Quasi-Metric Space I

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Abstract

Inspired by the work of Adhya and Ray, I provide my own proof of selected theorems and lemmas discussed in [1]. Original theorems should appear, in due course, in a future article.

Theorem 1. Every singleton set in a $\mu$-$T_1$ strong generalised topological space is $\mu$-closed.

Proof. All singletons in a singleton set are elements of the power set $\mathcal{P}(X)$. This means that there exist $B_1, B_2 \in \mu$, for each pair $x, y \in X$ (with $x \neq y$), such that $x \in \{x\}$, $y \notin \{x\}$ and $y \in \{y\}$, $x \notin \{y\}$. Because $x, y \in \mu$ and $X \in \mu$, due to $(X, \mu)$ being a strong generalised topological space, $\emptyset$ can only be in $\mu$ if $X$ is both $\mu$-closed and $\mu$-open. If each singleton set, $\{x\} \in X$, is not $\mu$-closed then $X = \bigcup_{x \in X} \{x\}$ is not $\mu$-closed. This is a contradiction. Using the same logic, every singleton set in a $\mu$-$T_1$ strong generalised topological space is also $\mu$-open. □

Theorem 2. A metric space is Lebesgue if and only if every pseudo-Cauchy sequence having distinct terms clusters in it.

Proof. ($\implies$) Let $x_m$ and $x_n$ both cluster to $x$. Let $d(x_m, x) < \delta_1$ when $m \in \mathbb{N} > k_1$ and let $d(x_n, x) < \delta_2$ when $n \in \mathbb{N} > k_2$. Assume that the function, $f$, is uniformly continuous - this means that $\forall \epsilon \exists \delta > 0$ such that if $x_1$ and $x_2 \in X$ with $d(x_1, x_2) < \delta$ then $d(f(x_1), f(x_2)) < \epsilon \forall x_1, x_2 \in X$. Let $k = \max\{k_1, k_2\}$ - this means that when $n, m > k$, $d(x_n, x_m) < d(x_n, x) + d(x_n, x) < \delta_1 + \delta_2 = \delta$. Therefore $d(f(x_n), f(x_m)) < f(\delta) = \epsilon$. Allowing $f(x) = y$ implies that both $f(x_m)$ and $f(x_n)$ cluster to $y$.

($\impliedby$) Let, $\forall \delta/2 > 0$, $d(x_n, x) < \delta/2 \forall n > k_x \in \mathbb{N}$. Therefore $d(x_n, x_m) < d(x_n, x) + d(x_m, x) < \delta/2 + \delta/2 = \delta \forall n, m \in \mathbb{N} > k_x$. Because $f$ is continuous, this implies that (for $f : X \to Y$ such that $x \mapsto f(x) = y$), $d(f(x_n), f(x_m)) < \epsilon, \forall n, m \in \mathbb{N} > k_y$ (for all $x \in X$). This implies that $d(f(x_n), f(x)) < \epsilon/2 \forall n > k_y \in \mathbb{N}$. This means that $f$ is also uniformly continuous. □

Lemma 3. Let $(X, d_X)$ and $(Y, d_Y)$ be $g$-quasi metric spaces of the same index $r$. A sequence $(x_n, y_n)$ is $G$-Cauchy in $(X \times Y, d_{XY})$ if and only if $(x_n)$ and $(y_n)$ are $G$-Cauchy in $(X, d_X)$ and $(Y, d_Y)$ respectively.
Proof. \( \implies \) Let \((x_n, y_n)\) by G-Cauchy in \((X \times Y, d_{XY})\). Choose \(\epsilon > r\). Then \(\exists k \in \mathbb{N}\) such that \(d_{XY}((x_n, y_n), (x_{n+1}, y_{n+1})) < \epsilon\), \(\forall n \geq k\). That is \(d_X(x_n, x_{n+1}), d_Y(y_n, y_{n+1}) < \epsilon\), \(\forall n \geq k\). Then \((x_n)\) and \((y_n)\) are G-Cauchy in \((X, d_X)\) and \((Y, d_Y)\), respectively.

\( \Leftarrow \) Let \((x_n)\) and \((y_n)\) be G-Cauchy in \((X, d_X)\) and \((Y, d_Y)\), respectively. Choose \(\epsilon > r\). Then \(\exists k_1, k_2 \in \mathbb{N}\) such that \(d_X(x_n, x_{n+1}) < \epsilon\), \(\forall n \geq k_1\) and \(d_Y(y_n, y_{n+1}) < \epsilon\), \(\forall n \geq k_2\). Set \(k = \max\{k_1, k_2\}\). Then \(d_X(x_n, x_{n+1}), d_Y(y_n, y_{n+1}) < \epsilon\), \(\forall n \geq k\). Hence \((x_n, y_n)\) is G-Cauchy in \((X \times Y, d_{XY})\).

Lemma 4. Let \((X, d_X)\) and \((Y, d_Y)\) be \(g\)-quasi metric spaces of the same index \(r\). A sequence \((x_n, y_n)\) is pseudo-Cauchy in \((X \times Y, d_{XY})\) if and only if \((x_n)\) and \((y_n)\) are pseudo-Cauchy in \((X, d_X)\) and \((Y, d_Y)\) respectively.

Proof. \( \implies \) Let \((x_n, y_n)\) by pseudo-Cauchy in \((X \times Y, d_{XY})\). Choose \(\epsilon > r\). Then \(\exists k \in \mathbb{N}\) such that \(d_{XY}((x_n, y_n), (x_{n+1}, y_{n+1})) < \epsilon\), \(\forall n \geq k\). That is \(d_X(x_n, x_{n+1}), d_Y(y_n, y_{n+1}) < \epsilon\), \(\forall p, q \geq k\). Then \((x_n)\) and \((y_n)\) are pseudo-Cauchy in \((X, d_X)\) and \((Y, d_Y)\), respectively.

\( \Leftarrow \) Let \((x_n)\) and \((y_n)\) be pseudo-Cauchy in \((X, d_X)\) and \((Y, d_Y)\), respectively. Choose \(\epsilon > r\). Then \(\exists k_1, k_2 \in \mathbb{N}\) such that \(d_X(x_n, x_{n+1}) < \epsilon\), \(\forall p, q \geq k_1\) and \(d_Y(y_n, y_{n+1}) < \epsilon\), \(\forall p, q \geq k_2\). Set \(k = \max\{k_1, k_2\}\). Then \(d_X(x_n, x_{n+1}), d_Y(y_n, y_{n+1}) < \epsilon\), \(\forall p, q \geq k\). Hence \((x_n, y_n)\) is pseudo-Cauchy in \((X \times Y, d_{XY})\).

References