Zeros of a sigma-additive set complex function. The case of the Fourier Transform
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Abstract
A non-trivial interpretation of Fourier integral theorem in the framework of measure spaces.

Let \( f \in L^1 (-\infty, +\infty) \) have constant sign and no zeros. By the Fourier integral theorem:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega t} d\omega
\]  

(1)

\[
\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt
\]

We define:

\[
\mu : \mathbb{R} \rightarrow (0, +\infty)
\]

(2)

\[
\mu(t) = \int_{-\infty}^{t} |f(\tau)| d\tau > 0, \forall t \in \mathbb{R}
\]

\[
\Sigma := \{ A = [t_0, t] | t_0, t_1 \in \mathbb{R} \}
\]

(3)

\( \Sigma \) is manifestly a \( \sigma \)-algebra on \( \mathbb{R} \). (2) defines a countably additive and positive set function on \( \Sigma \)

\[
\mu : \Sigma \rightarrow (0, +\infty)
\]

(4)

\[
\mu(A) = \int_{A} f(t) dt, \forall A \in \Sigma
\]

and is complete on \( \Sigma \) [1]. So \((\mathbb{R}, \Sigma, \mu)\) is a measurement space. The second of (1) becomes:

\[
\hat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} d\mu
\]

(5)

We define:

\[
\nu_\omega(A) := \int_{A} e^{-i\omega t} d\mu, \forall A \in \Sigma, \omega \in \mathbb{R}
\]

(6)

which is broken down into a real part and an imaginary part:

\[
\nu_\omega(A) = \underbrace{\text{Re} \nu_\omega(A)}_{\xi_\omega(A)} + i \underbrace{\text{Im} \nu_\omega(A)}_{\eta_\omega(A)}
\]

(7)

where

\[
\xi_\omega(A) = \int_{A} \cos(\omega t) d\mu, \eta_\omega(A) = \int_{A} [-\sin(\omega t)] d\mu, \forall A \in \Sigma
\]

(8)

are countably additive set functions.

Lemma 1 The set functions \( \xi_\omega(A) \), \( \eta_\omega(A) \) are absolutely continuous with respect to \( \mu \).

Proof.

\[
\mu(A) \equiv 0 \iff f(t) \equiv 0 \iff (\xi_\omega(A) \equiv 0 \iff \mu(A) \equiv 0
\]

Likewise for \( \eta_\omega(A) \). ■
Lemma 2

\[ \frac{d \xi}{d \mu} = \cos(\omega t), \quad \frac{d \eta}{d \mu} = \sin(\omega t) \]  

(9)

where \( \frac{d}{d \mu} \) is the Radon-Nikodym derivation operator.

Proof. The statement follows immediately from the Radon-Nikodym theorem \[1\]. ■

Definition 3 The \( \nu_{\omega}(A) \) defined by (6) is called set complex function.

From this it follows that the function:

\[ \rho_{\omega}(t) = e^{-i\omega t} \]  

(10)

is the Radon-Nikodym derivative of the set function \( \nu_{\omega}(A) \) with respect to the measure \( \mu \):

\[ \frac{d \nu_{\omega}}{d \mu} = \rho_{\omega}(t) \]  

(11)

Notation 4 \( \nu_{\omega}(\mathbb{R}) \) is the Fourier transform of \( f(t) \).

From the absolute continuity of \( \nu_{\omega}(A) \) with respect to \( \mu \), it follows that \( \nu_{\omega}(\mathbb{R}) \) can vanish only with respect to \( \omega \):

\[ \exists \omega_0 \in \mathbb{R} \mid \nu_{\omega_0}(\mathbb{R}) = 0 \]

We will therefore say that \( \omega_0 \) is a zero of \( \nu_{\omega}(\mathbb{R}) \). It follows

Lemma 5 If \( f(t) \) has definite parity \( \nu_{\omega}(\mathbb{R}) \) is devoid of zeros.

For example for the Gaussian

\[ f(t) = e^{-\frac{t^2}{2\alpha}} (\alpha > 0) \]  

(12)

we have \( \nu_{\omega}(\mathbb{R}) = e^{-\frac{\omega^2}{2}} \) which is devoid of zeros.

Notation 6 If \( f(t) \) has parity \( (+1) \), \( \nu_{\omega}(\mathbb{R}) \) has zero imaginary part. If \( f(t) \) has parity \( (-1) \), \( \nu_{\omega}(\mathbb{R}) \) has zero real part. In both cases, the function \( \hat{f}(\omega) \) preserves parity. Furthermore

\[ |\hat{f}(\omega)| \text{ has definite parity } \Rightarrow f(t) \text{ has definite parity} \]

The introduction of a parameter \( \alpha \) in \( f \) (eq. (12)) suggests extending the previous arguments to a real function of the two real variables \((a, t)\) defined on the strip

\[ S = [a, b] \times (-\infty, +\infty) \]  

(13)

for a given interval \([a, b]\) of \( \mathbb{R} \), limited or unlimited. We keep the previous hypothesis, i.e. \( f(\alpha, t) \) of class \( L^1(-\infty, +\infty) \) with respect to \( t \) and of constant sign. Fourier’s integral theorem returns the function of the complex variable \( \alpha + i\omega \)

\[ \hat{f}(\alpha + i\omega) = \int_{-\infty}^{+\infty} f(\alpha, t) e^{-i\omega t} dt \]

which for a given \( f(\alpha, t) \) can be holomorphic on the strip (13).
The function
\[ \mu_\alpha(t) := \int_{-\infty}^{t} |f(\alpha, t')| \, dt', \quad \forall \alpha \in [a, b] \] (14)
defines a one-parameter measure:
\[ \mu_\alpha : \Sigma \longrightarrow (0, +\infty) \] (15)
\[ \mu_\alpha : A \longrightarrow \mu_\alpha(A) = \int_{A} f(\alpha, t) \, dt, \quad \forall A \in \Sigma \]

The generalization of the previous definition follows
\[ \nu_{\alpha, \omega}(A) := \int_{A} e^{-i\omega t} \, d\mu_\alpha, \quad \forall A \in \Sigma, \quad (\alpha, \omega) \in S \] (16)

If \( \hat{f}(\alpha + i\omega) \) is holomorphic on \( S \), any accumulation points of the set of zeros of \( \nu_{\alpha, \omega}(\mathbb{R}) \) belong to \( \partial S \).

**References**