Collatz Conjecture: A countably infinite sequence

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1 Introduction

The Collatz conjecture, an unsolved problem in mathematics, has garnered attention over the years with various proven results. In this work, I present a new result that suggests the sequence can potentially have a countably infinite number of terms.

2 Discussion on result and proof

Consider \( n \) be a natural number on which we operate the Collatz Algorithm. We can observe that if \( n \) is even, the number would become \( \frac{n}{2} \) and eventually reach to 1 if we get an even number, i.e., any number of the form \( 2^k \), where \( k \) is an integer would reach 1 eventually.

So the thing that deviates \( n \) from reaching 1 is if \( n \) is an odd number but that is trivial to notice. Consider \( n \) to be an odd number, then we have:

\[
\begin{align*}
  n & \rightarrow 3n + 1 \rightarrow \frac{3n + 1}{2} \rightarrow s
\end{align*}
\]

Here, \( s \) can be an odd number or an even number depending on the number produced by the division \( \frac{3n + 1}{2} \). But here is something to observe, the number \( s \) is \( 1.5n + 0.5 \) which is greater than \( n \). Let’s assume that \( s \) is even. In that case writing the sequence again

\[
\begin{align*}
  n & \rightarrow 3n + 1 \rightarrow \frac{3n + 1}{2} \rightarrow \frac{3n + 1}{4}
\end{align*}
\]

We can observe that, \( \frac{3n + 1}{4} \) which is \( 0.75n + 0.25 \) is less than \( n \).

So, I conclude that if the transformation \( n \rightarrow 3n + 1 \) produces a number of the form \( 2^{2^k} \cdot a \) where \( k \) is a whole number and \( a \) is an odd number, then we get to a number less than \( n \).

Building on these observations (without proof), consider \( n = 8 \cdot 2^k - 1 \) for any whole number \( k \) which is an odd number. Applying the Collatz Algorithm
on such an $n$, we get:

$$(8 \cdot 2^k - 1)$$
$$(8 \cdot 2^k - 1) + 1$$
$$(3 \cdot (8 \cdot 2^k - 1) + 1)$$
$$\frac{3 \cdot (8 \cdot 2^k - 1) + 1}{2}$$
$$= \frac{3 \cdot 8 \cdot 2^k - 3 + 1}{2}$$
$$= (8 \cdot 3 \cdot 2^{k-1} - 1)$$
$$3 \cdot (8 \cdot 3 \cdot 2^{k-1} - 1) + 1$$
$$\frac{3 \cdot (8 \cdot 3 \cdot 2^{k-1} - 1) + 1}{2}$$
$$= \frac{8 \cdot 3^2 \cdot 2^{k-1} - 3 + 1}{2}$$
$$= (8 \cdot 3^2 \cdot 2^{k-2} - 1)$$
...
$$\rightarrow (3^{k+3} - 1) \quad (3)$$

We can henceforth observe from the above that the Collatz sequence for any $n = 8 \cdot 2^k - 1$ can have a partial sequence of the form given above. Carrying forward the Collatz Algorithm on $(3^{k+3} - 1)$ would give a smaller or bigger number but it’s at least possible to get the lower bound of the Collatz sequence length which depends on chosen $k$.

### 3 Conclusion

In this paper, I have explored the Collatz conjecture and presented a new result regarding the behavior of the sequence. The proof demonstrated that for natural numbers $n$ subjected to the Collatz Algorithm, the sequence can potentially have a countably infinite number of terms.