An Algebraic Structure of Music Theory

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Abstract We may define a binary relation. Then a nonempty finite set equipped with the binary relation is called a circle set. And we define a bijective mapping of the circle set, and the mapping is called a shift. We may construct a pitch structure over a circle set. And we may define a tonic and step of a pitch structure. Then the ordered pair of the tonic and step is called the key of the pitch structure. Then we define a key transpose along a shift. A key transpose is said to be regular if it consists of stretches, shrinks and a shift. A key transpose is regular if and only if it satisfies some hypotheses.

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1. INTRODUCTION

In definition 3.1, we define a binary relation ‘Φ’. Then a non-vacuous finite set P equipped with the binary relation Φ is called a circle set, and we define a bijective mapping δ that is called shift if the mapping is compatible with Φ, see definition 3.2.

A circle set has no heads, but we may select a member as a head. Hence we define a tonic(cf. [4]) τ of P in definition 3.3.

Let S := {λ, τ, S, Φ} be a set. The members of S is called scales(cf. [4]), and we define a function λ: P × P → S given by assigning to an ordered pair of P a scale, see definition 3.4 for more details.

Two unary relations ‘♯’ and ‘♭’ on a circle set P are defined in definitions 3.5 and 3.6, respectively.

Let Σ := {λ, τ, S, Φ} be a language. Then we may construct a partial structure M of the language Σ over a circle set P, and the partial structure M is called the pitch structure, see definition 3.7 for more details.

Then we obtain a sequence of the scales, the sequence is called the step of the pitch structure M, and denoted by SS_{τM}(M), see definition 3.8 for more details. The ordered pair (τM, SS_{τM}(M)) is called the key(cf. [4]) of M, see definition 3.9.

Suppose that M, N are two pitch structures over a circle set P. Then a bijective mapping κ: SS(M) → SS(N) is called a key transpose(cf. [4]) along a shift δ if the mapping κ satisfies the hypotheses of definition 3.10.

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We say that a key transpose $\kappa$ is regular if $\kappa$ consists of stretches, shrinks and a shift, see definitions 3.11 and 3.12 for details. And some members of $P$, that is invariant under $\kappa$, are called $\kappa$-invariant, see definition 3.13. A key transpose $\kappa$ is regular if and only if lemma 3.1 and lemma 3.2 holds, see proposition 3.2 for more details.

2. Preliminaries

We recall some definitions in universal algebra.

Definition 2.1 ([2, 3]). An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) language provided that

- $L$ is a nonempty set,
- $\sigma : L \rightarrow \mathbb{Z}$ is a mapping.

A language $\langle L, \sigma \rangle$ is denoted by $\mathcal{L}$. If $f \in \mathcal{L}$ and $\sigma(f) \geq 0$ then $f$ is called an operation symbol, and $\sigma(f)$ is called the arity of $f$. If $r \in \mathcal{L}$ and $\sigma(r) < 0$, then $r$ is called a relation symbol, and $-\sigma(r)$ is called the arity of $r$. A language is said to be algebraic if it has no relation symbols.

Definition 2.2 ([2]). Let $X$ be a nonempty class and $n$ a nonnegative integer. Then an $n$-ary partial operation on $X$ is a mapping from a subclass of $X^n$ to $X$. If the domain of the mapping is $X^n$, then it is called an $n$-ary operation. And an $n$-ary relation is a subclass of $X^n$ where $n > 0$. An operation(relation) is said to be unary, binary or ternary if the arity of the operation(relation) is $1$, $2$ or $3$, respectively. And an operation is called nullary if the arity is $0$.

Definition 2.3 ([2]). An ordered pair $A := \langle A, \mathcal{L} \rangle$ is said to be a structure of a language $\mathcal{L}$ if $A$ is a nonempty class and there exists a mapping which assigns to every $n$-ary operation symbol $f \in \mathcal{L}$ an $n$-ary operation $f^A$ on $A$ and assigns to every $n$-ary relation symbol $r \in \mathcal{L}$ an $n$-ary relation $r^A$ on $A$. If all operation on $A$ are partial operations, then $A$ is called a partial structure. A (partial)structure $A$ is said to be a (partial)algebra if the language $\mathcal{L}$ is algebraic.

Definition 2.4 ([2, 3]). Let $A, B$ be (partial)structures of a language $\mathcal{L}$. A mapping $\varphi : A \rightarrow B$ is said to be a homomorphism provided that

$\varphi(f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n))$ for every $n$-ary operation $f$;

$r^A(a_1, \ldots, a_n) \Rightarrow r^B(\varphi(a_1), \ldots, \varphi(a_n))$ for every $n$-ary relation $r$.

A homomorphism $\varphi$ is called an isomorphism if $\varphi$ is bijective.

3. An Algebraic Structure of Music Theory

Definition 3.1. Suppose that $P$ is a nonempty finite set. We may define a binary relation ‘$\Diamond$’ on $P$ as follows. For every $s \in P$,

- there is exactly one $u \in P$ such that $u \Diamond s$, and
- there is exactly one $v \in P$ such that $s \Diamond v$.

Remark. The binary relation ‘$\Diamond$’ is not an order relation.

Definition 3.2. A circle set is a nonempty finite set equipped with the binary relation ‘$\Diamond$’ defined in definition 3.1. Let $P$ be a circle set. Then a bijective mapping $\delta : P \rightarrow P$ is said to be a shift if $\delta$ preserves the order of $P$, i.e., $\delta(p_i) \Diamond \delta(p_j)$ if and only if $p_i \Diamond p_j$. 
**Example 3.1.** The set $X := \{ x \in \mathbb{N} \mid x \mod 7 \}$ can be regarded as a circle set.

![Diagram of a circle set](image)

And it is a shift that a mapping is defined by $i \mapsto (i + 1) \mod 7$ for $i \in X$.

**Example 3.2.** Let $A$ be a non-vacuous finite ordered set, $B$ a non-vacuous countable ordered set. Suppose that $(a_0, b_0), (a_1, b_1) \in A \times B$. If we define

$$
(a_0, b_0) \leq (a_1, b_1) \quad \text{if} \quad \begin{cases} a_0 \leq a_1, & \text{for } b_0 = b_1; \\ b_0 \leq b_1, & \text{for } b_0 \neq b_1. 
\end{cases}
$$

then $A \times B$ is an ordered set. Now, let $(a_0, b_0) \sim (a_1, b_1)$ if $a_0 = a_1$. It is clear that $\sim$ is an equivalence relation. Then the quotient set $(A \times B)/\sim$ can be regarded as a circle set.

A circle set $P$ has no head members. But we may select a member $\tau$ as a head.

**Definition 3.3.** Suppose that $P$ is a circle set. Let $\tau := p$ for an arbitrary $p \in P$. We call $\tau$ a tonic of $P$.

And we have the following important definitions.

**Definition 3.4.** Suppose that $P$ is a circle set. Let $\mathbb{S}$ be the set $\{\mathbb{I}, \mathbb{L}, \mathbb{O}\}$. We may define a function $\lambda: P \times P \to \mathbb{S}$ given by

$$
\lambda(p, p') = \begin{cases} \mathbb{I} \text{ or } \mathbb{L} & \text{if } p \not\sim p', \\ \mathbb{O} & \text{otherwise.} \end{cases}
$$

And the elements of the set $\mathbb{S}$ is called scales.

Recall the definition of unary relations which is defined in definition 2.2. And we have the following definitions.

**Definition 3.5.** Suppose that $P$ is a circle set. Let $\#$ be a unary relation on $P$ such that

$$
\lambda(\#(s), \#(p)) = \lambda(s, p);
$$

$$
\lambda(s, \#(p)) = \begin{cases} \mathbb{I} & \text{if } \lambda(s, p) = \mathbb{I}, \\ \mathbb{O} & \text{if } \lambda(s, p) = \mathbb{O}. \end{cases}
$$

$$
\lambda(\#(s), p) = \begin{cases} \mathbb{I} & \text{if } \lambda(s, p) = \mathbb{I}, \\ \mathbb{O} & \text{if } \lambda(s, p) = \mathbb{O}. \end{cases}
$$

for every $s, p \in P$ with $s \not\sim p$.

**Definition 3.6.** Suppose that $P$ is a circle set. Let $\flat$ be a unary relation on $P$ such that

$$
\lambda(\flat(s), \flat(p)) = \lambda(s, p);
$$

$$
\lambda(s, \flat(p)) = \begin{cases} \mathbb{I} & \text{if } \lambda(s, p) = \mathbb{I}, \\ \mathbb{O} & \text{if } \lambda(s, p) = \mathbb{O}. \end{cases}
$$

$$
\lambda(\flat(s), p) = \begin{cases} \mathbb{I} & \text{if } \lambda(s, p) = \mathbb{I}, \\ \mathbb{O} & \text{if } \lambda(s, p) = \mathbb{O}. \end{cases}
$$
(3) \[
\lambda(b(s), p) = \begin{cases} 
\otimes & \text{if } \lambda(s, p) = - , \\
\otimes & \text{if } \lambda(s, p) = \text{♭}
\end{cases}
\]
for every \( s, p \in P \) with \( s \circ p \).

**Assumption 3.1.** Let \( P \) be a circle set. For simplicity, we assume that
\[
\lambda(b(p), \#(q)) = \otimes; \\
\lambda(\#(p), b(q)) = \otimes,
\]
for all \( p, q \in P \). Since \( \# \) and \( b \) are unary relations, we have that \( \#(\#(p)), \#(b(p)), b(\#(p)) \) and \( b(b(p)) \) are invalid for all \( p \in P \). So we have not ‘double sharp’ and ‘flat flat’.

**Remark 3.1.** In fact, that \( \# \) and \( b \) are not real unary relations.

Let \( M = P \cup \{\text{♭}, \#\} \). By definitions 3.4 to 3.6, we have that \( \lambda \) is a partial binary operation on \( M \), and that \( \text{♭}, \# \) and \( \otimes \) are nullary operations. Hence we may define a partial structure[definition 2.3] of a language[definition 2.1] \( \mathfrak{L} \). Then we have the following definitions.

**Definition 3.7.** A partial structure \( M := \langle M, \mathfrak{L} \rangle \) of the language \( \mathfrak{L} \) is called a pitch structure over a circle set \( P \) provided that the underlying set \( M = P \cup \mathcal{S} \) where \( P \) equipped with \( \otimes \) is a circle set[definition 3.2], and the language is defined to be the set \( \mathfrak{L} := \{\tau, \mathcal{S}, \otimes\} \) where \( \lambda \) is a partial binary operation defined in definition 3.4, \( \otimes \) is a binary relation defined in definition 3.1, \( \tau \) is a nullary operation defined in definition 3.3, and \( \mathcal{S} = \{\text{♭}, \#\} \) is the set of nullary operations defined in definition 3.4.

Suppose that \( M \) is a pitch structure over a circle set \( P \). We may assume that \( |P| = n \) and \( \tau := m_0 \) for \( m_0 \in P \). If \( m_i \otimes m_{(i+1) \text{mod } n} \in P \), then \( \{\lambda(m_i, m_{(i+1) \text{mod } n})\} \) constitutes a scale sequence, e.g., \( \{\text{♭}, \#, \ldots, \text{♭}\} \).

**Definition 3.8.** Let \( M \) be a pitch structure over a circle set \( P, |P| = n \), and \( \tau := m_0 \) for \( m_0 \in P \). Then we define \( SS_{\tau_M}(M) \) to be the following sequence
\[
(3.3) \quad \{\lambda(m_0, m_1), \lambda(m_1, m_2), \ldots, \lambda(m_{n-2}, m_{n-1}), \lambda(m_{n-1}, m_0)\},
\]
if we have \( m_0 \otimes m_1 \otimes m_2 \otimes \cdots \otimes m_{n-2} \otimes m_{n-1} \otimes m_0 \in P \). And the sequence \( SS_{\tau_M}(M) \) is called a step of the pitch structure \( M \) at the tonic \( m_0 \).

**Remark.** For all pitch structure \( M \), we have \( \otimes \notin SS_{\tau}(M) \).

**Proposition 3.1.** Suppose that \( M, N \) are two pitch structures. We have that \( M \cong N \) implies \( SS_{\tau_M}(M) = SS_{\tau_N}(N) \).

**Proof.** Let \( \varphi: M \to N \) be an isomorphism. Since the scales in set \( \mathcal{S} = \{\text{♭}, \#, \otimes\} \) and \( \tau_M \) are nullary operations of \( M \), we have that \( \varphi|\mathcal{S} \) is an identity mapping of \( \mathcal{S} \) and \( \varphi(\tau_M) = \tau_N \). Observe that \( \lambda \) is a binary operation. By definition 2.4, it is obvious that \( SS_{\tau_M}(M) = SS_{\tau_N}(N) \).

**Remark 3.2.** Suppose that \( M, N \) are pitch structures. If there exists a homomorphism \( \varphi: M \to N \), then \( \varphi \) must be an isomorphism. This is an immediate consequence of definitions 2.4 and 3.1. The isomorphism \( \varphi \) is unique. If we assume that \( M, N \) have same underlying set \( M = \mathcal{S} \cup P \), then it is clear that \( \varphi|P \) is a shift. Suppose that \( M, N \) are pitch structures over a circle set \( P \). Let \( \tau_M = \tau_N \) and \( M \neq N \). Then it follows \( \lambda_M \neq \lambda_N \).
**Definition 3.9.** Suppose that \( M \) is a pitch structure over a circle set \( P \), and the tonic \( \tau = m_0 \). Then the ordered pair \( \langle \tau_M, SS_{\tau_M}(M) \rangle \) is called the **key** of \( M \).

**Definition 3.10.** Suppose that \( M, N \) are pitch structures over a circle set \( P \), and \( \tau_M = m_i, \tau_N = m_j \) for \( m_i, m_j \in P \). Let \( \delta \) be a shift[definition 3.2] which assigns \( m_j \) to \( m_i \). Then a bijective mapping \( \kappa : SS_{\tau_M}(M) \mapsto SS_{\tau_N}(N) \) is called a **key transpose** along \( \delta \) provided that \( \kappa \) assigns \( \lambda_N(\delta(m), \delta(m')) \) to \( \lambda_M(m, m') \) for every \( m, m' \in P \) with \( m \odot m' \).

**Remark.** We have that \( M \cong N \) implies that \( \kappa \) is an identity mapping.

**Example 3.3.** Suppose that \( P := \{ m_0, m_1, m_2, m_3, m_4 \} \) is a circle set, \( M \) is a pitch structure over \( P \), and \( \tau := m_0 \). Let \( SS(M) = \{ \#m_0, b(m_2) \} \). If we take \( \#, \) \( b \) on some members of \( M \), e.g., \( \#(m_0) \) and \( b(m_2) \), then we obtain a new sequence

\[
\{ \lambda(\#m_0), \lambda(m_1, b(m_2)), \lambda(b(m_2), m_3), \lambda(m_3, m_4), \lambda(m_4, \#m_0) \} = \{ -, \#b, \#b, \#b, - \},
\]

where the unary relations \( \# \) and \( b \) are defined in definitions 3.5 and 3.6.

**Example 3.4.** With the notations of example 3.3, if we change the value of \( \tau \), e.g., let \( \tau := m_2 \), then we also obtain a new sequence

\[
\{ \lambda(m_2, m_3), \lambda(m_3, m_4), \lambda(m_4, m_0), \lambda(m_0, m_1), \lambda(m_1, m_2) \} = \{ -, -, -, \#b, \#b \}.
\]

**Definition 3.11.** Suppose that \( M \) is a pitch structure over a circle set \( P \), and \( P := \{ m_0 \odot m_1 \odot \cdots \odot m_n \odot m_0 \} \).

Let \( m_i, m_j \in M \) with \( m_i \odot m_j \) for \( 0 \leq i \leq n - 1, j = (i + 1) \mod n \). The scale of \( \langle m_i, m_j \rangle \) is said to be **shrinkable** if \( \lambda(m_i, m_j) = \#b \). By definitions 3.5 and 3.6, we have that both \( \lambda(\#m_i), m_j) \) and \( \lambda(m_i, b(m_j)) \) are \( \neg \). Hence we call \( \lambda(\#m_i), m_j) \) and \( \lambda(m_i, b(m_j)) \) a \( \# \)-**shrink** and \( b \)-**shrink**, respectively. The scale of \( \langle m_i, m_j \rangle \) is said to be **stretchable** if \( \lambda(m_i, m_j) = - \). And we have that \( \lambda(m_i, \#m_j) \) and \( \lambda(b(m_i, m_j)) \) are a \( \# \)-**stretch** and \( b \)-**stretch**, respectively.

**Example 3.5.** Let the hypotheses be as in example 3.3. We have that the scale of \( \langle m_0, m_1 \rangle \) is shrinkable, the scale of \( \langle m_4, m_0 \rangle \) is stretchable. And we have that \( \lambda(\#m_0), m_1) \) and \( \lambda(m_4, \#m_0) \) are a \#-shrink and \#-stretch respectively, and \( \lambda(b(m_2), m_3) \) and \( \lambda(m_1, b(m_2)) \) are a \( b \)-stretch and \( b \)-shrink respectively.

We may take the two classes of the transposition in examples 3.3 and 3.4 on a pitch structure \( M \) simultaneously.

**Example 3.6.** Let the notations be as in examples 3.3 and 3.4. Suppose that \( N \) is a pitch structure over the circle set \( P \), and \( \tau_N := m_2 \). Let \( \delta \) be a shift which assigns \( m_2 \) to \( m_0 \), and \( \kappa : SS_{\tau_M}(M) \mapsto SS_{\tau_N}(N) \) a key transpose along \( \delta \). If we assume that

\[
SS_{\tau_M}(N) := \{ \#, \#b, -b, b, - \},
\]

then it is clear that \( \kappa \) is equivalent to the process which is defined as follows:

1. Take \( \# \) and \( b \) on \( m_0 \) and \( m_2 \) respectively, as described in example 3.3.
2. Let \( \tau_M = m_2 \), as described in example 3.4.

Therefore, we may say that the key transpose \( \kappa \) consists of a stretch, shrink and shift. And the order of the process is not important.
Definition 3.12. Suppose that \( M, N \) are pitch structures over a circle set \( P \). If \( SS(M) \) is transposed to \( SS(N) \) by a key transpose \( \kappa \) in such a way that is described in examples 3.3, 3.4 and 3.6, that is, the key transpose consists of stretches[definition 3.11], shrinks[definition 3.11] and a shift[definition 3.2], then we say that the key transpose \( \kappa \) is regular.

Remark. A key transpose may be not regular.

Example 3.7. Suppose that \( M \) is a pitch structure over a circle set \( P \), and \( |P| = n \). For every \( 0 \leq i \leq n - 1 \), there are two trivial key transposes. One is
\[
\{ \#(m_i), \#(m_{i+1} \mod n), \ldots, \#(m_{(i+n-1) \mod n}), \#(m_i) \},
\]
and the other is
\[
\{ b(m_i), b(m_{i+1} \mod n), \ldots, b(m_{(i+n-1) \mod n}), b(m_i) \}.
\]
They are regular. And there are no changes on all of scales in the case of the trivial key transpose.

Definition 3.13. Suppose that \( M, N \) are two pitch structures over a circle set \( P \). Let \( \kappa : SS_{\tau_i}(M) \rightarrow SS_{\tau_i}(N) \) be a nontrivial regular key transpose and \( m \in P \). The element \( m \) is said to be \( \kappa \)-invariant if there exist \( \#(m) \) and \( b(m) \) under the key transpose \( \kappa \).

Definition 3.14. Suppose that \( M \) is a pitch structure over a circle set \( P \). Let \( P := \{ m_0 \odot m_1 \odot \cdots \odot m_{n-1} \odot m_0 \} \). Then the directions 3.4 and 3.5 are called clockwise and anticlockwise, respectively.

\[
\begin{align*}
(3.4) & \quad m_0 \odot m_1 \odot \cdots \odot m_{n-1} \odot m_0 \\
(3.5) & \quad \text{\textarrow{\rightarrow}}
\end{align*}
\]

We shall see what properties a key transpose satisfies if it is regular.

Lemma 3.1 (\#-shrink \iff \#-stretch). Suppose that \( M, N \) are pitch structures over a circle set \( P \), and
\[
P := \{ p_0 \odot p_1 \odot \cdots \odot p_{n-1} \odot p_0 \}.
\]
Let \( \kappa : SS_{\tau_i}(M) \rightarrow SS_{\tau_i}(N) \) be a nontrivial key transpose along a shift \( \delta \) which assigns to \( \tau_M \tau_N \) and \( \lambda_M(p_i, p_j) = \lambda_N(p_i, p_j) = \text{for} \ p_i \odot p_j \in P \). Then the scale of \( \langle p_i, p_j \rangle \) is transformed from \( \lambda_M(p_i, p_j) \) to \( \lambda_N(p_i, p_j) \) under the key transpose \( \kappa \) via a \#-shrink, i.e., \( \lambda_M(p_i, p_j) \) if and only if there exist \( p_r \odot p_s \in P \) with \( \lambda_M(p_r, p_s) = \text{,} \lambda_N(p_r, p_s) = \text{such that}
\]
\begin{enumerate}
\item \( j' = (j + d) \mod n \) with \( d \leq 0 \), i.e., in the anticlockwise,
\item the scale of \( \langle p_r, p_j' \rangle \) is transformed from \( \lambda_M(p_r, p_j) \) to \( \lambda_N(p_r, p_j) \) under \( \kappa \) via a \#-stretch, i.e., \( \lambda_M(p_r, \#(p_j)) \), hence \( p_r \) is \( \kappa \)-invariant, and
\item \( \kappa \) makes no changes on the scales of the consecutive members pairs in \( \{ p_r \odot \cdots \odot p_i \} \) if \( p_r \neq p_i \).
\end{enumerate}

Proof. We assume \( p_r \odot p_i \odot p_j \). Since \( \lambda_M(\#(p_i), p_j) \) and assumption 3.1, we have that either
\[
\begin{align*}
(3.6) & \quad \lambda_N(p_r, p_i) = \lambda_M(p_r, p_i), \\
\text{or} & \quad \lambda_N(p_r, p_i) = \text{,} \\
(3.7) & \quad \lambda_M(p_r, p_i) = \text{.}
\end{align*}
\]
Hence if (3.7) holds, then the proof is complete. Now we assume that equation (3.6) holds, and observe assumption 3.1. Then there exists a \( p_r \in P \) such that \( \kappa \) makes no changes on the scales of the consecutive members pairs in \( \{ p_r \otimes \cdots \otimes p_j \} \) by induction. Hence we have that \( \kappa \) takes \( \delta \) on all of elements in \( \{ p_r \otimes \cdots \otimes p_j \} \). It follows that there exists a \( p_r \) with \( p_r \otimes p_j \) such that \( \lambda_M(p_r, p_j) = \lambda_N(p_r, p_j) = \kappa \), and the scale of \( \langle p_r, p_j \rangle \) is transformed from the former to the latter under \( \kappa \) via a \( \delta \)-stretch, i.e., \( \lambda_M(p_r, p_j) = \kappa \). Otherwise, the nontrivial key transpose hypotheses would not hold. Hence it is clear that \( p_r \) is \( \kappa \)-invariant. On the other hand, we may assume \( p_r \otimes p_j \). Then the proof of the converse is similar. This completes the proof.

Remark 3.3. Let \( \kappa \) be a key transpose along \( \delta \). Then we have that \( \kappa \) sends \( \lambda_M(p_i, p_j) \) to \( \lambda_N(\delta(p_i), \delta(p_j)) \) for \( p_i \otimes p_j \in P \), cf. definition 3.10. But in lemmas 3.1 and 3.2, we observe \( \lambda_M(p_i, p_j) \) and \( \lambda_N(p_i, p_j) \).

We have the following lemma that is similar to lemma 3.1.

**Lemma 3.2** \((b\text{-shrink} \iff b\text{-stretch})\). Suppose that \( M, N \) are pitch structures over a circle set \( P \), and

\[
P := \{ p_0 \otimes p_1 \otimes \cdots \otimes p_{n-1} \otimes p_0 \}.
\]

Let \( \kappa: SS_{\tau_M}(M) \to SS_{\tau_N}(N) \) be a nontrivial key transpose along a shift \( \delta \) which assigns to \( \tau_M \tau_N \), and \( \lambda_M(p_i, p_j) = \kappa \), \( \lambda_N(p_i, p_j) = \delta \) for \( p_i \otimes p_j \in P \). The scale of \( \langle p_i, p_j \rangle \) is transformed from \( \lambda_M(p_i, p_j) \) to \( \lambda_N(p_i, p_j) \) under the key transpose \( \kappa \) via a \( b\text{-shrink} \), i.e., \( \lambda_M(b(p_i), p_j) \) if and only if there exist \( p_r \otimes p_j \) with \( \lambda_M(p_r, p_j) = \delta \), \( \lambda_N(p_r, p_j) = \kappa \).

**Proof.** This is similar to the proof of lemma 3.1.\( \square \)

**Remark 3.4.** Let \( \kappa \) be a nontrivial key transpose. We observe lemmas 3.1 and 3.2. We shall find that a \( \delta \)-shrink must be adjoint to a \( \delta \)-stretch, and a \( b\text{-shrink} \) must be adjoint to a \( b\text{-stretch} \). And we have that \( \kappa \) makes no changes on the scales of the consecutive members pairs between an adjoint pair.

**Proposition 3.2.** Suppose that \( M, N \) are two pitch structures over a circle set \( P \). Let \( \kappa: SS(M) \to SS(N) \) be a non-trivial key transpose. Then the key transpose \( \kappa \) is regular if and only if \( \delta \) is not an identity mapping, cf. definition 3.10 and lemma 3.2 hold.

**Proof.** Immediate from definitions 3.10 and 3.12 and lemmas 3.1 and 3.2.\( \square \)

**Remark 3.5.** Suppose that \( M, N \) are pitch structures over a circle set \( P \). Let \( \varphi: M \to N \) be a homomorphism. Observe remark 3.2. We have that \( \varphi \) is an isomorphism. By proposition 3.1, we have \( SS_{\tau_M}(M) = SS_{\tau_N}(N) \). And it is clear that \( \delta := \varphi|P \) is a shift[definition 3.2] which assigns \( \tau_N \) to \( \tau_M \). If \( \kappa \) is a key transpose along \( \delta \) then \( \kappa \) is an identity mapping, since we have definition 3.10. And if \( \kappa \) is regular then \( \kappa \) consists of shrinks, stretches and a shift, even if \( \kappa \) is an identity mapping, cf. remark 3.3.

**Corollary 3.2.1.** Suppose that \( M, N \) are two pitch structures over a circle set \( P \). Let \( \kappa: SS(M) \to SS(N) \) be a non-trivial key transpose. Then the key transpose \( \kappa \) is regular if and only if the key transpose \( \kappa^{-1} \) is regular.
Proof. It is clear that lemma 3.1 and lemma 3.2 hold for $\kappa^{-1}$ if the lemmas hold for $\kappa$, and vice versa. \qed

References


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