How the effect of the uncertainty principle might be in simultaneous measurements

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Abstract

We respect the uncertainty principle (the Schrödinger–Robertson uncertainty relation) in describing properly physical phenomena even more in today's or future experiments with opposite results. We explain such a question about the uncertainty principle this: We perform an experiment by simultaneous measurements in order to test the effect of the uncertainty principle. Only commuting observables must be measured thinking in the experimental situation. Thus, the effect of the uncertainty principle cannot be seen by simultaneous measurements. This explanation of the experimental data is our main assertion in this paper. However, what we dare to say is not to disturb the uncertainty principle, but to notice our policy in using the principle, respecting itself. Therefore, we respect the Schrödinger–Robertson uncertainty relation as the famous mathematical form of the uncertainty principle. We show the Schrödinger–Robertson uncertainty relation has naturally the understandable upper limit (the Bloch sphere) and the meaningful lower limit (exactly zero). We expect our discussions give some insight for future studies for the uncertainty principle.

1 Introduction

Quantum mechanics (cf. [1, 2, 3, 4, 5, 6, 7]) gives accurate and at-times-remarkably accurate numerical predictions and much experimental data has fit to quantum predictions for long time. In quantum mechanics, the uncertainty principle is any of the variety of mathematical inequalities asserting a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position x and momentum p, can be known simultaneously. For instance, in 1927, Werner Heisenberg stated that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [8]. The formal inequality relating to the standard deviation of position σ_x and the standard deviation of momentum σ_p was derived by Earle Hesse Kennard [9] later that year and by Hermann Weyl [10] in 1928.

Maccone and Pati discuss stronger uncertainty relations for all incompatible observables [11]. Quantum dynamics of simultaneously measured non-commuting observables is discussed [12]. Dynamics of a qubit while simultaneously monitoring its relaxation and dephasing are also discussed [13]. The upper limit of the Schrödinger–Robertson uncertainty relation in a two-level system (e.g., electron spin, photon polarizations, and so on) is discussed in [14]. This is certified by the Bloch sphere when we would measure $\hat{\sigma_x}$ and $\hat{\sigma_y}$. How about the lower limit of the uncertainty relation? In the authors' knowledge, nobody derives the lower limit of the Schrödinger–Robertson uncertainty relation (exactly zero).

The motivations behind this work to be discussed in this paper are this: We respect profoundly the uncertainty principle (the Schrödinger–Robertson uncertainty relation) for describing properly physical phenomena even more in today's or future experiments with opposite results. For example, there is a paper [15] by Werner A. Hofer titled "Heisenberg, uncertainty, and the scanning tunneling microscope" which leads us to a question about the uncertainty principle. The assertion is that the density of electron charge is a physically real, i.e., in principle precisely measurable quantity.

In this paper, we respect the uncertainty principle (the Schrödinger–Robertson uncertainty relation) in describing properly physical phenomena even more in today's or future experiments with opposite results. We explain such a question about the uncertainty principle this: We perform an experiment by simultaneous measurements in order to test the effect of the uncertainty principle. Only commuting observables must be measured thinking in the experimental situation. Thus, the effect of the uncertainty principle cannot be seen by simultaneous measurements. This explanation of the experimental data is our main assertion in this paper. However, what we dare to say is not to disturb the uncertainty principle, but to notice our policy in using the principle, respecting itself. Therefore, we respect the Schrödinger–Robertson uncertainty relation as the famous mathematical form of the uncertainty principle. We show the Schrödinger–Robertson uncertainty relation has naturally the understandable upper limit (the Bloch sphere) and the meaningful lower limit (exactly zero). We expect our discussions give some insight for future studies for the uncertainty principle.

This paper is organized this:

In Sec. 2, we propose the symmetry of observables as a new interpretation for some exceptional experimental results while accepting the uncertainty principle. In Sec. 3, we show that the effect of the uncertainty principle cannot be seen by simultaneous measurements. This explanation of the experimental data is our main assertion in this paper. In Sec. 4, we respect the Schrödinger–Robertson uncertainty relation. In Sec. 5, the upper limit of the Schrödinger–Robertson uncertainty relation is given in qubits handling. In Sec. 6, the lower limit of the Schrödinger–Robertson uncertainty relation is also given. Section 7 deals with conclusions in this paper.

2 Symmetry of observables

The symmetry of observables is worth considering, in order to obtain an honest interpretation of the uncertainty principle.

It can be said that the symmetry of two observables and the commutativity of the two are equivalent. To prove this, let us investigate that when the observables are noncommutative, the observables are not symmetric using spin's behavior.

1. Trying to measure the spin observable σ_z in some eigenstate with the eigenvalue $+1$.

$$\sigma_z|\uparrow\rangle = +1|\uparrow\rangle. \tag{2.1}$$

2. Then, we have $+1$ as the result of the measurement spin observable σ_z with a probability 1.

3. The result of the spin observable σ_x is -1 with the probability 0.5.

$$\sigma_x|\uparrow\rangle = \pm 1|\uparrow\rangle. \tag{2.2}$$

4. Even though the results are ± 1, these measurements are depend on the order of these two.

5. It happens that the first measurement of spin observables σ_x might obtain $+1$ with the probability 0.5.

$$\sigma_x|\uparrow\rangle = \pm 1|\uparrow\rangle. \tag{2.3}$$

6. This means that if the two observables are non-commutative, the result of measurements is not symmetric, which means that the fact depends on the order of these two measurements.

7. Let us make the contraposition of the two above. We can obtain that when the two observables are symmetric, namely the case not concerning the two measurements' order, these observables are commutative.

8. Obviously if the two observables are commutative, the results of the two measurements are symmetric independent of the order of the measurements.

As a result, two observables are symmetric, independent of the order of the two measurements if and only if the two observables are commutative.

3 Effect of the uncertainty principle cannot be seen by simultaneous measurements

There is a paper [15] by Werner A. Hofer titled "Heisenberg, uncertainty, and the scanning tunneling microscope" which leads us to a question about the uncertainty principle. The assertion is that the density of electron charge is a physically real, i.e., in principle precisely measurable quantity. This explanation of the experimental data is our main assertion in this paper.

Let us explain the question about the uncertainty principle this:

1. They perform an experiment by simultaneous measurements, i.e., symmetric measurements (they are free from the order of measurements themselves) [16] in order to test the effect of the uncertainty principle.

2. Only commuting observables must be measured thinking in the experimental situation.

3. Thus, the effect of the uncertainty principle cannot be seen by simultaneous measurements.

We explain more the experimental situation this: Clearly, the effect of the uncertainty principle cannot be seen when we measure only commuting observables. The experimental case that the results of measurements are symmetric measurements (they are free from the order of measurements themselves) is equivalent to the case that we measure commuting observables. Therefore, the effect of the uncertainty principle cannot be seen when the results of measurements are symmetric measurement, i.e., simultaneous measurements.

It could be said that the mathematical character of the uncertainty principle can not work by simultaneous measurements. Thus, the experiment leads us to a question about the uncertainty principle.

As a result, the effect of the uncertainty principle cannot be seen by simultaneous measurements. However, what we dare to say is not to disturb the uncertainty principle, but to notice our policy in using the principle, respecting itself.

4 Schrödinger–Robertson uncertainty relation

In this section, we respect the Schrödinger–Robertson uncertainty relation. The detail derivation is shown in Robertson [17], Schrödinger [18], and standard textbooks such as Griffiths [19]. As for the discussion of the Schrödinger–Robertson uncertainty relation, the main point is the Cauchy-Schwarz inequality [20] as shown below: Why the uncertainty relation is derived is due to the fact that, in the matrix theory, there is non-commutativeness when we consider Multiplications. Addition says only commutativeness in the theory.

For any Hermitian operator $\hat{A}$, based upon the definition of variance, we have

$$\sigma_A^2 = \langle \Psi(\hat{A} - \langle \hat{A} \rangle) | (\hat{A} - \langle \hat{A} \rangle)\Psi \rangle, \tag{4.1}$$

where $\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle$. We let $|f\rangle = |(\hat{A} - \langle \hat{A} \rangle)\Psi\rangle$ and thus

$$\sigma_A^2 = \langle f | f \rangle. \tag{4.2}$$

Similarly, for any other Hermitian operator $\hat{B}$ in the state $|\Psi\rangle$

$$\sigma_B^2 = \langle \Psi(\hat{B} - \langle \hat{B} \rangle) | (\hat{B} - \langle \hat{B} \rangle)\Psi \rangle = \langle g | g \rangle, \tag{4.3}$$

for $|g\rangle = |(\hat{B} - \langle \hat{B} \rangle)\Psi\rangle$ and $\langle \hat{B} \rangle = \langle \Psi | \hat{B} | \Psi \rangle$. Thus, the product of the two variances can be expressed as

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle. \tag{4.4}$$

In order to relate the two vectors $|f\rangle$ and $|g\rangle$ with each other, we use the Cauchy-Schwarz inequality [20] which is defined as

$$\langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2, \tag{4.5}$$

and thus Eq. (4.4) can be written as

$$\sigma_A^2 \sigma_B^2 \geq |\langle f | g \rangle|^2. \tag{4.6}$$

Since $\langle f | g \rangle$ is generally a complex number, we use the fact that the modulus squared of any complex number z is defined as $|z|^2 = zz^*$, where z^* is the complex conjugate of z. The modulus squared can also be expressed as

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 = \left(\frac{z + z^*}{2} \right)^2 + \left(\frac{z - z^*}{2i} \right)^2. \tag{4.7}$$

We let $z = \langle f | g \rangle$ and $z^* = \langle g | f \rangle$ and substitute these into the equation above in giving

$$|\langle f | g \rangle|^2 = \left(\frac{\langle f | g \rangle + \langle g | f \rangle}{2} \right)^2 + \left(\frac{\langle f | g \rangle - \langle g | f \rangle}{2i} \right)^2. \tag{4.8}$$

The inner product $\langle f | g \rangle$ is written out explicitly as

$$\langle f | g \rangle = \langle \Psi(\hat{A} - \langle \hat{A} \rangle) | (\hat{B} - \langle \hat{B} \rangle)\Psi \rangle, \tag{4.9}$$

and using the fact that $\hat{A}$ and $\hat{B}$ are Hermitian operators, we find, after some algebra,

$$\langle f | g \rangle = \langle \Psi | \hat{A}\hat{B}\Psi \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle. \tag{4.10}$$

Similarly, it can be shown that $\langle g | f \rangle = \langle \Psi | \hat{B}\hat{A}\Psi \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$. For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Thus we have

$$\langle f | g \rangle - \langle g | f \rangle = \langle \Psi | [\hat{A}, \hat{B}] | \Psi \rangle \tag{4.11}$$

and

$$\langle f|g\rangle + \langle g|f\rangle = \langle\Psi|\{\hat{A},\hat{B}\}|\Psi\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{4.12}$$

where we may introduce the anticommutator $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. We now substitute the above two equations into Eq. (4.8) in giving

$$|\langle f|g\rangle|^2 = \left(\frac{1}{2}\langle\Psi|\{\hat{A},\hat{B}\}|\Psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right)^2. \tag{4.13}$$

Substituting the above into Eq. (4.6), we have **the Schrödinger–Robertson uncertainty relation** this:

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\Psi|\{\hat{A},\hat{B}\}|\Psi\rangle - \langle\Psi|\hat{A}|\Psi\rangle\langle\Psi|\hat{B}|\Psi\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right)^2}. \tag{4.14}$$

As a result, the Schrödinger–Robertson uncertainty relation is given by (4.14). For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. And we may introduce the anticommutator $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$.

5 Upper limit of the Schrödinger–Robertson uncertainty relation

In this section, we discuss the fact that the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation. We derive the Schrödinger–Robertson uncertainty relation by using the Bloch sphere in the specific case.

The upper limit of the Schrödinger–Robertson uncertainty relation in a two-level system (e.g., electron spin, photon polarizations, and so on) is derived by [14]. This is certified by the Bloch sphere when we would measure $\hat{\sigma_x}$ and $\hat{\sigma_y}$. Therefore, the Bloch sphere imposes the upper limit of the Schrödinger–Robertson uncertainty relation.

As a result here, the Schrödinger–Robertson uncertainty relation in a two-level system has the upper limit in the Bloch sphere.

6 Lower limit of the Schrödinger–Robertson uncertainty relation

In the authors' knowledge, nobody derives the lower limit of the Schrödinger–Robertson uncertainty relation. We suppose that $\hat{A},\hat{B}$ are two Hermitian operators on an N-dimensional unitary space. Let us consider a simultaneous pure eigenstate $|\Psi_i\rangle, (i = 1, 2, ..., N)$, that is, $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$, for the two Hermitian operators $\hat{A},\hat{B}$ such that $\langle\Psi_i|\hat{A}|\Psi_i\rangle = a_i, \langle\Psi_i|\hat{B}|\Psi_i\rangle = b_i$.

The Schrödinger–Robertson uncertainty relation is as shown in (4.14).

Statement

When $[\hat{A},\hat{B}] = 0$, the Schrödinger–Robertson uncertainty relation becomes

$$\sigma_A\sigma_B \geq \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{6.1}$$

and the lower bound is zero.

Proof: We consider the Schrödinger–Robertson uncertainty relation in the case where $[\hat{A},\hat{B}] = 0$

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2}. \tag{6.2}$$

Thus, we have

$$\sigma_A\sigma_B \geq \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle. \tag{6.3}$$

On the other hand, we have

$$\begin{aligned}&\langle\Psi_i|\hat{A}\hat{B}|\Psi_i\rangle = a_i b_i,\\&\langle\Psi_i|\hat{A}|\Psi_i\rangle\langle\Psi_i|\hat{B}|\Psi_i\rangle = a_i b_i,\end{aligned}\tag{6.4}$$

where $[\hat{A},\hat{B}]=0$ and a_i, b_i are respectively eigenvalues of the two Hermitian operators $\hat{A}$ and $\hat{B}$.

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We show that the lower bound of the Schrödinger–Robertson uncertainty relation is exactly zero. The Schrödinger–Robertson uncertainty relation says a precise measurement on commuting observables, symmetric measurement [16], is possible.

As a result, the lower bound of the Schrödinger–Robertson uncertainty relation is exactly zero.

In summary, we have respected the Schrödinger–Robertson uncertainty relation. Our discussion of the uncertainty relation has asserted a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position ($\hat{x}$) and momentum ($\hat{p}$), can be known simultaneously. Additionally, it has turned out that the relation says the natural understandable upper limit in the Bloch sphere, in qubits handling, and the meaningful lower limit (exactly zero).

7 Conclusions

In conclusions, we have respected the uncertainty principle (the Schrödinger–Robertson uncertainty relation) in describing properly physical phenomena even more in today's or future experiments with opposite results. We have explained such a question about the uncertainty principle this: We have performed an experiment by simultaneous measurements in order to test the effect of the uncertainty principle. Only commuting observables must have been measured thinking in the experimental situation. Thus, the effect of the uncertainty principle cannot have been seen by simultaneous measurements. This explanation of the experimental data is our main assertion in this paper. However, what we dare to say has not been to disturb the uncertainty principle, but to notice our policy in using the principle, respecting itself. Therefore, we have respected the Schrödinger–Robertson uncertainty relation as the famous mathematical form of the uncertainty principle. We have shown the Schrödinger–Robertson uncertainty relation has naturally the understandable upper limit (the Bloch sphere) and the meaningful lower limit (exactly zero). We have expected our discussions give some insight for future studies for the uncertainty principle.

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Declarations

Ethical Approval

The authors are in an applicable thought to ethical approval.

Competing Interests

The authors state that there is no conflict of interest.

Author Contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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Data Availability

No data associated in the manuscript.

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