For calculating Nontrivial Zeros of Riemann Zeta function-ζ, the definition

\[ \xi(s) = \frac{s}{2} (s-1) \pi^{s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) \]

of Riemann Xi function-ζ is not appropriate.

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ABSTRACT

We show that for calculating nontrivial zeros of the Riemann Zeta function $\zeta$, the form of the definition $\xi(s) = (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $s \in \mathbb{C}$ of the function $\xi$ and the followed deduction that nontrivial zeros of functions $\zeta(s)$ and $\xi(s)$ are identical is not appropriate. The definition of function $\xi$ in which both functions $\xi$ and $\zeta$ are functions of same complex variable $s$ and the assumption of identicalness of nontrivial zeros of $\xi$ and $\zeta$ is ambiguous, so may be the deep reason, the Riemann hypothesis could not be resolved yet. However, the definition $\xi(t) = (s/2)(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $t = \alpha + i\beta$ and $s = 1/2 + it$, introduced by B. Riemann (1859) leads the results: (i) when $\xi(\alpha + i\beta) = 0$, $\beta = 0$, $\alpha \in \mathbb{R}$, corresponding nontrivial zero of the function $\zeta(s)$ are of the form $s = 1/2 + i\alpha$ and (ii) when $t = \alpha + i\beta$ and $\xi(\alpha + i\beta) = 0$, nontrivial zeros of the function $\zeta(s)$ are of the form $s = (1/2 - \beta) + i\alpha$ which lie on both sides of the line $\alpha = 1/2$. Here, we sketch the zeros of the function $\zeta(s)$ those correspond to real zeros of the function $\xi(s)$ that shows the Riemann hypothesis is true only when nontrivial zeros of functions $\xi(s)$ and $\zeta(s)$ lie on two perpendicular lines.

Keywords: Zeta function, Riemann’s Xi Function, nontrivial zeros, critical strip, critical line.
1 INTRODUCTION

In 1859, B. Riemann [1] in his research report introduced a function $\zeta(s) = \sigma + it, \sigma, t \in \mathbb{R}$ known as the Riemann’s zeta function $\zeta(s)$ with the definition,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \ldots (1)$$

Riemann further created another function known as the Riemann Xi-function $\xi(t), t = \alpha + i\beta$ defined as:

$$\xi(t) = \frac{(s/2)(s-1)}{\pi^{s/2}}\zeta(s), s = \frac{1}{2} + it \quad \ldots (2)$$

The definition (2) is the original definition of the function $\xi$. But in Mathematics literature present day authors e.g. [2], [3], [4] and other use an alternative definition of function $\xi$ as

$$\xi(s) = \frac{(s/2)(s-1)}{\pi^{s/2}}\zeta(s) \quad \ldots (3)$$

With the definition (3) authors claim that nontrivial zeros of functions $\xi(s)$ and $\zeta(s)$ are identical.

In this research article, we show that the use of the definition (3) of the function $\xi$ cannot be justified as it creates mathematical ambiguities. However, the original definition (2) of the function $\xi$ corroborated with Riemann’s statement: “it is clear that $\xi(t)$ can vanish only if the imaginary part of $t$ lies between $i/2$ and $-i/2$.” which indicates that $t$ is a complex number produces the results: (i) Corresponding to each complex zero $t = \alpha + i\beta$ of the function $\xi(t)$, there exists a complex zero $s = (1/2 - \beta) + i\alpha$ of the function $\zeta(s)$, i.e., zeros of functions $\xi(t)$ and $\zeta(s)$ are a distance apart (not identical). (ii) Corresponding to each real zero $t = \alpha, \alpha \in \mathbb{R}$ of the function $\xi(t)$, there exists a complex zero $s = \frac{1}{2} + i\alpha$ of the function $\zeta(s)$. Perceived zeros of results (i) and (ii), are shown in Fig. 1(a) and 1(b) on the last page.
2 RESULTS

Recall the definition (3) connecting functions $\xi$ and $\zeta$ both of same complex variable $s$,

$$\xi(s) = (s/2)(s-1)(\pi^{s/2})\zeta(s), \quad s = \mu + i\lambda$$

\[ \text{... (4)} \]

Clearly, $\xi(0) = 0$ and $\xi(1) = 0$, so $s = 0, s = 1$ are real zeros of $\xi(s)$. Suppose zeros functions $\xi(s)$ and $\zeta(s)$ are identical, then $s = 0, s = 1$ must also be zeros of $\zeta(s)$ but according to definition (1) of $\zeta(s)$, $\zeta(0) = \infty$ and $\zeta(1) = \infty$, therefore, $s = 0, s = 1$ are not zeros of $\zeta(s)$, so not of the function $\xi(s)$. That is ambiguity in definition (3). Actually, when $s$ is a real number, all zeros of $\zeta(s)$ necessarily are zeros of the function $\xi(s)$ but when $s$ is a complex number zeros of functions $\xi(s)$ and $\zeta(s)$ may be different can be shown as:

Suppose $\xi = G + iH, (s/2)(s-1)(\pi^{s/2}) = C + iD$ and $\zeta(s) = A + iB$, then from result (4),

$$G + iH = (CA - DB) + i(AD + BC)$$

\[ \text{... (5)} \]

Zeros of $\xi(s)$ can be obtained choosing $G=0$ and $H=0$ which means $CA - DB = 0$ and $AD + BC = 0$. This system of equations produces $A = 0$, $B = 0$, $A = iB$, $C = 0$, $D = 0$ and $C = iD$. Moreover, the function $\zeta(s)$ can be written as $\zeta(s) = \sqrt{A^2 + B^2} (cos\phi + isin\phi)$ with

$$\phi = tan^{-1}\left(\frac{B}{A}\right).$$

Now, if $\xi(s) = 0, (s/2)(s-1)(\pi^{s/2}) \neq 0$, then $\sqrt{A^2 + B^2} (cos\phi + isin\phi) = 0$ which implies the equation $\zeta(s) = 0$ is unsolvable as when $sin\phi = 0 \Rightarrow cos\phi \neq 0$.

Further, suppose that $s = a_i, a_i \in \mathbb{R} or \mathbb{C}, \ i = 1, 2, 3, \ldots, n$ are zeros of the function $\xi(s)$ and $s = b_j, b_j \in \mathbb{R} or \mathbb{C}, \ j = 1, 2, 3, \ldots, m$ zeros of the function $\zeta(s)$, i.e., $\xi(s) = \prod_{i=1}^{n}(s - a_i)$ and $\zeta(s) = \prod_{j=1}^{m}(s - b_j)$. Therefore, the result (4) can be expressed as,
\[
\prod_{i=1}^{n} (s-a_i) = (s/2)(s-1)\pi^{\alpha/2}\Gamma(s/2) \prod_{j=1}^{m} (s-b_j) \quad \ldots (6)
\]

There are two cases:

**Case I:** At least one zero \( s = a \) (say) is common to both functions \( \xi \) and \( \zeta \) then,

\[
\xi(s) = (s-a) \prod_{i=1}^{n-1} (s-a_i), \quad \text{and} \quad \zeta(s) = (s-a) \prod_{i=1}^{m-1} (s-b_i),
\]

therefore,

\[
(s-a) \prod_{i=1}^{n-1} (s-a_i) = (s/2)(s-1)\pi^{\alpha/2}\Gamma(s/2)(s-a) \prod_{i=1}^{m-1} (s-b_i).
\]

Further, write \( \prod_{i=1}^{n-1} (s-a_i) = \xi_1(s) \) and \( \prod_{i=1}^{m-1} (s-b_i) = \zeta_1(s) \), then

\[
(s-a) \left[ \xi_1(s) - (s/2)(s-1)\pi^{\alpha/2}\Gamma(s/2)\xi_1(s) \right] = 0
\]

\[
\xi_1(s) - (s/2)(s-1)\pi^{\alpha/2}\Gamma(s/2)\xi_1(s) \neq 0
\]

\[
\left[ \xi_1(s) - (s/2)(s-1)\pi^{\alpha/2}\Gamma(s/2)\xi_1(s) \right]_{s=a} = \left\{ 0/(s-a) \right\}_{s=a}
\]

\[
\left[ \xi_1(a) - (a/2)(a-1)\pi^{\alpha/2}\Gamma(a/2)\xi_1(a) \right] = 0/0
\]

\[
\ldots (7)
\]

Thus there exists at least one case that for \( s = a \): \( \left[ \xi_1(a) - (a/2)(a-1)\pi^{\alpha/2}\Gamma(a/2)\xi_1(a) \right] \) is not non-zero but indeterminate. But, in general \( \left[ \xi_1(a) - (a/2)(a-1)\pi^{\alpha/2}\Gamma(a/2)\xi_1(a) \right] \) is considered a non-zero.

**Case II:** Functions \( \xi(s) \) and \( \zeta(s) \) have same number say \( p \) of identical zeros. Let \( \gamma \) be one of such zeros, then

\[
\prod_{i=1}^{p} (s-\gamma)^p \left[ 1-(1/2)s(s-1)\pi^{\alpha/2}\Gamma(s/2) \right] = 0
\]

\[
\Rightarrow 1-(1/2)\gamma(\gamma-1)\pi^{\gamma/2}\Gamma(\gamma) = 0/0 \quad \ldots (8)
\]
If \( 1-(1/2)\gamma(\gamma-1)\pi^{\gamma/2}\Gamma(\gamma) \) is nonzero then from result (8), either \( 0 = 0 \) or

\[
1-(1/2)\gamma(\gamma-1)\pi^{\gamma/2}\Gamma(\gamma)
\]
is indeterminate. Also, if \( \gamma \) equals 1, then \( 1 = 0/0 \) and if

\[
1-(1/2)\gamma(\gamma-1)\pi^{\gamma/2}\Gamma(\gamma)
\]
equals zero then \( 0 = 0/0 \), i.e. 0 is itself indeterminate.

Whatever be the case I or II discussed above but even one common zero \( s = \alpha \) (say) results

\[
[\xi_\alpha(\alpha) - (1/2)\alpha(\alpha-1)\pi^{-\alpha/2}\Gamma(\alpha/2)\xi_\alpha(\alpha)] = 0/0
\]
which shows 0 is not a free number, its use is conditional. Thus, from the above discussion it can be concluded that (i) the definition (3) of the function \( \xi \) is not a proper definition for calculating nontrivial zeros of the function \( \zeta(s) \) and (ii) to solve an equation like \( f(x) \times g(x) = 0 \), \( f(x) \) or \( g(x) \in \mathbb{C} \), the definition of zero requires investigation because the conclusion from the equation \( X + iY = 0 \), \( X,Y \in \mathbb{R} \)
implies \( X = 0 \) and \( Y = 0 \) is not always true. The consideration \( \zeta(s) = X + iY = 0 \) implies \( X = 0 \) and \( Y = 0 \) is the foremost reason; the Riemann hypothesis could not have been resolved yet, also the claimed nontrivial zeros \( 14.134725142, 21.022039639, 25.010857580 \)
and so on may not be nontrivial zeros of the function \( \zeta(s) \) but are of some other function.

That we will show elsewhere.

Now, using the definition \( \xi(t) = (s/2)(s-1)(\pi^{-s/2})\xi(s) \), we establish a relation between nontrivial zeros of function \( \xi(t) \) and \( \zeta(s) \), \( s = 1/2 + it \).

Riemann states: “It is clear that \( \xi(t) \) can vanish only if the imaginary part of \( t \) lies between \( i/2 \) and \( -i/2 \)” That suggests \( t \) is a complex variable. Suppose \( t = \mu + i\lambda \) (say) and

\( \zeta(s), s = 1/2 + it \). Therefore, from the definition (3),
\[ \xi(\mu + i\lambda) = (1/2)(1/2 - \lambda + \mu i)(1/2 - \lambda + \mu i - 1)\pi^{(1/2 - \lambda + \mu i)^2} \Gamma[(1/2)(1/2 - \lambda + \mu i)] \xi(1/2 - \lambda + \mu i) \]

Substitute, 0 for \( \mu \) and 1/2 for \( \lambda \) (or \( t = i/2 \))

\[ \xi(i/2) = (1/2)(0)(-1 + 0i)\pi^0 \Gamma(0)\xi(0) = 0 \quad \ldots (9) \]

Substitute, 0 for \( \mu \) and \(-1/2\) for \( \lambda \) (or \( t = -i/2 \))

\[ \xi(-i/2) = (1/2)(1)(0)\pi^{1/2} \Gamma[(1/2)(1)]\xi(1) = 0 \quad \ldots (10) \]

That shows \( t = -i/2 \) and \( t = i/2 \) are nontrivial zeros of the function \( \xi(t) \) but corresponding to \( \xi(-i/2) \) and \( \xi(i/2) \) the values \( \zeta(0) \) and are \( \zeta(1) \) undefined. To avoid this ambiguity Riemann states: “\( \xi(t) \) can vanish only if the imaginary part of \( t \) lies between \( i/2 \) and - \( i/2 \)”.

The results (9) and (10) show if nontrivial zeros of function \( \xi(t) \) lie between \( t = -i/2 \) to \( t = i/2 \), then corresponding zeros of the function \( \zeta(s) \) lie between \( s = 1 \) to \( s = 0 \). Thus, the range of nontrivial zeros of the function \( \zeta(s) \) is \( s \in [0,1] \) which is the critical strip for nontrivial zeros of function \( \zeta(s) \).

The critical strip for nontrivial zeros of \( \zeta(s) \) can also be determined as:

Suppose \( t = \alpha \pm i\beta \) are zeros of the function \( \xi(t) \), then according to Riemann’s statement,

\[-i/2 \leq t \leq i/2 \]
\[ \Rightarrow i^21/2 \leq -it \leq -i^21/2 \Rightarrow -1/2 \leq -i \alpha \pm i\beta \leq 1/2 \]
\[ \Rightarrow 1/2 \leq -i\alpha \mp \beta \leq 1/2 \Rightarrow 1/2 \geq i\alpha \pm \beta \geq -1/2 \]
\[ \Rightarrow 1 \geq 1/2 \pm \beta + i\alpha \geq 0 \Rightarrow 0 \leq 1/2 \pm \beta + i\alpha \leq 1 \]

But \( 1/2 \pm \beta + i\alpha \) is variable of the function \( \zeta(s) \) corresponding to \( t = \alpha \pm i\beta \). Thus, if zeros of function \( \xi(t) \) lie between \( t = -i/2 \) to \( t = i/2 \), then zeros of the function \( \zeta(s) \) lie between \( s = 0 \) to \( s = 1 \).
Thus, nontrivial zeros of the function $\zeta(s)$ are of the form $1/2 + i \alpha$ that lie in the region $0 \leq 1/2 + \beta \leq 1$ that verbalize the Riemann hypothesis. Further, if $\beta$ equals zero, i.e. all zeros of function $\xi(t = \alpha \pm i\beta)$ are real then zeros of $\zeta(s)$ are of the form $1/2 + i\alpha$ that lie in the region $0 \leq 1/2 \leq 1$ on the line $a = 1/2$. Clearly, the functions $\xi(t = \alpha \pm i\beta)$ and $\zeta(s = 1/2 + it)$ have same number of zeros and there is one-to-one correspondence between real zeros of the function $\xi(t)$ and nontrivial complex zeros of the function $\zeta(s)$.

The perceived (not calculated) nontrivial zeros of functions $\xi(t)$ and $\zeta(s)$ when (i) $t \in \mathbb{C}$ a complex number, and (ii), when $t$ is real number are shown in Fig. 1(a) and Fig. 1(b) respectively. **Note:** Here, for to show the relative locations of zeros of the function $\zeta(s)$, zeros of the function $\xi(t)$ are perceived (not calculated).
Thus, if \( t = \alpha \pm i\beta \) is zero of the function \( \xi(t) \), then corresponding zero of the function \( \zeta(s) \) is \( s = \left( \frac{1}{2} \mp \beta \right) \pm i\alpha \). That show zeros of functions \( \xi \) and \( \zeta \) cannot have same form and same variable and in the context of the Riemann hypothesis the form of definition of function \( \xi \)

\[
\xi(s) = \left( \frac{s}{2} \right) (s-1) \pi^{-s/2} \zeta(s), \ s = \mu + i\lambda
\]

is ambiguous.

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**References**


Declaration

The Author does not have any compelling interest writing this research article. The Author communicates this research article through this pre-print repository to share the knowledge to the interested audience.

Additional Information: Corresponding to non-trivial zero $\alpha + i\beta$ of the function $\xi$, non-trivial zero of the function $\zeta$ is $\left(\frac{1}{2} - \beta\right) + i\alpha$.

Fig. 1(a): Zeros of functions $\xi(t)$ and $\zeta(s)$ when $t$ is a complex variable

Fig. 1(b): Zeros of functions $\xi(t)$ and $\zeta(s)$ when $t$ is a real variable