Some notes on shadows
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Jul 21, 2023

Abstract: These notes were taken whilst thinking about the monotone-light factorization, which lead to the productive idea of a shadow category, or a certain kind of category with a preordered structure.

§0 Preamble
Conventions

By an isomorphism we will always mean a bijective n-categorical isomorphism. By a pushout or pullback, we will mean an n-pushout or n-pullback. We avoid working with the finer details of n-categories, but appreciate their relationship to one-categories for the purposes of localization.

Let \( K = \partial K^2 \) be a boundary-forming and simply connected space. Let the convex portion of space about \( K \) have an inner product

\[
\text{Sets} \times \text{Top} \to \text{DispMfld}
\]

\[
\downarrow
\]

\[
\text{SSets}
\]

so that there is a projection onto a curve within \( \mathcal{U}(K) \).

We will call \( K \) an \((\varepsilon, \delta)\)-chain, and we will call its pushout into \( \text{SSets} \) a “shadow.” We will operate using the commutative fusion rule

\[
\varepsilon \star \delta = \delta \star \varepsilon = \text{Hom}(\text{Sets} \times \text{Top}, -)
\]

We characterize each unique geometric fibration \( \theta \to \{-\} \) according to a “length spectrum,” which “records” information about the number of objects with maps into identical tangent categories.

Definition 0.0.1 A \( \delta \)-transitive connection is a first-degree connection on an \((\varepsilon, \delta)\)-chain.

Let \( A \) be a geometric series with a least element \( \alpha \). Then, there is a relationship

\[
\text{sup}(A) R \alpha
\]

of rank \( p \), which accords with the \( p \)-weight\(^1\) of an associated module in \( A \).

§1 Chain Transitivity

A sequence \( Q \) of weakly chained ind-spaces give a precise Fourier projection onto the interior of a topological space as determined by the display maps which foliate \( Q \). We define \( Q \) to be a chain-connected space; that is, for any two elements \( \{p, q\} \) in \( Q \), they may be compared by

\[
p R q = (q R p)^{-1}
\]

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\(^1\) See [ChainTr]; we must cherish the authors of this paper for their marvelous account of the succession function, which had yet to make its more operadic appearance. [GSP] gives a similar indispensable account.

\(^2\) See [Calc], pg. 13
We say a chain-connected space has the property of chain transitivity if, for any two sections
\[ \{p_i \to q_i, p_i \to p_i\} \simeq \text{fib}(Q), \]
there are the retracts
\[ \{q_i \to p_i, p_i \to p_i\} \simeq \text{cofib}(Q^p) \]

This definition has been precipitated slightly beforehand, but now receives its precise incarnation.

**Definition 1.0.1.** A display map
\[ \phi_i : ((p \star q) \lor (q \star p)) \to ((p \lor q) \lor (q \lor p)) \]
is a fusion-distributive logical connective.

**Example**

Let \( \phi_\tau (\mathcal{B}) \to \phi_\tau^{-1} \) be a “twist” on an algebra \( \mathcal{A} \). We call this map of displays \( \phi_\tau (-) \to \phi_\tau (-) \) a display block if it covers a frame \( / \) in the \( \varpi_i \)-cocycle of an arbitrary system of involutions.

Here, an involution means a coefficient \( q \) attached to a function field \( f \)
\[ \varpi_i : q(f) \to (q(-f)) \lor (q(f^p)) \]
Following [Beardsley], we will write \( L_n \) for the appropriate localization functor \( \varpi_i \zeta \to \zeta \), where
\[ \zeta = q(f) \to (-) \]

Let \( g \) be an arbitrary geodesic in some presentable category. Then, we have a restriction from the higher bundle of \( g \) to the \( g \)-action on objects. This is a localization from the \( \infty \)-Cats form of a representation to the geometric form.

**Definition 1.1.0** A **Jordan form** is the kernel of some \( \zeta \)-object. An object with a Jordan form satisfies the idempotent criterium on pg. 4 of [Jord].

**Definition 1.1.1** Let \( S : X \to X \) be a continuous map. A sequence \( \{x_n\} \) in \( X \) is an asymptotic pseudo-orbit of \( S \) if
\[ \lim_{n \to \infty} d(S^n(x), x_n + 1) = 0 \]

**Remark** It would be interesting, at least to the author, if every \( \zeta \)-object object had an asymptotic pseudo-orbit.

For two rank one isomorphisms \( xRy \) and \( x'Ry' \), we take the difference
\[ \{x \cup y\} \setminus \{x' \cup y'\} = \omega \]

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\(^3\) [Strong], definition 2.3
to be a pseudo-orbit of an \( R \)-algebra \( \mathcal{A} \). If we let \( \{x \approx x', y \approx y'\} \) be distinct equivalence classes, then we can take the immersion

\[
\gamma: \omega \to \omega
\]

from the outer sum of a pair and the inner hom of the other.

§ 2 Shadows

Definition 2.0.1 A po-category is a category \( \mathcal{P} \) with a 2-isomorphism into the bi-category\(^4\) \( \mathcal{S} \mathcal{S}ets \times \mathcal{I} \).

Definition 2.0.2 A shadow category is a po-category

\[
\mathcal{S}hado = \text{Push(}\mathcal{P} \text{)}
\]

which is locally a pushout.

A shadow is a compactly generated object in \( \mathcal{S}hado \), which is globally presentable.

Proposition 2.1.0 \( \mathcal{S}hado \) is Cartesian-closed.

Proof Let \( \omega_1, \omega_2 \) be two real cones, with the inclusions

\[
\begin{align*}
\omega_1 & \in \mathcal{P} \\
\omega_2 & \in \text{Push(}\mathcal{P} \text{)}
\end{align*}
\]

The bijection

\[
\omega_1 \leftrightarrow \omega_2
\]

automatically makes \( \mathcal{S}hado \) symmetric monoidal, which means it is Cartesian closed.

Let \( \mathcal{L} \) be a diagram with a unique factorization \( f: y' \to y \), where \( \text{proj}(y, y') = z \). Assuming each object is a po-category, we have an isomorphism

\[
\mathcal{S}hado = z
\]

Definition 2.1.1 An E-chain is a strictly modellable set of integrands with morphisms into \( \mathcal{S}hado \).

Here, we have explicitly defined the boundary-forming parts of an orbifold to be those series of fibrations which admit connections into the shadow category.

Proposition Any map \( E \to \mathcal{S}hado \) is at least a d-shadowing.\(^5\)

A “shadow” is, in some sense, any continued fraction which has stable approximation as an algebraic symbol \( \sigma \).

\[
\text{Map}(E, \mathcal{S}hado) = \partial \sigma \to \sigma
\]

Here, the left adjoint is a totally disconnected free variable.

We can compute this more simply by putting

\[^4\text{Where } \mathcal{I} \text{ is the category of intervals.}\]

\[^5\text{See [ChainTr], pg. 3.}\]
\[ \sigma_{\text{DISC}} \rightarrow \sigma_{\text{FIN}} = \sum_{i=0}^{\infty} (d(\partial i, i)) = \pi_0(\tau). \]

**Axiom 2.1.2 Adjointness**

For two stable objects \( \pi_0(\tau_0) \rightarrow \pi_0(\tau_{k+1}) \), with adjoint morphisms \( f \) and \( f' \), these morphisms shall be called **mutually orthogonal** and **metric-forming**.

We take a simple comparison triangle, \( \tilde{\Delta} = \angle abc = \sum_{c} p^c \), and calculate, say, the standard deviation of each angle from the arithmetic mean. Then we obtain a measure of the hyperbolicity (as measured by the \( \text{Cat}(k) \)-number) of a space, specifically a positive or negative sectional curvature.

**References**


[ChainTr] W.R. Brian, J. Meddaugh, B.E. Raines *Chain transitivity and variations of the shadowing property*
