Cantor's illusion

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abstract

This analysis shows Cantor's diagonal argument published in 1891 did not prove the cardinality of his infinite set M is greater than the set of integers N.

the argument

Translation from Cantor's 1891 paper [1]:

Namely, let \( m \) and \( n \) be two different characters, and consider a set [Inbegriff] \( M \) of elements

\[ E = (x_1, x_2, \ldots, x_v, \ldots) \]

which depend on infinitely many coordinates \( x_1, x_2, \ldots, x_v, \ldots \), and where each of the coordinates is either \( m \) or \( w \). Let \( M \) be the totality [Gesamtheit] of all elements \( E \).

To the elements of \( M \) belong e.g. the following three:

\[ E^I = (m, m, m, m, \ldots), \]
\[ E^{II} = (w, w, w, w, \ldots), \]
\[ E^{III} = (m, w, m, w, \ldots). \]

I maintain now that such a manifold [Mannigfaltigkeit] \( M \) does not have the power of the series \( 1, 2, 3, \ldots, v, \ldots \).

This follows from the following proposition:

"If \( E_1, E_2, \ldots, E_v, \ldots \) is any simply infinite [einfach unendliche] series of elements of the manifold \( M \), then there always exists an element \( E_0 \) of \( M \), which cannot be connected with any element \( E_v \)."

For proof, let there be

\[ E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,v}, \ldots) \]
\[ E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,v}, \ldots) \]
\[ E_v = (a_{v,1}, a_{v,2}, \ldots, a_{v,v}, \ldots) \]

\[ \ldots \]

where the characters \( a_{u,v} \) are either \( m \) or \( w \). Then there is a series \( b_1, b_2, \ldots, b_v, \ldots \), defined so that \( b_v \) is also equal to \( m \) or \( w \) but is different from \( a_{v,v} \).

Thus, if \( a_{v,v} = m \), then \( b_v = w \).

Then consider the element

\[ E_0 = (b_1, b_2, b_3, \ldots) \]

of \( M \), then one sees straight away, that the equation
\[ E_0 = E_u \]
cannot be satisfied by any positive integer u, otherwise for that u and for all values of v.

\[ b_v = a_{u,v} \]
and so we would in particular have

\[ b_0 = a_{u,u} \]

which through the definition of \( b_v \) is impossible. From this proposition it follows immediately that the totality of all elements of \( M \) cannot be put into the sequence \([Reihenform]: E_1, E_2, \ldots, E_v, \ldots\) otherwise we would have the contradiction, that a thing \([Ding]\) \( E0 \) would be both an element of \( M \), but also not an element of \( M \).

(End of translation)

**list**

The issue is the geometric form of a list, thus this analysis begins with real world finite lists instead of speculative infinite lists. The list \( L \) is a visual aid to comprehend the properties of a finite set of sequences. The list extends vertically and is finite in length. The list is also a finite array of characters, each with a unique \((u, v)\) coordinate location, with \( u \) and \( v \) from the set of integers \( N \). Only the diagonal and its horizontal counterpart and its negation are shown for clarity.

**sequence**

A sequence is defined as a one dimensional pattern of characters using the set \( \{m, w\} \). Each sequence occupies one row \( u \), and extends horizontally with a finite length. The key factor is the independent property of each sequence. Each is formed independently of the others, and entered at random locations within the list. The character \( m \) or \( w \), for each position \( v \) is determined by a random process such as a coin toss, thus there is no rigid rule of formation. An alphabetical order could be imposed on the list, but for \( v>3 \), there would be no diagonal sequence of a repeating character beginning at \( u=1 \).

If \( v \) is the number of characters in a sequence, and \( c \) the number of characters used to form the sequence then the number of unique sequences \( s \) in a list \( L_v \) is a function of \( v \).

\[ s = c^v \]

For \( c=2 \) the negation of a sequence results from interchanging all characters \( m \) and \( w \). Analyzing the progression of finite lists for \( c=2 \):

if \( v=1 \), then \( s=2 \).
In fig.1 the diagonal \( b \) contains \( m \) and does not extend the length of the list, and the square portion cannot contain \( E_0 \) its negation. In reality, it is contained in row 2.

In fig.2 the diagonal \( b \) does not extend the length of the list, and the square portion cannot contain \( E_0 \) its negation. In reality, it is contained somewhere within the 27 missing rows (\( u_{30} \)), and a horizontal \( b \) can appear anywhere in the list (\( u_{8} \)). The diagonal \( b \) is not new.

**observation**

In all cases of a list \( L_v \), where \( v \) is an integer from the set \( N \), the diagonal \( b \) does not extend the length of the list. While \( v \) increases linearly, \( u \) increases exponentially.

\[
\Delta s = 2^{v+1} - 2^v = 2^v .
\]

The list doubles in length for each increment of \( v \), and is never square.
In a truly random list, all sequences are independent of each other, and there is no factor that imposes any degree of order. The concept of an infinite dimension is a contradiction of terms. It removes the very thing that allows measurement, a boundary. An infinite sequence has no measurement. The list, being a sequence of rows is also not measurable for the same reason. There is no magic v where the list appears in a square form. Since c=2, all sequences have a corresponding negation and appear in pairs. Each Lv can be divided into two subsets. One containing sequences beginning with m, and one containing sequences beginning with w. Thus a list cannot have one missing sequence.

**conclusion**

Cantor envisions an extended diagonal sequence (red) with u=v, i.e. a geometric square. The real world form is a narrowing strip (black). The contradiction Cantor describes is of his own making, using a distorted incomplete list.

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