# Two visualizing proofs of the Pythagorean theorem

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**Abstract**

The Pythagorean theorem is one of the most proved theorem of all time, most of the proofs use manipulation of areas to prove that the square of the hypotenuse is indeed the sum of the squares of the two legs. I often felt disconnected with the proofs, it was one of the situations where one could prove something, but could not quite see why. So here I present two dynamical and visualizing methods for proving the Pythagorean theorem.

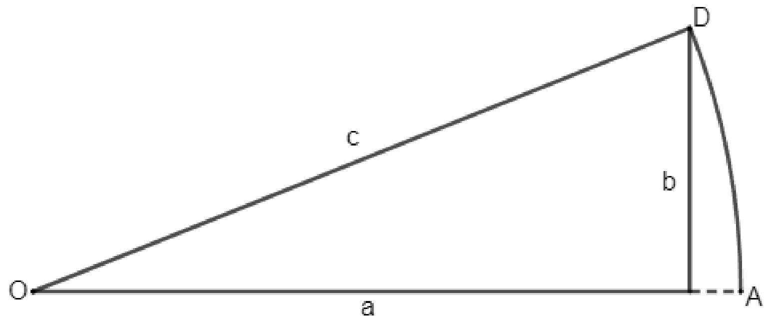
# Contents

# 1 The question

The problem to solve here is quite straight forward: given two known sides $a$ and $b$, let them form a right-angled triangle, with $a$ and $b$ the legs. Then what is the length of the hypotenuse $c$?

# 2 The first method

Our approach will be a dynamical method. Although we do not know the length of $c$, we can nevertheless draw a graph containing it. Given a right-angled triangle $abc$ with $c$ as the hypotenuse, and point $O$ as the vertex between $a$ and $c$,

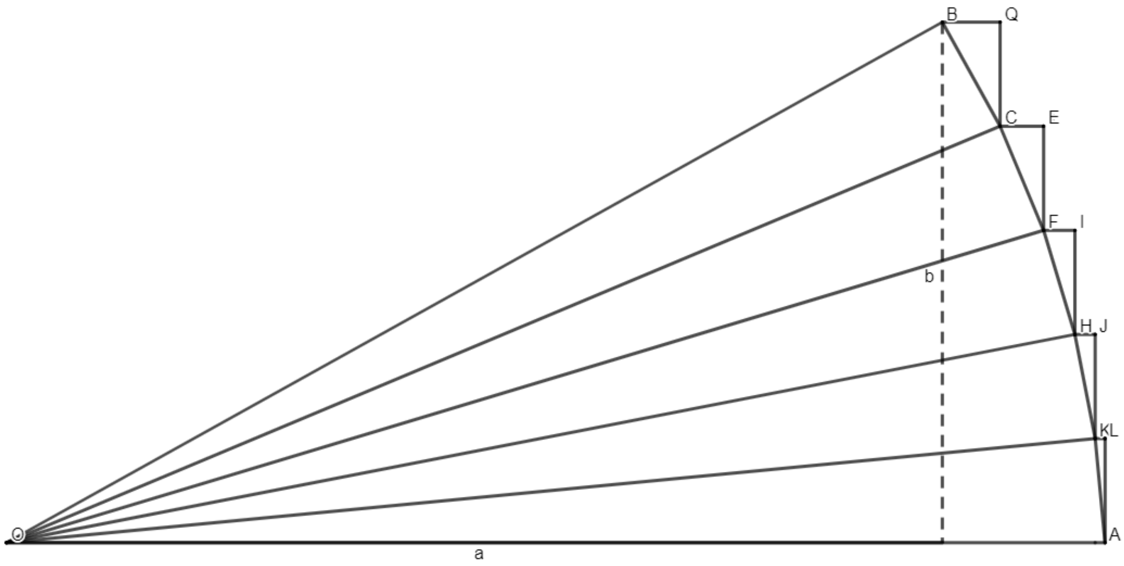



let $c$ rotate clockwise about $O$, until $c$ overlaps with $a$(so rotate from $OD$ to $OA$). During this rotation it is our goal to trace the horizontal projection of $c$, or the projection of $c$ onto the line containing $a$. Clearly, before the rotation, the horizontal projection of $c$ is just equal to $a$, and as $c$ rotates towards $a$, its projection grows larger, then finally, as $c$ overlaps with $a$, its projection will be equal to itself, $c$, which is exactly what we are trying to calculate. The dotted line in the figure shows the total horizontal increment gained from this rotation.

## 2.1 The setup

Rotation or circular movement can be approximated by some finite amount of linear motions. For example, if an object always move in a straight line for a short distance, before readjusting its direction of motion such that the direction of motion is always perpendicular to the radius at that point, then such movement can be a good approximation for a circular motion. The more frequent it adjusts its direction of motion, the closer to a circular motion it will be.

Now let us approximate the rotation of hypotenuse $c$ with 5 linear motions, or the 5-step-rotation, in the following figure,




the same triangle $abc$ is shown with $OB$ as hypotenuse $c$. Moreover, $BC$, $CF$, $FH$, $HK$ and $KA$ are all line segments, they represent the 5 linear motions to approximate rotation. As always the direction of motion is perpendicular to the "radius", so,

$$BC \perp OB \quad CF \perp OC \quad FH \perp OF \quad HK \perp OH \quad KA \perp OK \tag{1}$$

let us call these 5 line segments "tangent segment" or "tangent movement". The horizontal component and vertical component of each tangent segment are also drawn on the graph, for example, $BQ$ and $QC$ are respectively the horizontal component and vertical component of tangent segment $BC$, $CE$ and $EF$ are respectively the horizontal component and vertical component of the tangent segment $CF$, and so on. It is trivial to state that the horizontal component of every tangent segment is parallel to $a$, and the vertical component of every tangent segment is parallel to $b$. The length of these tangent segments are defined in the way that their vertical components are always equal to each other,

$$QC = EF = IH = JK = LA \tag{2}$$

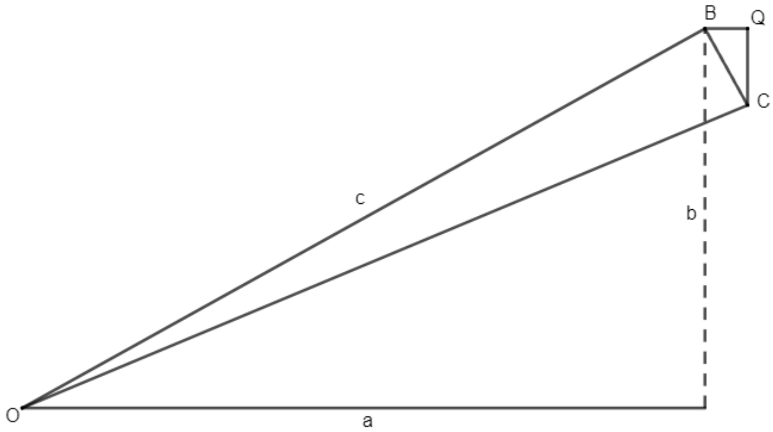
as the sum of these vertical components is $b$, each vertical component of these tangent segments is $b/5$.

So $OB$ took 5 linear motions to "rotate" downward to "become" $OA$. Of course $OA$ does not equal to $OB$ as it was not a perfect rotation, so radius is not conserved. But it was still a decent approximation, and the final horizontal projection after 5 steps of linear motion(in this case would be $OA = a + BQ + CE + FI + HJ + KL$), is the approximated hypotenuse by this 5-step-rotation.

Now let us generalize this setup by defining the number of linear motions, or the number of tangent segments to be $N$(the vertical component of each tangent segment then will be $b/N$), so the larger is $N$ the closer it is to a circular motion. If $N \to \infty$, this movement of $OB$ to $OA$ becomes a perfect rotation, the infinite amount of tangent segments together become an arc, and the final horizontal projection would exactly be equal to the hypotenuse $c$.

## 2.2 Find the pattern

For the general case $N$, let us go ahead and calculate the horizontal projection. To make things easier, as the downward components of all tangent segments are $b/N$, let us just define $\Delta = b/N$. Now look at the first tangent movement,




the original horizontal projection before the first tangent movement is just $a$, then the first tangent movement adds $BQ$ to the horizontal projection. As triangle $BQC$ is similar to triangle $abc$, the first horizontal increment $BQ$ is,

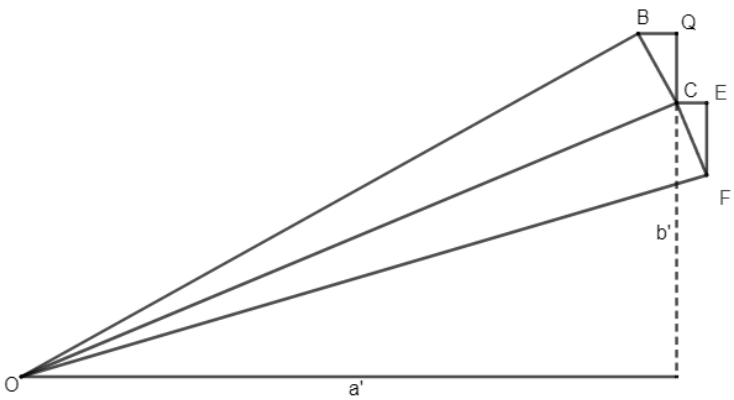
$$BQ = \frac{b}{a} * QC = \frac{b}{a} * \Delta \tag{3}$$

now after the first movement, at the new point $C$, let us denote the vertical component and horizontal component of $OC$ by $b'$ and $a'$ respectively,

$$b' = b - QC = b - \Delta \tag{4}$$

$$a' = a + BQ = a + \frac{b}{a} * \Delta \tag{5}$$

so the horizontal projection after the first step is $a'$. Now let us look at the second tangent movement,




the horizontal increment $CE$ can again be obtained by similar triangles,

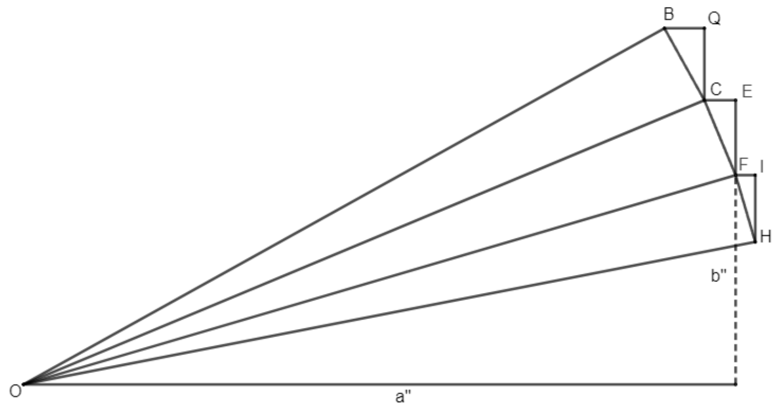
$$CE = \frac{b'}{a'} * EF = \frac{b'}{a'} * \Delta \tag{6}$$

again at the new point $F$, denote the vertical and horizontal components of $OF$ by $b''$ and $a''$,

$$b'' = b' - EF = b' - \Delta \tag{7}$$

$$a'' = a' + CE = a' + \frac{b'}{a'} * \Delta \tag{8}$$

so the horizontal projection after the second step is $a''$. Now to the third tangent movement,




by similar triangles, the third horizontal increment $FI$ is,

$$FI = \frac{b''}{a''} * IH = \frac{b''}{a''} * \Delta \tag{9}$$

at the new point $H$, denote the vertical and horizontal components of $OH$ by $b'''$ and $a'''$,

$$b''' = b'' - IH = b'' - \Delta \tag{10}$$

$$a''' = a'' + FI = a'' + \frac{b''}{a''} * \Delta \tag{11}$$

so the horizontal projection after the third tangent movement is $a'''$. No need to continue with calculating more tangent movements, one should be able to see the pattern at this stage: the next horizontal increment is always $\Delta$ times the ratio between the vertical component and horizontal component at that point. Inductively:

$$(a', b') = (a + \frac{b}{a}\Delta, b - \Delta) \tag{12}$$

$$(a'', b'') = (a' + \frac{b'}{a'}\Delta, b' - \Delta) \tag{13}$$

$$(a''', b''') = (a'' + \frac{b''}{a''}\Delta, b'' - \Delta) \tag{14}$$

$$\dots$$

Now let us observe the series of these horizontal increments:

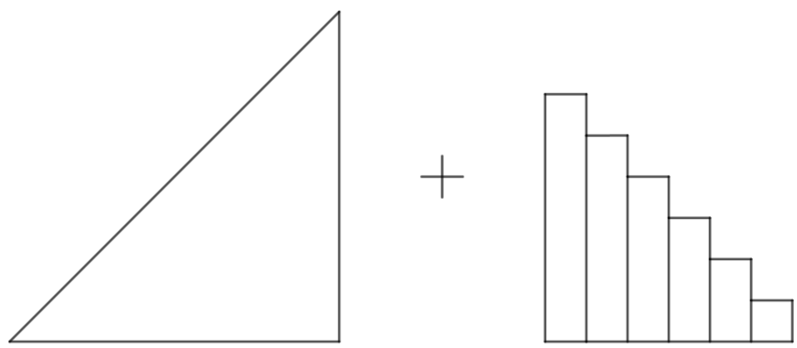
$$\frac{b}{a}\Delta \quad \frac{b'}{a'}\Delta \quad \frac{b''}{a''}\Delta \quad \dots \tag{15}$$

ignore the $\Delta$ for now as every term has one. The series clearly is decreasing as the numerator is decreasing while the denominator is increasing. The numerator is pretty straight forward as the it is just linearly decreasing by $\Delta$ every term. While the denominator, is always the horizontal projection of the last term. In other words, in every step, a horizontal increment is added to the original horizontal projection, and this added increment makes the next horizontal increment smaller because of enlarged denominator by itself.

## 2.3 The merging triangles model

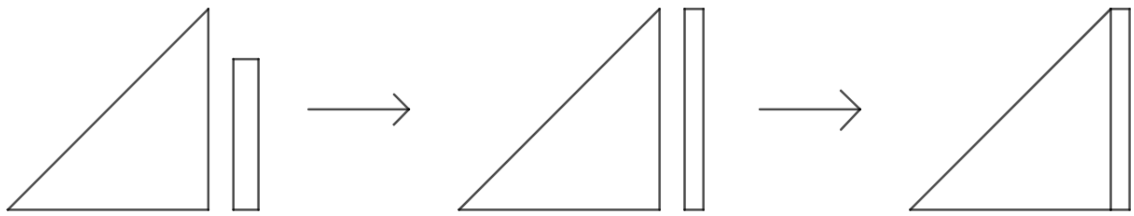
Now should the behavior of the denominator in some way remind us of wrapping elastic bands on to a tube? If there are some identical elastic bands to be wrapped layer by layer onto a large tube, as more elastic bands are put onto the tube, the resulting radius of the tube becomes larger as these elastic bands add thickness to the tube. At the same time, because the radius of the tube is larger, the next elastic band must be stretched further to be put onto the tube, as a result the elastic band must be thinner, hence will increase the radius of the tube by less amount. By the same reason the next elastic bands put on will only contribute less and less radius increments.

This analogy mimic the behavior of the denominator in some way, but we still need the numerator to decrease linearly by $\Delta$. The correct model, is the merging triangles model, that is to merge two isosceles right-angled triangles into a bigger one. Like before, we could approximate this process by an $N$-step approximation.

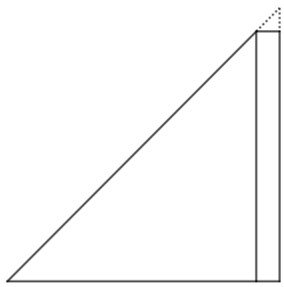


The figure above shows the $N = 6$ case, the triangle to the left, call it the base triangle, has side length $a$, and the "triangle" to the right has "side length" $b$. Of course the triangle to the right is not a real triangle, it is actually made of $N$ rectangular slices(6 slices in the figure). These slices all have width $\Delta = b/N$, and their height, from left to right, are $b$, $b - \Delta$, $b - 2\Delta$,...,$b - (N - 1)\Delta$, let us also denote the height of these slices, from left to right by $b$, $b'$, $b''$,...,$b^{(N-1)\prime}$, it is worth noting that $b^{(i+1)\prime} = b^{i\prime} - \Delta$. Now back to the problem of the "triangle" to the right being just $N$ slices. Without doing the proof in details, it should be obvious that if the number of slices, or simply $N$, tends to $\infty$, all the slices together form a perfect isosceles right-angled triangle with side length $b$, so it will be a real triangle in the limiting case.

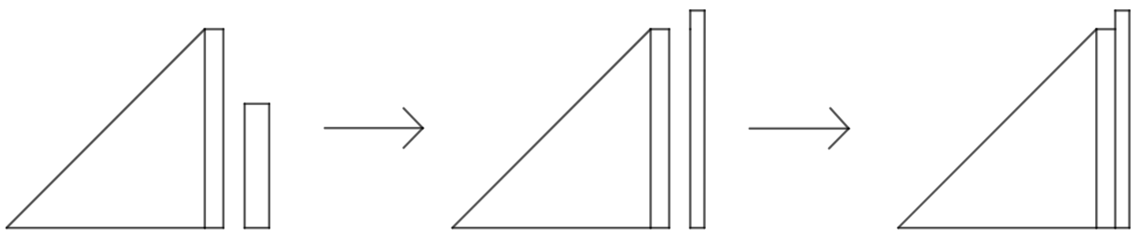
Now to merge the two triangles together into a bigger one, we will do it slice by slice. First, merge the first slice(the tallest slice) to the base triangle.



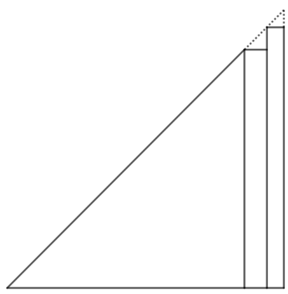
This slice has height $b$ but the base triangle has height $a$, to glue this slice on, the height of the slice first needs to be stretched to $a$, as the area of the slice must be preserved, the width of the slice must be shrink to $(b/a)\Delta$. The resulting shape,



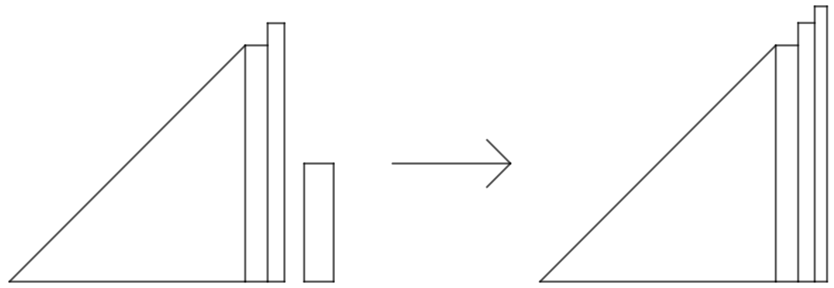
is almost an isosceles right-angled triangle, apart from a little triangle missing at the top. No need to worry about it now, just ignore that and proceed. So the base "triangle" now has side length $a + (b/a)\Delta$, let us denote it as $a'$. Now merge the second slice, this slice has height $b - \Delta$ or $b'$, and the base triangle now has side length $a + (b/a)\Delta$ or $a'$, so the slice needs to be stretched from $b'$ in height to $a'$ in height, then glued on:



as a consequence, the slice's width is shrinked to $(b'/a')\Delta$, and the resulting "isosceles right angled triangle" now has side length $a' + (b'/a')\Delta$, let's call it $a''$,

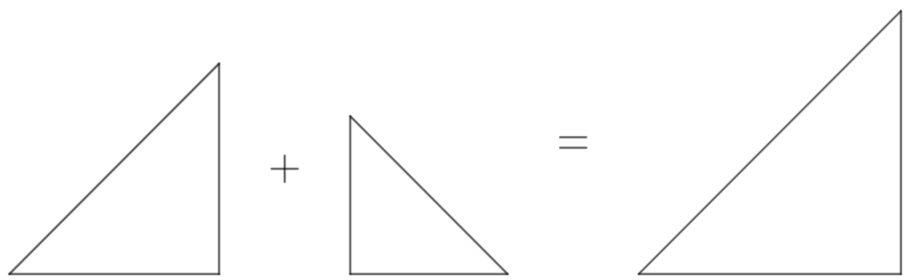


again another smaller triangle is missing at the top, but ignore it for now again. To the third slice,



the third slice has height $b' - \Delta$ or $b''$, and the base triangle now has side length $a''$, so the slice must be stretched from $b''$ in height to $a''$ in height, hence its width is shrinked to $(b''/a'')\Delta$. After it is glued on, the resulting "triangle" now has side length $a'' + (b''/a'')\Delta$, let us call it $a'''$.

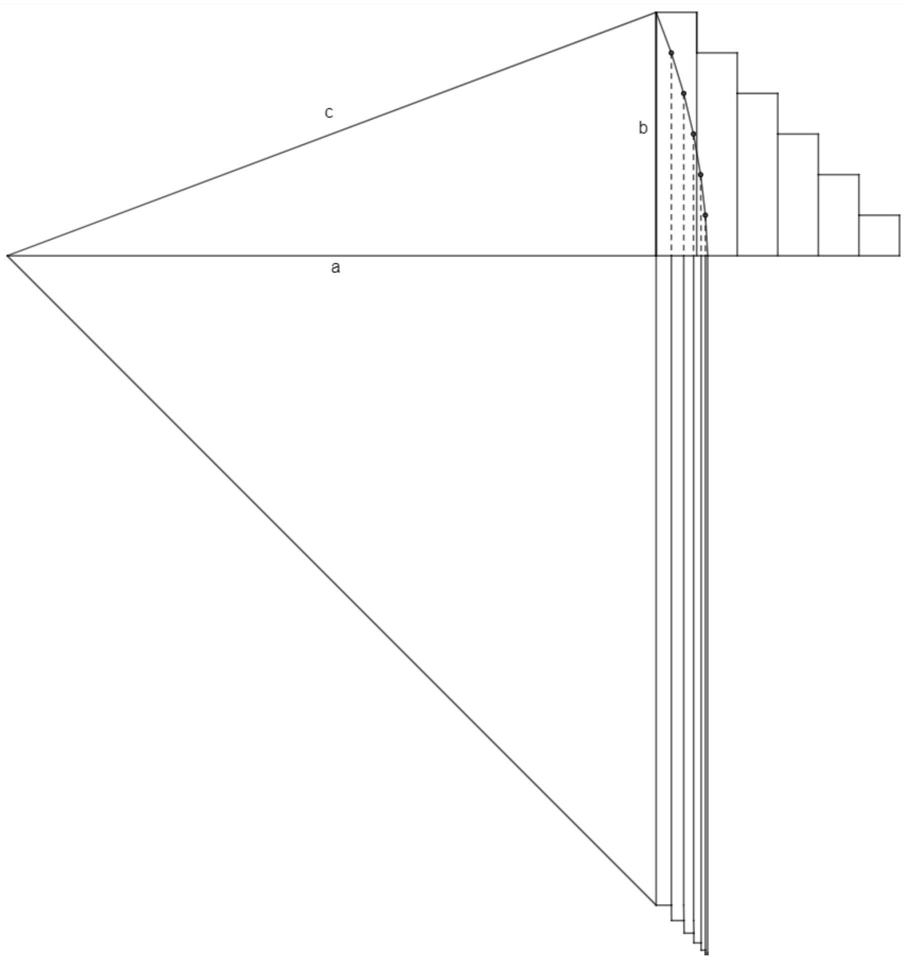
Needless to merge more slices in specific, the pattern should be clear by now. Review the rotating hypotenuse case, one should be able to see the variables and maths expressions of this merging trianlges model being exactly the same as those of the rotating hypotenuse model, in a way that the side length of the resulting right angled-triangle evolves exactly the same way as the horizontal component of the hypotenuse, which means that the horizontal projection of the hypotenuse after the $N$-step rotation can alternatively be found by calculating the side length of the resulting triangle, after $N$-step merging. Finally, for the merging triangles model, if $N \to \infty$, then $\Delta \to 0$, and there would be no missing little triangles anymore as the area of those little trianges are $\sim \Delta^2$,



i.e. the resulting triangle is now a legitimate isosceles right-angled triangle, let us denote its side length by $c$. As mentioned before, the "triangle" to the right also becomes a real right-angled triangle with side length $b$ as $N \to \infty$. Finally

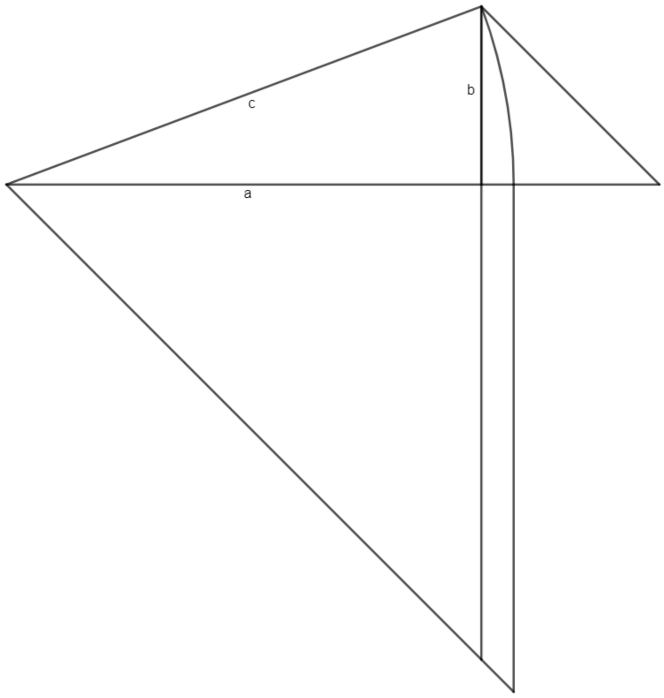
$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2 \tag{16}$$

or $c = \sqrt{a^2 + b^2}$. On the other hand, when $N \to \infty$, the horizontal projection of the hypotenuse equals to the hypotenuse itself after the rotation is complete , that means the hypotenuse is also $c = \sqrt{a^2 + b^2}$. What interesting about the square root operation here is that it compromises the infinite amount of merging slice actions into just one nice operation, in our case anyway. It is also probably one of the reasons why the square root of a number is usually an irrational number. Now a conclusive figure to put the two models together:




This figure shows the $N = 6$ case, by a glance the 6 tangent segments together already look much like an arc. Here triangle $abc$ itself is shown, along $a$, the base triangle with area $\frac{1}{2}a^2$ is drawn, pointing downward; to the left of $b$, the approximated "triangle" cut up in 6 slices is shown. While for the rotation dynamics, $c$ takes 6 tangent segments to rotate clockwise to $a$, in these 6 steps, $c$'s horizontal projection after each tangent movement are also shown by the dotted lines. At the same time, the 6 slices are as well merged onto the right side of the base triangle. One can see that in every step, the "side length" of the resulting triangle is exactly the same as the horizontal component of the hypotenuse, which proves the equivalence of these two models.

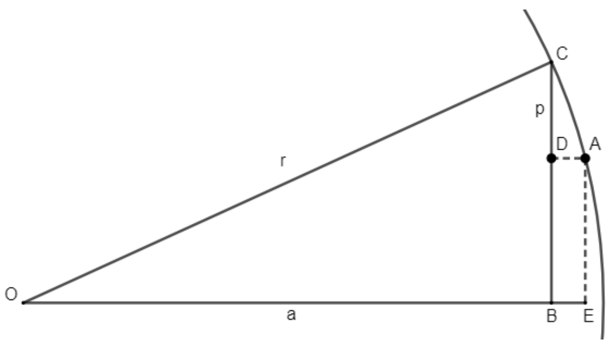
The limiting case where $N \to \infty$ it would look like:



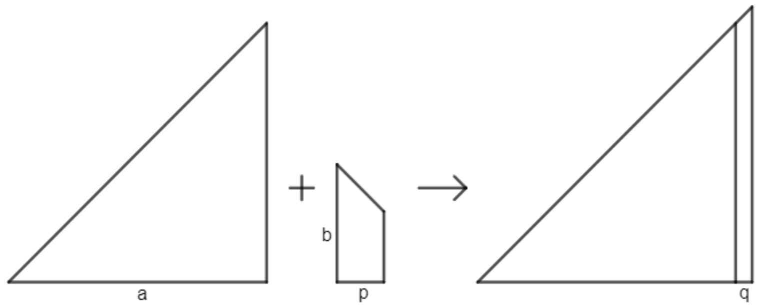

## 2.4 Conclusion and generalization

Now the proof is complete, we find that the square root operation is in some way connected with rotation, not in the sense of $\pi$, as the actual length of the tangent segments(or the arc length) was never in our calculation, but rather, the square root operation is well connected with the change of the horizontal and vertical components swiped by an arc.

To generalize this result a little more, if one asks a question: an object is continuously doing a circular motion about the origin $O(0,0)$, it was at $C$ with coordinate $(a,b)$, some time later we measured its y-coordinate and find it was decreased by $p$, then how much did its x-coordinate change compare to $a$?




here $CB = b$. So the object traveled from $C$ to $A$ along the circle and its y-coordinate had decreased by $p$. The answer to this question is, rather than let the hypotenuse rotate all the way down, stop it at point $A$, and the increment in horizontal projection which is $BE$, would be the desired result. Equivalently, in terms of merging triangles, merge from the tallest slices until the next slice is $b-p$ in height. So it is really just using part of the triangle rather than the whole triangle to merge to the base triangle,



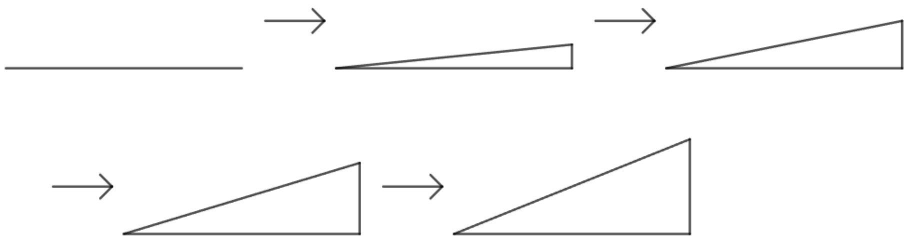

taking the square root of the sum of the areas of the base triangle and the trapezoid, would give us the horizontal projection at $A$, then subtract it by $a$ we would get the horizontal increment $p$ and the question is done. The exact numerical answer to this question and other related results will not be discussed here as we mainly focus on the proofs of the Pythagorean theorem.

# 3 The second method

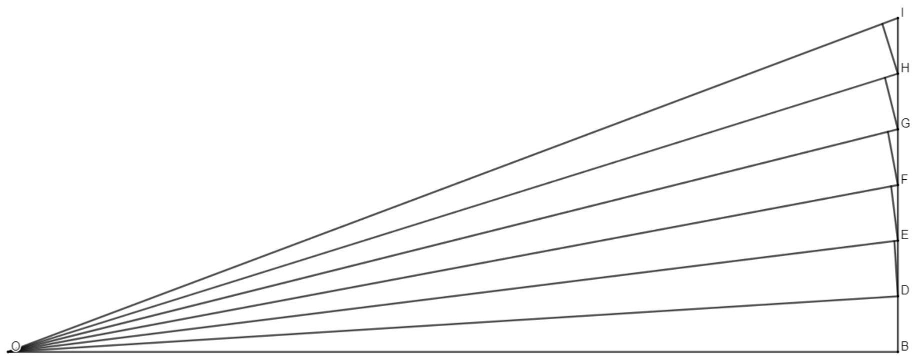
Now, if one still feels that the first method is not quite direct and intuitive enough, as one could claim we did not find the actual hypotenuse, we only found the horizontal projection of the hypotenuse after rotating it down manually, well the second method might just be the choice as it is more from the scratch and more continuous in a sense. The analogy and techniques of the second methods are quite similar to the first one, so we will focus more on the setup side and omit some of the details.

## 3.1 The setup

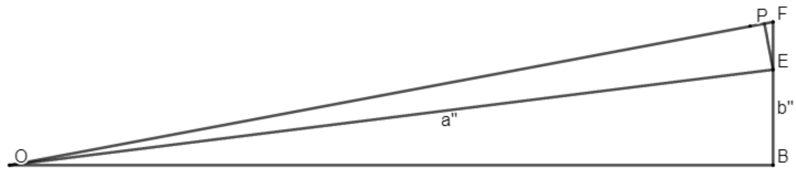
The approach of this method is still a dynamical one,



starting from side $a$ lying down horizontally, this time, let $a$ be the hypotenuse and rotate upward in $N$ steps, each step by $\Delta = b/N$ in height. The hypotenuse in every step is then calculated inductively(like the horizontal projection in the first method), and the hypotenuse after $N$ steps would be the desired hypotenuse for triangle $abc$, or at least an $N$-step approximation. So in the first method, we let the vertical component start from $b$ then decrease $\Delta$ per step, and become 0 at the end. While in this method, it is exactly the opposite, i.e. the vertical component starts from 0, increases $\Delta$ per step and becomes $b$ at the end. The graph for the $N = 6$ case would look like:




so $OB = a$, it is also the initial hypotenuse, and $IB = b$. Moreover, $IH = HG = ... = DB = \Delta$. In each step, the hypotenuse will do a tangent movement, then pick up some extra length, and the next step it will be exactly the same but with a longer hypotenuse. To be more specific, assume at some stage of the rotation,




the hypotenuse at that time is $OE$ or call it $a''$, and the vertical component is $EB$ or call it $b''$. Now $PE$ is the tangent segment, $PE \perp a''$ as direction of motion is always perpendicular to the radius. The new hypotenuse after this tangent movement is $OF$, which is $OP + PF$. Here we have to make an assumption that $OP = OE$(it becomes true in the limiting case), hence $OP = a''$. Now let us calculate $PF$, the increment of hypotenuse. Different from the first method, in right-angled triangle $FPE$, $FE$ is now the hypotenuse while tangent segment $PE$ is one of the leg. By similar triangles, as $FE = \Delta$, $PF = \frac{b''}{a''}\Delta$, so the new hypotenuse $OF$,

$$OF = a'' + \frac{b''}{a''}\Delta \tag{17}$$

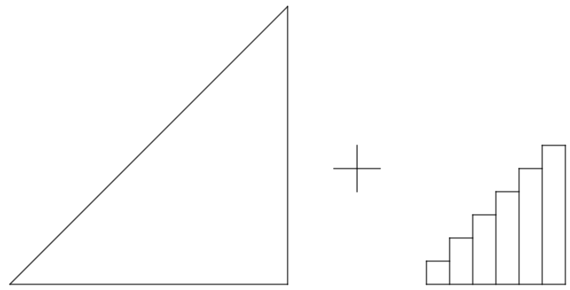
and the new vertical component $BF$,

$$BF = b'' + \Delta \tag{18}$$

Unsurprisingly, if we denote the new hypotenuse $OF$ as $a'''$, and the new vertical component $BF$ as $b'''$, when we do the next tangent movement the procedure is exactly the same, there is no need to go into details anymore.

Of course, one might have already noticed another problem, that is $PF$ and $OP$ are not really on the same line. $PF$ is parallel to $OE$, but $OP$ is not parallel to $OE$ as it is the hypotenuse of the right-angled triangle $OEP$. And again, in the limiting case where triangle $OEP$ is no difference from a straight line, $OE$ would be parallel to $OP$, consequently $PF$ and $OP$ would together form a line segment.

The corresponding merging triangles model would be:

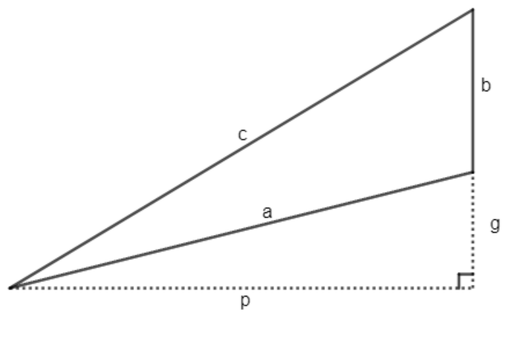


so instead of merging slices from tallest to lowest, it is now from the lowest to tallest. The equivalency is that for a given $N$, at any step, the side length of the resulting triangle is exactly same as the length of the hypotenuse. This equivalency is no more different than the equivalency in the first method, so no further illustration will be given here.

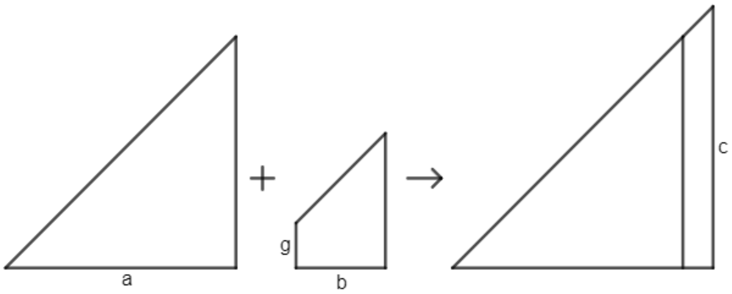
Finally when $N \to \infty$, there will be no bugs in either models, and the final hypotenuse, which is the hypotenuse of triangle $abc$, must be equal to the final side length of the resulting triangle, which is again $\sqrt{a^2 + b^2}$.

## 3.2 Conclusion and generalization

Now the second proof is complete, comparing it to the first method, it is more straightforward as the end product after the rotation is exactly the hypotenuse of triangle $abc$, at where it originally belongs. The rotation in this method is not really a classical rotation as its radius always changes, while in the first method the radius stays the same. The second method is also continuous in the sense that at any step of the rotation, there is always a calculated hypotenuse for some right-angled triangle. For example, at the stage that a third of the slices are merged, the hypotenuse at that stage is $\sqrt{a^2 + (b/3)^2}$, it actually represents the hypotenuse of a right-angled triangle with legs $a$ and $\frac{1}{3}b$. In this spirit, we can also try not to start from the first slice,




here the starting point is $a$, which is already the hypotenuse of a right-angled triangle, and we want to rotate the hypotenuse further, until it is $b + g$ in height, to get a larger hypotenuse $c$. More importantly the starting point is not a right-angled triangle anymore. The merging process here is to start merging from the slice with height $g$, and stop at the slice with height $g + b$.




Of course, this situation is in some way similar to the cosine law, while the difference being we are not provided with an angle, but instead of more known sides. To discuss its relation with the cosine law would involve trigonometry functions and angles, they are beyond the scope of this paper so will not be discussed here.

# 4 Final note

Both proofs could give one the sense that we did not do too much in terms of proving something, we rather just found two procedures of merging two triangles together into a bigger one(there can be infinite ways of merging two triangles together), only that these two procedures could explain the dynamics of a hypotenuse in a human-understandable way. But that was exactly the scope here, which is to dig into the Pythagorean theorem as it is, without clever manipulations of areas. Only that the theorem says $a^2 + b^2 = c^2$, we rather found $\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$ could explain better in terms of the two methods we used.

# References

As this paper is relatively fundamental and uses only common and basic mathematics, no citations were used.