Can the effect of the uncertainty principle be seen by simultaneous measurements?

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Abstract

We respect profoundly the uncertainty relation in describing properly physical phenomena more even in today's or future experiments with opposite results. We explain such a conflict with the uncertainty principle as follows: Through a feature of matrix theory the results of some special experiment should be regarded as exceptional, but theoretically symmetric in using only diagonal matrices for observables. More concretely, we perform an experiment by simultaneous measurements. Only commuting observables must be measured in such a physical situation. Thus, the effect of the uncertainty principle can not be seen. This explanation of the experimental data is our main assertion in this paper. And, the time evolution of the uncertainty relation is also proposed here. As a result, the effect of the uncertainty relation can be seen by no-simultaneous measurements. That is, the effect indeed appears in the different times t and t'.

1 Introduction

Quantum mechanics (cf. [1, 2, 3, 4, 5, 6, 7]) gives accurate and at-times-remarkably accurate numerical predictions and much experimental data has fit to quantum predictions for long time. In quantum mechanics, the uncertainty principle is any of the variety of mathematical inequalities asserting a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position $\hat{x}$ and momentum $\hat{p}$, can be known simultaneously. For instance, in 1927, Werner Heisenberg stated that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [8]. The formal inequality relating to the standard deviation of position σ_x and the standard deviation of momentum σ_p was derived by Earle Hesse Kennard [9] later that year and by Hermann Weyl [10] in 1928.

Maccone and Pati discuss stronger uncertainty relations for all incompatible observables [11]. Quantum dynamics of simultaneously measured non-commuting observables is discussed [12]. Dynamics of a qubit while simultaneously monitoring its relaxation and dephasing are also discussed [13]. The upper limit of the Schrödinger–Robertson uncertainty relation in a two-level system (e.g., electron spin, photon polarizations, and so on) is discussed in [14]. This is certified by the Bloch sphere when we would measure $\hat{\sigma_x}$ and $\hat{\sigma_y}$. How about the lower limit of the uncertainty relation? In the authors knowledge, nobody derives the lower limit of the Schrödinger–Robertson uncertainty relation (exactly zero).

The motivations behind this work to be discussed in this paper are as follows: First, we respect profoundly the Schrödinger–Robertson uncertainty relation for describing properly physical phenomena more even in today's or future experiments with opposite results. For example, there is a paper [15] by Werner A. Hofer titled "Heisenberg, uncertainty, and the scanning tunneling microscope" which leads us to a conflict with the uncertainty principle. The assertion is that the density of electron charge is a physically real, i.e., in principle precisely measurable quantity. Let us explain the conflict with the uncertainty principle as follows: They perform this experiment by simultaneous measurements. Only commuting observables must be measured in such a physical situation. Thus, the effect of the uncertainty principle can not be seen. This explanation of the experimental data is our main assertion in this paper.

Next, we derive correcter and more precise uncertainty relation for describing properly physical phenomena, being useful for analyzing several systems in condensed matter and certain atomic nuclei that the time evolution of quantum transitions may be observed. Thus, we derive the time evolution of the Schrödinger–Robertson uncertainty relation and the uncertainty relation is indeed valid for different times t and t'. We can take into account the time evolution for the Schrödinger–Robertson uncertainty relation, deriving new formulae. Moreover, we understand the fact that the new formulae are natural from the understandable upper limit (the Bloch sphere) and the meaningful lower limit (exactly zero) by virtue of the convex argumentation.

In this paper, we respect profoundly the Schrödinger–Robertson uncertainty relation in describing properly physical phenomena more even in today's or future experiments with opposite results. We explain such a conflict with the uncertainty principle as follows: Through a feature of matrix theory the results of some special experiment should be regarded as exceptional, but theoretically symmetric in using only diagonal matrices for observables. More concretely, we perform an experiment by simultaneous measurements. Only commuting observables must be measured in such a physical situation. Thus, the effect of the Schrödinger–Robertson uncertainty relation can not be seen. This explanation of the experimental data is our main assertion in this paper. And, the time evolution of the Schrödinger–Robertson uncertainty relation is also proposed here. As a result, the effect of the uncertainty relation can be seen by no-simultaneous measurements. That is, the effect indeed appears in the different times t and t'.

We expect the discussion is useful for analyzing several systems in condensed matter and certain atomic nuclei in which the time evolution of transitions may be observed.

This paper is organized as follows:

In Sec. 2, we propose the symmetry of observables as a new interpretation for some exceptional experimental results while accepting the uncertainty principle. In Sec. 3, we show that the effect of the uncertainty principle can not be seen by simultaneous measurements. In Sec. 4, we respect profoundly the uncertainty principle. In Sec. 5, we discuss the Schrödinger–Robertson uncertainty relation. In Sec. 6, the upper limit of the Schrödinger–Robertson uncertainty relation is given in qubits handling. In Sec. 7, the lower limit of the Schrödinger–Robertson uncertainty relation is also given. In Sec. 8, the time-dependent Schrödinger–Robertson uncertainty relation in the Schrödinger representation is discussed. In Sec. 9, the time-dependent Schrödinger–Robertson uncertainty relation in the Heisenberg representation is also discussed. In Sec. 10, we discuss quantum measurement theory for commuting observables based on functions. Section 11 deals with discussion and conclusion in this paper.

2 Symmetry of observables

The symmetry of observables is worth considering, in order to obtain an honest interpretation of the uncertainty principle.

It can be said that the symmetry of two observables and the commutativity of the two are equivalent. To prove this, let us investigate that when the observables are noncommutative, the observables are not symmetric using spin's behavior.

1. Trying to measure the spin observable σ_z in some eigenstate with the eigenvalue $+1$.

$$\sigma_z|\uparrow\rangle = +1|\uparrow\rangle. \tag{2.1}$$

2. Then, we have $+1$ as the result of the measurement spin observable σ_z with a probability 1.

3. The result of the spin observable σ_x is -1 with the probability 0.5.

$$\sigma_x|\uparrow\rangle = \pm 1|\uparrow\rangle. \tag{2.2}$$

4. Even though the results are ± 1, these measurements are depend on the order of these two.

5. It happens that the first measurement of spin observables σ_x might obtain $+1$ with the probability 0.5.

$$\sigma_x|\uparrow\rangle = \pm 1|\uparrow\rangle. \tag{2.3}$$

6. This means that if the two observables are non-commutative, the result of measurements is not symmetric, which means that the fact depends on the order of these two measurements.

7. Let us make the contraposition of the two above. We can obtain that when the two observables are symmetric, namely the case not concerning the two measurements' order, these observables are commutative.

8. Obviously if the two observables are commutative, the results of the two measurements are symmetric independent of the order of the measurements.

As a result, if two observables are symmetric independent of the order of the two measurements means if and only if the two observables are commutative.

3 Effect of the Schrödinger–Robertson uncertainty relation can not be seen by simultaneous measurements

There is a paper [15] by Werner A. Hofer titled "Heisenberg, uncertainty, and the scanning tunneling microscope" which leads us to a conflict with the uncertainty principle. The assertion is that the density of electron charge is a physically real, i.e., in principle precisely measurable quantity.

Let us explain the conflict with the uncertainty principle as follows:

1. They perform this experiment by simultaneous measurements, i.e., symmetric measurements [16].
2. Only commuting observables must be measured in such a physical situation.
3. Thus, the effect of the Schrödinger–Robertson uncertainty relation can not be seen.

This explanation of the experimental data which says the conflict with the uncertainty principle is our main assertion in this paper.

4 Respecting profoundly the uncertainty principle

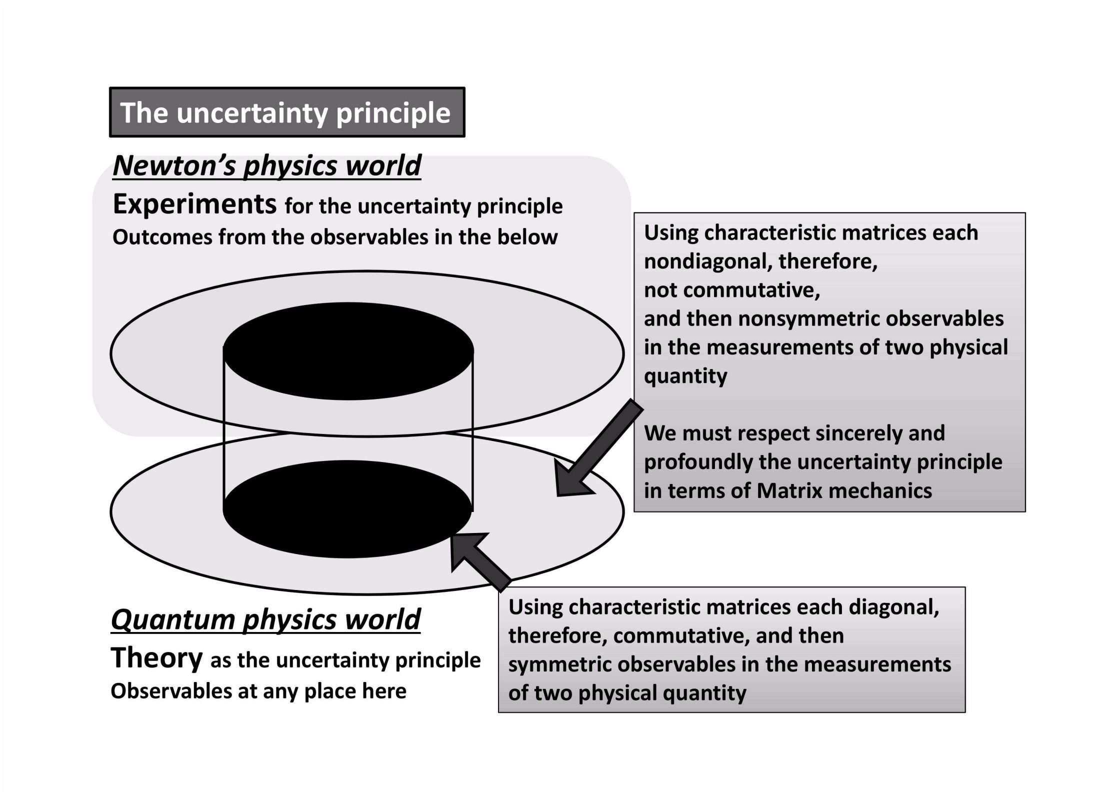

Figure 1: How to learn the uncertainty principle

Quantum physics is highly developed with its applications since longtime ago. One of the central points of the theory, as it is called, the uncertainty principle which encompasses some kind of mathematical inequalities for the threshold of precision of physical simultaneous measurements of pairs of physical observables in physical quantities. For instance, in 1927, Werner Heisenberg stated that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [8]. We study the uncertainty principle by using Figure 1.

It seems that when the uncertainty principle had disclosed, the experiment was first and then the principle followed the results to explain. As for the experiment, in observables the position of the particle is probably and naturally at least related to the momentum of it in terms of some natural environment. It means that the two physical quantities are disclosed by means of the matrices not diagonal in two characteristic equations for solving each of the physical quantities. Therefore, since the two results are not correctly measured, the use of two matrices not diagonal is naturally correct.

As these days the precision of measurement equipments became higher, we could have the case where two observables are equal, and "the two matrices in their theoretical analyses correspond to the diagonal" (See [15]), exceptionally. That means measurements are independent each other and then symmetrical to the order of these observables by virtue of "two characteristic matrices diagonal." Here we do not have the uncertainty principle because we have two correct observables simultaneously under "two diagonal matrices" for the two measurements.

While the two characteristic matrices are not diagonal and then not commutative, the two observables are not independent mathematically. So, the order of measurements gives the different results and the observables are not simultaneous and depending on the order of measurements.

As a result, the uncertainty principle does not work when two observables are independent on the order of the measurements, namely symmetric measurements [16] and then commutative in these observables with their theoretical characteristic matrices diagonal. Once more in the measurements [15], they perform this experiment by simultaneous measurements. Only commuting observables must be measured in such a physical situation. Thus, the uncertainty principle does not hold. However, the uncertainty relation is valid for different times t and t'. What we dare to say is not to disturb the uncertainty principle in physics, but to notice the feature in using the principle, respecting itself. We hope the discussion is useful for analyzing several systems in condensed matter and certain atomic nuclei in which time-dependent transitions may be observed.

5 Schrödinger–Robertson uncertainty relation

In this section, we discuss the Schrödinger–Robertson uncertainty relation. The detail derivation is shown in Robertson [17], Schrödinger [18], and standard textbooks such as Griffiths [19]. As for the discussion of the Schrödinger–Robertson uncertainty relation, the main point is the Cauchy-Schwarz inequality [20] as shown below: Why the uncertainty relation is derived is due to the fact that, in the matrix theory, there is non-commutativeness when we consider Multiplications. Addition says only commutativeness in the theory.

For any Hermitian operator $\hat{A}$, based upon the definition of variance, we have

$$\sigma_A^2 = \langle \Psi(\hat{A} - \langle \hat{A} \rangle) | (\hat{A} - \langle \hat{A} \rangle)\Psi \rangle, \tag{5.1}$$

where $\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle$. We let $|f\rangle = |(\hat{A} - \langle \hat{A} \rangle)\Psi\rangle$ and thus

$$\sigma_A^2 = \langle f | f \rangle. \tag{5.2}$$

Similarly, for any other Hermitian operator $\hat{B}$ in the state $|\Psi\rangle$

$$\sigma_B^2 = \langle \Psi(\hat{B} - \langle \hat{B} \rangle) | (\hat{B} - \langle \hat{B} \rangle)\Psi \rangle = \langle g | g \rangle, \tag{5.3}$$

for $|g\rangle = |(\hat{B} - \langle \hat{B} \rangle)\Psi\rangle$ and $\langle \hat{B} \rangle = \langle \Psi | \hat{B} | \Psi \rangle$. Thus, the product of the two variances can be expressed as

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle. \tag{5.4}$$

In order to relate the two vectors $|f\rangle$ and $|g\rangle$ with each other, we use the Cauchy-Schwarz inequality [20] which is defined as

$$\langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2, \tag{5.5}$$

and thus Eq. (5.4) can be written as

$$\sigma_A^2 \sigma_B^2 \geq |\langle f | g \rangle|^2. \tag{5.6}$$

Since $\langle f | g \rangle$ is generally a complex number, we use the fact that the modulus squared of any complex number z is defined as $|z|^2 = zz^*$, where z^* is the complex conjugate of z. The modulus squared can also be expressed as

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 = \left(\frac{z + z^*}{2} \right)^2 + \left(\frac{z - z^*}{2i} \right)^2. \tag{5.7}$$

We let $z = \langle f|g\rangle$ and $z^* = \langle g|f\rangle$ and substitute these into the equation above in giving

$$|\langle f|g\rangle|^2 = \left(\frac{\langle f|g\rangle + \langle g|f\rangle}{2}\right)^2 + \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2. \tag{5.8}$$

The inner product $\langle f|g\rangle$ is written out explicitly as

$$\langle f|g\rangle = \langle \Psi(\hat{A} - \langle \hat{A}\rangle)|(\hat{B} - \langle \hat{B}\rangle)\Psi\rangle, \tag{5.9}$$

and using the fact that $\hat{A}$ and $\hat{B}$ are Hermitian operators, we find, after some algebra,

$$\langle f|g\rangle = \langle \Psi|\hat{A}\hat{B}\Psi\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle. \tag{5.10}$$

Similarly, it can be shown that $\langle g|f\rangle = \langle \Psi|\hat{B}\hat{A}\Psi\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle$. For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Thus we have

$$\langle f|g\rangle - \langle g|f\rangle = \langle \Psi|[\hat{A}, \hat{B}]|\Psi\rangle \tag{5.11}$$

and

$$\langle f|g\rangle + \langle g|f\rangle = \langle \Psi|\{\hat{A}, \hat{B}\}|\Psi\rangle - 2\langle \hat{A}\rangle\langle \hat{B}\rangle, \tag{5.12}$$

where we may introduce the anticommutator $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. We now substitute the above two equations into Eq. (5.8) in giving

$$|\langle f|g\rangle|^2 = \left(\frac{1}{2}\langle \Psi|\{\hat{A}, \hat{B}\}|\Psi\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle \Psi|[\hat{A}, \hat{B}]|\Psi\rangle\right)^2. \tag{5.13}$$

Substituting the above into Eq. (5.6), we have **the Schrödinger–Robertson uncertainty relation** as follows:

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle \Psi|\{\hat{A}, \hat{B}\}|\Psi\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle \Psi|[\hat{A}, \hat{B}]|\Psi\rangle\right)^2}. \tag{5.14}$$

As a result, the Schrödinger–Robertson uncertainty relation is given by (5.14). For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. And we may introduce the anticommutator $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$.

6 Upper limit of the Schrödinger–Robertson uncertainty relation

In this section, we discuss the fact that the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation. We derive the Schrödinger–Robertson uncertainty relation by using the Bloch sphere in the specific case.

The upper limit of the Schrödinger–Robertson uncertainty relation in a two-level system (e.g., electron spin, photon polarizations, and so on) is derived by [14]. This is certified by the Bloch sphere when we would measure $\hat{\sigma_x}$ and $\hat{\sigma_y}$. Therefore, the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation.

As a result here, the Schrödinger–Robertson uncertainty relation in a two-level system has the upper limit in the Bloch sphere.

7 Lower limit of the Schrödinger–Robertson uncertainty relation

In the authors knowledge, nobody derives the lower limit of the Schrödinger–Robertson uncertainty relation. We suppose that $\hat{A}, \hat{B}$ are two Hermitian operators on an N-dimensional unitary space. Let us consider a simultaneous pure eigenstate $|\Psi_i\rangle, (i = 1, 2, ..., N)$, that is, $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$, for the two Hermitian operators $\hat{A}, \hat{B}$ such that $\langle\Psi_i|\hat{A}|\Psi_i\rangle = a_i, \langle\Psi_i|\hat{B}|\Psi_i\rangle = b_i$.

The Schrödinger–Robertson uncertainty relation is as shown in (5.14).

Statement

When $[\hat{A}, \hat{B}] = 0$, the Schrödinger–Robertson uncertainty relation becomes

$$\sigma_A \sigma_B \geq \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{7.1}$$

and the lower bound is zero.

Proof: We consider the Schrödinger–Robertson uncertainty relation in the case where $[\hat{A}, \hat{B}] = 0$

$$\sigma_A \sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\{\hat{A}, \hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2}. \tag{7.2}$$

Thus, we have

$$\sigma_A \sigma_B \geq \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle. \tag{7.3}$$

On the other hand, we have

$$\begin{aligned} &\langle\Psi_i|\hat{A}\hat{B}|\Psi_i\rangle = a_i b_i, \\ &\langle\Psi_i|\hat{A}|\Psi_i\rangle\langle\Psi_i|\hat{B}|\Psi_i\rangle = a_i b_i, \end{aligned} \tag{7.4}$$

where $[\hat{A}, \hat{B}] = 0$ and a_i, b_i are respectively eigenvalues of the two Hermitian operators $\hat{A}$ and $\hat{B}$.

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We show that the lower bound of the Schrödinger–Robertson uncertainty relation is exactly zero. The Schrödinger–Robertson uncertainty relation says a precise measurement on commuting observables, symmetric measurement [16], is possible.

From the above, we conclude the lower bound of the Schrödinger–Robertson uncertainty relation is exactly zero.

In conclusions, we have reviewed the Schrödinger–Robertson uncertainty relation. Our discussion of the uncertainty relation has asserted a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position ($\hat{x}$) and momentum ($\hat{p}$), can be known. Additionally, it has turned out that the relation says the natural understandable upper limit in the Bloch sphere, in qubits handling, and the meaningful lower limit (exactly zero).

8 Time-dependent Schrödinger–Robertson uncertainty relation in the Schrödinger representation

In this section, we derive the time-dependent Schrödinger–Robertson uncertainty relation in the Schrödinger representation. Our general uncertainty relation asserts, in different times t and t', a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position at time t ($\hat{x}(t)$) and momentum at time t' ($\hat{p}(t')$), can be known. Here, we introduce the notation t which relates to time. The quantum transition probability between a quantum state at time t $|\Psi(t)\rangle$ and another quantum state at time t' $|\Psi(t')\rangle$ is given by $|\langle\Psi(t)|\Psi(t')\rangle|^2$.

For any Hermitian operator $\hat{A}$, based upon the definition of variance, we have

$$\sigma_A^2(t) = \langle\Psi(t)(\hat{A} - \langle\hat{A}\rangle(t))|(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)\rangle, \tag{8.1}$$

where $\langle\hat{A}\rangle(t) = \langle\Psi(t)|\hat{A}|\Psi(t)\rangle$. We let $|f(t)\rangle = |(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)\rangle$ and thus

$$\sigma_A^2(t) = \langle f(t)|f(t)\rangle. \tag{8.2}$$

Similarly, for any other Hermitian operator $\hat{B}$ in the state $|\Psi(t')\rangle$

$$\sigma_B^2(t') = \langle\Psi(t')(\hat{B} - \langle\hat{B}\rangle(t'))|(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle = \langle g(t')|g(t')\rangle, \tag{8.3}$$

for $|g(t')\rangle = |(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle$ and $\langle\hat{B}\rangle(t') = \langle\Psi(t')|\hat{B}|\Psi(t')\rangle$. Thus, the product of the two variances can be expressed as

$$\sigma_A^2(t)\sigma_B^2(t') = \langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle. \tag{8.4}$$

In order to relate the two vectors $|f(t)\rangle$ and $|g(t')\rangle$ with each other, we use the Cauchy-Schwarz inequality [20] which is defined as

$$\langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle \geq |\langle f(t)|g(t')\rangle|^2, \tag{8.5}$$

and thus Eq. (8.4) can be written as

$$\sigma_A^2(t)\sigma_B^2(t') \geq |\langle f(t)|g(t')\rangle|^2 = \left(\frac{\langle f(t)|g(t')\rangle + \langle g(t')|f(t)\rangle}{2}\right)^2 + \left(\frac{\langle f(t)|g(t')\rangle - \langle g(t')|f(t)\rangle}{2i}\right)^2. \tag{8.6}$$

The inner product $\langle f(t)|g(t')\rangle$ is written out explicitly as

$$\langle f(t)|g(t')\rangle = \langle\Psi(t)(\hat{A} - \langle\hat{A}\rangle(t))|(\hat{B} - \langle\hat{B}\rangle(t'))\Psi(t')\rangle, \tag{8.7}$$

and using the fact that $\hat{A}$ and $\hat{B}$ are Hermitian operators, we find, after some algebra,

$$\begin{aligned} \langle f(t)|g(t')\rangle &= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle - \langle\hat{B}\rangle(t')\langle\Psi(t)|\hat{A}\Psi(t')\rangle \\ &\quad - \langle\hat{A}\rangle(t)\langle\Psi(t)|\hat{B}\Psi(t')\rangle + \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle. \end{aligned} \tag{8.8}$$

Similarly, it can be shown that

$$\begin{aligned} \langle g(t')|f(t)\rangle &= \langle\Psi(t')|\hat{B}\hat{A}\Psi(t)\rangle - \langle\hat{A}\rangle(t)\langle\Psi(t')|\hat{B}\Psi(t)\rangle \\ &\quad - \langle\hat{B}\rangle(t')\langle\Psi(t')|\hat{A}\Psi(t)\rangle + \langle\hat{B}\rangle(t')\langle\hat{A}\rangle(t)\langle\Psi(t')|\Psi(t)\rangle. \end{aligned} \tag{8.9}$$

Let define A_1, A_2 as:

$$\langle f(t)|g(t')\rangle - \langle g(t')|f(t)\rangle = 2iA_1 \tag{8.10}$$

and

$$\langle f(t)|g(t')\rangle + \langle g(t')|f(t)\rangle = 2A_2. \tag{8.11}$$

We now substitute the above two equations into Eq. (8.6) in giving

$$|\langle f(t)|g(t')\rangle|^2 = (A_1)^2 + (A_2)^2. \tag{8.12}$$

Thus, we get **the time-dependent Schrödinger–Robertson uncertainty relation in the Schrödinger representation** as follows:

$$\sigma_A(t)\sigma_B(t') \geq \sqrt{(A_1)^2 + (A_2)^2}. \tag{8.13}$$

Also, as a result, the time-dependent Schrödinger–Robertson uncertainty relation in the Schrödinger representation is given by (8.13).

The time-independent Schrödinger–Robertson uncertainty relation is a part of our general formula where $t = t'$, i.e.,

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\Psi|\{\hat{A},\hat{B}\}|\Psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right)^2}. \tag{8.14}$$

For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. And we may introduce the anticommutator $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$.

9 Time-dependent Schrödinger–Robertson uncertainty relation in the Heisenberg representation

In this section, we derive the time-dependent Schrödinger–Robertson uncertainty relation in the Heisenberg representation. Here,

$$\hat{A}(t) = e^{iHt/\hbar}\hat{A}e^{-iHt/\hbar}, \hat{B}(t') = e^{iHt'/\hbar}\hat{B}e^{-iHt'/\hbar}, \tag{9.1}$$

where H is a Hamiltonian of the system.

For any Hermitian operator $\hat{A}(t)$, based upon the definition of variance, we have

$$\sigma^2_{A(t)} = \langle \Psi(\hat{A}(t) - \langle \hat{A}(t)\rangle)|(\hat{A}(t) - \langle \hat{A}(t)\rangle)\Psi\rangle, \tag{9.2}$$

where $\langle \hat{A}(t)\rangle = \langle \Psi|\hat{A}(t)|\Psi\rangle$. We let $|f(t)\rangle = |(\hat{A}(t) - \langle \hat{A}(t)\rangle)\Psi\rangle$ and thus

$$\sigma^2_A(t) = \langle f(t)|f(t)\rangle. \tag{9.3}$$

Similarly, for any other Hermitian operator $\hat{B}(t')$ in the state $|\Psi\rangle$

$$\sigma^2_{B(t')} = \langle \Psi(\hat{B}(t') - \langle \hat{B}(t')\rangle)|(\hat{B}(t') - \langle \hat{B}(t')\rangle)\Psi\rangle = \langle g(t')|g(t')\rangle, \tag{9.4}$$

for $|g(t')\rangle = |(\hat{B}(t') - \langle \hat{B}(t')\rangle)\Psi\rangle$ and $\langle \hat{B}(t')\rangle = \langle \Psi|\hat{B}(t')|\Psi\rangle$. Thus, the product of the two variances can be expressed as

$$\sigma^2_{A(t)}\sigma^2_{B(t')} = \langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle. \tag{9.5}$$

After some algebra similar to them described in the section 5, we find **the time-dependent Schrödinger–Robertson uncertainty relation in the Heisenberg representation** as follows:

$$\sigma_{A(t)}\sigma_{B(t')} \geq \sqrt{\left(\frac{1}{2}\langle \Psi|\{\hat{A}(t), \hat{B}(t')\}|\Psi\rangle - \langle \hat{A}(t)\rangle\langle \hat{B}(t')\rangle\right)^2 + \left(\frac{1}{2i}\langle \Psi|[\hat{A}(t), \hat{B}(t')]|\Psi\rangle\right)^2}. \tag{9.6}$$

As a result, the time-dependent Schrödinger–Robertson uncertainty relation in the Heisenberg representation is given by (9.6) and the lower limit of the uncertainty relation is exactly zero for different times t and t' when

$$[\hat{A}(t), \hat{B}(t')] = \mathbf{0}. \tag{9.7}$$

10 Quantum measurement theory for commuting observables based on functions

The relation between quantum physics and Newton physics is based on observables in measurements from quantum physics to Newton's, which fact is likely to be important in progressing today's physics, mostly in quantum's. Therefore, we try to formalize observables as functional values from in quantum physics, where the function is to obtain the eigenvalue of some eigenvalue problem shown by its corresponding matrix.

In common practice, eigenvalues usually evolve as representative observables. The series/order of measurements seem to be regarded as some function to show the two eigenvalues as observables. Here the measurements consist of two observables in obtaining eigenvalues of two eigensystems based upon the matrices whose structures are diagonal matrices. As a result, we may introduce the function to obtain observables from the eigenvalue problem shown some matrix.

When the function is working for two matrices A_1 and A_2 eigenvalues problem with the order of measurements, these matrices are treated in product rule between these matrices, the measurement operation seems to be this.

$$f(A_1) \cdot f(A_2) = f(A_1 \cdot A_2). \tag{10.1}$$

And the following relation as functions is very important: $f(g(O)) = g(f(O))$, where O means A_1 or A_2, shows the order of the two measurements f and g.

In practice, the two measurements are usually depending on the order of these operations because we do not have commutative between these matrices except for diagonal matrices for two eigenvalue problem in product rule operation.

In short, at the same time we can obtain the observables of two eigenvalue problem with only diagonal matrix cases. This mathematical proof in matrix physics is obvious and rather should be usable in such quantum problems and their analyses. Therefore, the theoretical fact is even in the uncertainty principle. In fact, the principle has the exception where the two observables are true at the same time because these matrices related to eigenvalues are commutative and then we get two eigenvalues at the same time.

11 Discussion and Conclusion

Basically the uncertainty principle needs the case $[\sigma_x, \sigma_z]$ not equal 0, which theoretically causes the different results in each of the two observables in calculations. Such a model is very much suitable for the practical experiments in Newton's world capturing the practical measurement results including errors in the experiments.

Extending our understanding the uncertainty principle to much more, the uncertainty principle is based upon the matrix mechanics in terms of the beauty of mathematical explanation of it. In fact the above relation does not explain the rare but true case in observables.

It seems that most observables with their real physical experiment errors, even very small, admit the above relation with matrices not diagonal. While the measurement precision with the equipment is not so high, the uncertainty principle is reasonable with the feature of matrices mathematics. And most cases in measurements in Newton's world are allowable as their experimental errors. So far so good!

Proportional to the development of experimental equipment, the measurements are more precisional, and the mathematical analyses of the observables are not always correct. There could be the case that we cannot use the matrix mechanics in all the cases because there could be the one where the matrices diagonal were needed.

In conclusion, we, with our sincere desire, explain that the uncertainty principle is convenient with its own property due to matrix mechanics' power limit, if measurements are done with not so high precision equipment to make results including errors. However, if the observables are from on the experiment with correct measurements in high precision, and it happens that if the matrix analyses with the matrices diagonal are used to the correct results, the uncertain principle could seem to be used reasonably and correctly. But, this interpretation seems to be hard in its validity.

And, we have derived the time-dependent uncertainty relations. The uncertainty relations have been valid for different times t and t'. We have expected the discussion is useful for analyzing several systems in condensed matter and certain atomic nuclei in which time-dependent transitions may be observed.

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Declarations

Ethical Approval

The authors are in an applicable thought to ethical approval.

Competing Interests

The authors state that there is no conflict of interest.

Author Contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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No data associated in the manuscript.

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