) \frac{C^{i+1}_n}{i+1}
\]

\[ S'_p = \sum_{i=0}^{p} \frac{D^{p-i} p_i r^i n i}{(p-i)!} C^{i+1}_n. \]

5 ARITHMETIC SEQUENCES PRODUCTS

5.1 Arithmetic sequence of type \( A^p_n \)

5.1.1 Definition

\((U_{n,p})\) is an arithmetic sequence of type \( A^p_n \) if there exists a real \( r \) called reason such that, for all natural numbers \( n \) and \( p \):

\[ \frac{U_{n+1,p} - U_{n,p}}{pU_{n+1,p-1}} = r \text{ or } U_{n+1,p} - U_{n,p} = prU_{n+1,p-1} \]

5.1.2 Expression of \( U_{n,p} \) in terms of \( n \)

If the sequence \( U_{n,p} \) is arithmetic of type \( A^p_n \) of first term, \( U_{k,p} = \prod_{k=1}^{p} (m+(i-1)r) \)

and reason \( r \) then:

\[ U_{n,p} = \sum_{i=0}^{p} C^i_p U_{k,p-ir^i} A^{i+1}_{n-k} \quad U_{k,p-ir^i} = \prod_{j=i+1}^{p} (m+(j-1)r) \]

with \( U_{k,0} = 1 \)

5.1.3 Sum of consecutive terms

\[ S_{n,p} = \sum_{i=0}^{p} C^i_p U_{k,p-ir^i} A^{i+1}_{\text{number of terms}} \frac{A^{i+1}_{n-k}}{i+1} \]

Remark: \( p = 1, S_{n,1} \) is the sum of a classical arithmetic sequence
5.2 Example

Let be the sequence \((V_{n,3})\) defined by : 
\[ V_{n,3} = \left(\frac{2}{3} + \frac{1}{4}n\right)^3 - \frac{1}{4} \left(\frac{2}{3} + \frac{1}{4}n\right), \quad n \in \mathbb{N}^* \]

1a) Express \(V_{n,3}\) as a product of consecutive factors in terms of \(n\).

b) Prove that \(V_{n,3}\) is an arithmetic sequence of type \(A_p^n\) whose reason will be determined.

2a) Calculate \(V_{1,3}\) and give a new expression of \(V_{n,3}\) as a function of \(n\).

b) Express the sum \(S_{n,3}\) of the first \(n\) terms of \(V_{n,3}\) as a function of \(n\).

3) Let \(U_{n,3} = \left(\frac{2}{3} + \frac{1}{4}n\right)^3, \quad n \in \mathbb{N}^*\) and \(S'_{n,3}\) sum of the first \(n\) terms of \(U_{n,3}\). Express \(S'_{n,3}\) as a function of \(S_{n,3}, S_{n,1}\) and \(n\).

5.2.1 Resolution:

1a) Expression of \(V_{n,3}\) as a product of factor

\[
V_{n,3} = \left(\frac{2}{3} + \frac{1}{4}n\right)^3 - \frac{1}{4} \left(\frac{2}{3} + \frac{1}{4}n\right)
= \left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}n^2 - \frac{1}{4}\right)
= \left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}n - \frac{1}{4}\right)\left(\frac{2}{3} + \frac{1}{4}n + \frac{1}{4}\right)
V_{n,3} = \left(\frac{2}{3} + \frac{1}{4}(n-1)\right)\left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)
\]

b) Let us prove \(V_{n,3}\) is an arithmetic sequence of type \(A_p^n\) whose reason will be determined.

\[
\frac{V_{n+1,3}-V_{n,3}}{3V_{n+1,2}} = \frac{\left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)\left(\frac{2}{3} + \frac{1}{4}(n+2)\right) - \left(\frac{2}{3} + \frac{1}{4}(n-1)\right)\left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)}{3\left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)}
= \frac{\left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)\left(\frac{2}{3} + \frac{1}{4}(n+2)\right) - \left(\frac{2}{3} + \frac{1}{4}(n-1)\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)}{3\left(\frac{2}{3} + \frac{1}{4}n\right)\left(\frac{2}{3} + \frac{1}{4}(n+1)\right)}
\]

\[
\frac{V_{n+1,3}-V_{n,3}}{3V_{n+1,2}} = \frac{1}{4},
\]

Hence \(V_{n,3}\) is an arithmetic sequence of type \(A_p^n\) of reason \(r = \frac{1}{4}\)

2a) The value of \(V_{1,3}\)
\[ V_{n,3} = \left( \frac{2}{3} + \frac{1}{4}(n - 1) \right) \left( \frac{2}{3} + \frac{1}{4}n \right) \left( \frac{2}{3} + \frac{1}{4}(n + 1) \right) \]

\[ V_{1,3} = \frac{2}{3} \times \frac{11}{12} \times \frac{7}{6} \]

\[ V_{1,3} = \frac{77}{108} \]

**New expression of \( V_{n,3} \) as a function of \( n \)**

\[ V_{n,3} = \sum_{i=0}^{3} C_i^3 V_{1,3} - i \left( \frac{1}{4} \right)^i A_{n-1}^i \]

\[ = C_0^3 V_{1,3} \left( \frac{1}{4} \right)^0 A_{n-1}^0 + C_3^3 V_{1,3} \left( \frac{1}{4} \right)^3 A_{n-1}^3 \]

\[ = \frac{2}{3} \times \frac{11}{12} \times \frac{7}{6} + 3 \left( \frac{11}{12} \times \frac{7}{6} \right)^2 (n - 1) + 3 \left( \frac{7}{6} \right)^2 (n - 1)(n - 2) + \left( \frac{1}{4} \right)^3(n - 1)(n - 2)(n - 3) \]

\[ V_{n,3} = \frac{77}{108} + \frac{77}{96} (n - 1) + \frac{7}{32} (n - 1)(n - 2) + \frac{1}{64} (n - 1)(n - 2)(n - 3) \]

b) Let’s express the \( S_{n,3} \) sum of the first \( n \) terms of \( V_{n,3} \) as a function of \( n \)

\[ S_{n,3} = \sum_{i=0}^{3} C_i^3 V_{1,3} - i \left( \frac{1}{4} \right)^i A_{n}^i + 1 \]

\[ = C_0^3 V_{1,3} \left( \frac{1}{4} \right)^0 A_{n}^0 + C_3^3 V_{1,3} \left( \frac{1}{4} \right)^3 A_{n}^3 \]

\[ S_{n,3} = \frac{77}{108} n + \frac{77}{96} n(n - 1) + \frac{7}{96} n(n - 1)(n - 2) + \frac{1}{256} n(n - 1)(n - 2)(n - 3) \]

3) Let’ express \( S_{n,3}' \) as a function of \( S_{n,3} \) and \( S_{n,1} \)

\[ U_{n,3} = \left( \frac{2}{3} + \frac{1}{4}n \right)^3, \; n \in \mathbb{N}^* \]

\[ V_{n,3} = \left( \frac{2}{3} + \frac{1}{4}n \right)^3 - \frac{1}{42} \left( \frac{2}{3} + \frac{1}{4}n \right) \]

\[ \left( \frac{2}{3} + \frac{1}{4}n \right)^3 = V_{n,3} + \frac{1}{42} \left( \frac{2}{3} + \frac{1}{4}n \right) \]

\[ U_{n,3} = V_{n,3} + \frac{1}{42} \left( \frac{2}{3} + \frac{1}{4}n \right)(n - 1) \]

\[ = V_{n,3} + \frac{1}{42} \left( \frac{2}{3} + \frac{1}{4}n \right)(n - 1) + \frac{1}{4} \]

\[ U_{n,3} = V_{n,3} + \frac{1}{42} V_{1,1} + \frac{1}{4} \]
$$S_{n,3}' = U_{1,3} + U_{2,3} + U_{3,3} + \ldots + U_{n,3}$$

$$= (V_{1,3} + \frac{1}{4^2} V_{1,1} + \frac{1}{4^3} V_{1,1}) + (V_{2,3} + \frac{1}{4^2} V_{2,1} + \frac{1}{4^3} V_{2,1}) + (V_{3,3} + \frac{1}{4^2} V_{3,1} + \frac{1}{4^3} V_{3,1}) + \ldots + (V_{n,3} + \frac{1}{4^2} V_{n,1} + \frac{1}{4^3} V_{n,1})$$

$$= (V_{1,3} + V_{2,3} + V_{3,3} + \ldots + V_{n,3}) + \frac{1}{4^2} (V_{1,1} + V_{2,1} + V_{3,1} + \ldots + V_{n,1}) + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \ldots + \frac{1}{4^3}$$

$$S_{n,3}' = S_{n,3} + \frac{1}{4^2} S_{n,1} + \frac{n}{4^3}$$

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