Contraction of Ramanujan Formulas in the Letter to Hardy.

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0- Abstract:

In this paper we show an approach to the Ramanujan summation of series formulas, proving that it is possible a contracted version of them.

1- Introduction.

Srinivasa Ramanujan (1887-1920), the Hindu genius send to G. H. Hardy a letter in 1903 [1]. In this letter were a few discoveries and advanced (for that time) questions that he made by himself. In this paper we will focus on part “V: Theorems of summation of series”. We will do a more modern contraction of the equations with a calculus approach. We will distinguish between “Ramanujan notation” and “Contracted notation”.

I will use to contract negative parts of sequences my own operator (Subtractory), if you want to know more about negative-summation operator you can see [2].

As warning I will say that I do not test the veracity of any Ramanujan’s equality so it can be wrong as we understand the mathematics in a numeric form today.

2- Section V contractions:

(1.1) Ramanujan notation

$$\frac{1}{1^3}2^1 + \frac{1}{2^3}2^1 + \frac{1}{3^3}2^1 + \frac{1}{4^3}2^4 + ... = \frac{1}{6} (\log 2)^3 - \frac{\pi^2}{12} \log 2 + \frac{1}{1^3}3^1 + \frac{1}{3^3}3^1 + \frac{1}{5^3}3^1 + ...$$

(1.2) Contracted notation

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{2^n} = \frac{1}{6} \log(2)^3 - \frac{\pi^2}{12} \log(2) + \sum_{n=1}^{\infty} \frac{1}{1^3} (2n-1)^3$$

(2.1) Ramanujan notation

$$1 + 9 \left( \frac{1}{4} \right)^4 + 17 \left( \frac{1 \cdot 5}{4 \cdot 8} \right)^4 + 25 \left( \frac{1 \cdot 5 \cdot 9}{4 \cdot 8 \cdot 12} \right)^4 + ... = \frac{2 \sqrt{2}}{\sqrt{\pi} \left( \Gamma \left( \frac{3}{4} \right) \right)}$$

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(2.2) Contracted notation

\[ \sum_{n=1}^{\infty} 1 + 8n \left( \prod_{m=0}^{\infty} \frac{1+4m}{4m} \right) = \frac{2\sqrt{2}}{\sqrt{\pi} \{ \Gamma \left( \frac{3}{4} \right) \}} \]

(3.1) Ramanujan notation

\[ 1 - 5 \left( \frac{1}{2} \right)^3 + 9 \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^3 - ... = \frac{2}{\pi} \]

(3.2) Contracted notation

\[ 1 + 2 \left( \sum_{n=0}^{\infty} 8n \left( \prod_{m=1}^{\infty} \frac{2m-1}{2m} \right) \right) + 4 \left( \sum_{n=1}^{\infty} 4n \left( \prod_{m=1}^{\infty} \frac{2m-1}{2m} \right) \right) = \frac{2}{\pi} \]

(4.1) Ramanujan notation

\[ \frac{1}{e^{2\pi} - 1} + \frac{2}{e^{4\pi} - 1} + \frac{3}{e^{6\pi} - 1} + ... = \frac{1}{24} \]

(4.2) Contracted notation

\[ \sum_{n=1}^{\infty} \frac{n^{13}}{e^{(2n)\pi} - 1} = \frac{1}{24} \]

(5.1) Ramanujan notation

\[ \frac{\coth \pi}{1^7} + \frac{\coth 2\pi}{2^7} + \frac{\coth 3\pi}{3^7} + ... = \frac{19\pi^7}{56700} \]

(5.2) Contracted notation

\[ \sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56700} \]

(6.1) Ramanujan notation

\[ \frac{1}{1^5 \cosh \frac{\pi}{2}} - \frac{1}{3^5 \cosh \frac{3\pi}{2}} + \frac{1}{5^5 \cosh \frac{5\pi}{2}} - ... = \frac{\pi^5}{768} \]
(6.2) Contracted notation
\[ \left( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5 \cosh \frac{(2n-1)\pi}{2}} \right) + 2 \left( \sum_{n=1}^{\infty} \frac{1}{(4n-1)^5 \cosh \frac{(4n-1)\pi}{2}} \right) = \frac{\pi^5}{768} \]

(7.1) Ramanujan notation
\[ \frac{1}{(1^2+2^2)(\sinh 3\pi - \sinh \pi)} + \frac{1}{(2^2+3^2)(\sinh 5\pi - \sinh \pi)} + \frac{1}{(3^2+4^2)(\sinh 7\pi - \sinh \pi)} + \ldots = \frac{1}{2 \sinh \pi} \left( \frac{1}{\pi} + \coth \pi - \frac{\pi}{2} \tanh^2 \frac{\pi}{2} \right) \]

(7.2) Contracted notation
\[ \sum_{n=1}^{\infty} \frac{1}{(n^2+(n+1)^2)(\sinh (2n+1)\pi - \sinh \pi)} = \frac{1}{2 \sinh \pi} \left( \frac{1}{\pi} + \coth \pi - \frac{\pi}{2} \tanh^2 \frac{\pi}{2} \right) \]

(8.1) Ramanujan notation
\[ \frac{1}{(25+\frac{1^4}{100})(e^\pi + 1)} + \frac{3}{(25+\frac{3^4}{100})(e^{3\pi} + 1)} + \frac{5}{(25+\frac{5^4}{100})(e^{5\pi} + 1)} + \ldots = \frac{\pi}{8} \coth^2 \frac{5\pi}{2} - \frac{4689}{11890} \]

(8.2) Contracted notation
\[ \sum_{n=1}^{\infty} \frac{(2n-1)}{(25+\frac{(2n-1)^4}{100})(e^{(2n-1)\pi} + 1)} = \frac{\pi}{8} \coth^2 \frac{5\pi}{2} - \frac{4689}{11890} \]

(9.1) Ramanujan notation
\[ \frac{1}{1^7 \cosh \frac{\pi}{2} \sqrt{3}} - \frac{1}{3^7 \cosh \frac{3\pi}{2} \sqrt{3}} + \ldots = \frac{\pi^7}{23040} \]

(9.2) Contracted notation
\[ \left( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^7 \cosh \frac{(2n-1)\pi}{2} \sqrt{3}} \right) + 2 \left( \sum_{n=1}^{\infty} \frac{1}{(4n-1)^7 \cosh \frac{(4n-1)\pi}{2} \sqrt{3}} \right) = \frac{\pi^7}{23040} \]

(10.1) Ramanujan notation
\[ \{ 1+\left( \frac{n}{1} \right)^3 \} \{ 1+\left( \frac{n}{2} \right)^3 \} \{ 1+\left( \frac{n}{3} \right)^3 \} \ldots \]
Can always be exactly found if \( n \) is any integer positive or negative.

(10.2) Contracted notation

\[
\prod_{m=1}^{\infty} \left\{ 1 + \left( \frac{n}{m} \right)^3 \right\}
\]

Can always be exactly found if \( n \) is any integer positive or negative.

(11.1) Ramanujan notation

\[
\frac{2}{3} \int_{0}^{1} \frac{\tan^{-1} x}{x} \, dx - \int_{0}^{2-\sqrt{3}} \frac{\tan^{-1} x}{x} \, dx = \frac{\pi}{12} \log 2 + \sqrt{3}
\]

3- Conclusions.

As you can see almost every summation series from Ramanujan (except integral one) can be expressed as calculus contracted notation. I think, and this is just a comment, that nowadays we can do a more technical mathematics, with more precision in our calculus expressions.

4- References.

https://www.qedcat.com/misc/ramanujans_letter.jpg