A simple Markov chain for the Collatz problem

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Abstract

We show that the iteration of the Collatz function is represented by a simple three states Markov chain. This simple model is implemented to show the probabilistic convergence of the algorithm to the equilibrium point set \{1,2\}.

Introduction

Define the iterating function introduced by R. Terras[1]:

\[ a_{n+1} = \frac{(3^b a_n + b)}{2} \]  \hspace{1cm} (1)

where \( b = 1 \) when \( a_n \) is odd and \( b = 0 \) when \( a_n \) is even. The Collatz conjecture asserts that by starting with any positive integer \( a_0 \), there exists a natural number \( k \) such that \( a_k = 1 \).

1. The Markov Chain and the transition probability

The Eq. (1) can be represented by a Markov chain with three states.

Let partition positive natural numbers \( N \) in three sets (states):

A : \{3,5,7,9,……………\}

B : \{4,6,8,10,…………..\}

C : \{1,2\}

Then, consider the following Markov chain:

\[ X_{i+1} = PX_i \]  \hspace{1cm} (2)

Where \( X_i \) is a vector with three components each of which represents the probability of a number to belong to one of the above defined sets, i.e. \( X_i (1,1) = \text{Prob.\{a number is in A\}}, X_i (2,1) = \text{Prob.\{a number is in B\}} \) and \( X_i (3,1) = \text{Prob.\{a number is in C\}} \). \( P \) is a 3x3 real transition matrix whose element, i.e. \( p_{ij} \) is the probability of transition from state \( j \) to state \( i \). The Markov chain of the Collatz problem is shown in Figure 1.
Figure 1. Markov chain of the Collatz problem.

\[
P = \begin{bmatrix} 
\frac{1}{2} & p & 0 \\
\frac{1}{2} & p & 0 \\
0 & 1 - 2p & 1 
\end{bmatrix}; \quad p < \frac{1}{2}
\]

2. The probabilistic convergence of the dynamic system

The limiting of \( X_i \) can be defined as

\[
X_\infty = \lim_{n \to \infty} P^n X_0
\] (3)
Let
\[ P = S V S^{-1} \quad (4) \]
where \( S \) and \( V \) are a 3x3 eigen vector and eigen value matrix, respectively.

\[
V = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & p + 1/2
\end{bmatrix}
\]

\[
S = \begin{bmatrix}
0 & 2/c & -1/d \\
0 & -1/cp & -1/d \\
1 & (1-2p)/cp & (2-4p)/d(1-2p)
\end{bmatrix}
\]

where
\[
c^2 = \frac{(8p^2 - 4p + 2)}{p^2}
\]

\[
d^2 = \frac{(6p^2 - 6p + 1.5)}{(0.5 - p)^2}
\]

and \( S^{-1} \) is shown to represent in a matrix form as

\[
S^{-1} = \begin{bmatrix}
1 & 1 \\
[M]_{2 \times 3}
\end{bmatrix}
\]

where \([M]_{2 \times 3}\) is a real 2x3 matrix.

As \( n \to \infty \),
\[
\lim_{n \to \infty} P^n = S \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} S^{-1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

Then a convergence of eq. (3) follows as
The matrix structure which yields the above condition, show that the Markov chain has an absorbing state which is a state C. Once entered in state C it remains in state C, i.e. a number will alternate between 1 and 2.

3. **Conclusions**

In the paper, it has been shown that using a simple structured Markov chain, eq. (1) representing the Collatz iteration, converges to 1 with probability 1.

**References**