\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \] all are irrational numbers

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Abstract

It is proved that \(\sqrt{3} - \sqrt{2}\) and \(\sqrt{3} + \sqrt{2}\), \(e\) and \(\pi - e\), \(e + \pi\), \(\pi e\) and \(\frac{\pi}{e}\), all are irrational numbers. It is an argument by contradiction.

Notation and reminder

\[\pi:\text{ known as Archimedes constant, is the ratio of a circle's circumference to its diameter and } 3 < \pi < 4.\]

\[e = \sum_{m=0}^{\infty} \frac{1}{m!}:\text{ known as Euler's number and } 2 < e < 3.\]

\[\mathbb{N}^* := \{1,2,3,4,\ldots\} \text{ the natural numbers}.\]

\[\mathbb{Z} := \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\} \text{ the integers and } \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}.\]

\[\mathbb{Q} := \left\{ \frac{p}{q}: (p,q) \in \mathbb{Z} \times \mathbb{Z}^* \text{ and } p \land q = 1 \right\} \text{ the set of rational numbers.}\]

\[\mathbb{R} : \text{ the set of real numbers}.\]

\[\mathbb{R} \setminus \mathbb{Q} := \{x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}: \mathbb{Q} \subset \mathbb{R}\} \text{ the set of irrational numbers.}\]

\[p \land q := \max\{d \in \mathbb{N}^* : d/p \text{ and } d/q\} \text{ the greatest common divisor of } p \text{ and } q.\]

\[\forall: \text{ the universal quantifier and } \exists: \text{ the existential quantifier}.\]

Introduction

Irrational numbers are the type of real numbers that cannot be expressed in the rational form \(\frac{p}{q}\), where \(p, q\) are integers and \(q \neq 0\). In simple words, all the real numbers that are not rational numbers are irrational. In this paper we show that \(\sqrt{3} - \sqrt{2}\) and \(\sqrt{3} + \sqrt{2}\), \(e\) and \(\pi\), \(\pi - e\), \(\pi + e\), \(\pi e\) and \(\frac{\pi}{e}\), all are irrational numbers. It is an argument by contradiction.
\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers} \]

**Theorem 1.** \( \sqrt{6} \in \mathbb{R} \setminus \mathbb{Q} \). In other words, \( \sqrt{6} \) is an irrational number.

**Proof.** An argument by contradiction. Suppose that \( \sqrt{6} \in \mathbb{Q} \), and as \( \sqrt{6} > 0 \) then \( \exists p, q \in \mathbb{N}^* \) such that \( \sqrt{6} = \frac{p}{q} \) and \( p \wedge q = 1 \), then \( (\sqrt{6})^2 = \left( \frac{p}{q} \right)^2 \), then \( 6 = \frac{p^2}{q^2} \) and \( 6q^2 = p^2 \Rightarrow p^2 \) is even and \( p \in \mathbb{N}^* \Rightarrow p \) is even or \( p = 2k: k \in \mathbb{N}^* \Rightarrow 6q^2 = (2k)^2 = 4k^2 \Rightarrow 3q^2 = 2k^2 \) and \( 3 \wedge 2 = 1 \Rightarrow 2 \) divides \( q^2 \) and \( 2 \) is prime \( \Rightarrow 2 \) divides \( q \) and \( q \in \mathbb{N}^* \Rightarrow q \) is even or \( q = 2k' : k' \in \mathbb{N}^* \), hence \( p \wedge q \geq 2 \), and we get a contradiction because \( p \wedge q = 1 \).

**Main Theorem 1.** \( \sqrt{3} - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \) and \( \sqrt{3} + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \).

In other words, \( \sqrt{3} - \sqrt{2} \) and \( \sqrt{3} + \sqrt{2} \) both are irrational numbers.

**Proof.** An argument by contradiction. Suppose that \( \sqrt{3} - \sqrt{2} \in \mathbb{Q} \), then \( \exists r \in \mathbb{Q} \) such that \( \sqrt{3} - \sqrt{2} = r \) implies that \( (\sqrt{3} - \sqrt{2})^2 = r^2 \in \mathbb{Q} \Rightarrow 5 - 2\sqrt{6} = r^2 \in \mathbb{Q} \Rightarrow \sqrt{6} = \frac{5 - r^2}{2} \in \mathbb{Q} \), and we get a contradiction.

On the other hand, suppose that \( \sqrt{3} + \sqrt{2} \in \mathbb{Q} \), then \( \exists r \in \mathbb{Q} \) such that \( \sqrt{3} + \sqrt{2} = r \) implies that \( (\sqrt{3} + \sqrt{2})^2 = r^2 \in \mathbb{Q} \Rightarrow 5 + 2\sqrt{6} = r^2 \in \mathbb{Q} \Rightarrow \sqrt{6} = \frac{r^2 - 5}{2} \in \mathbb{Q} \), and we get a contradiction.

**Lemma 2.** We have \( \lim_{n \to +\infty} \sum_{m=n+1}^{\infty} \frac{n!}{m!} = 0 \) and \( \lim_{n \to +\infty} n \cdot \sum_{m=n+1}^{\infty} \frac{n!}{m!} = 1 \).

**Proof.** \( \forall n \in \mathbb{N}^* \), \( \sum_{m=n+1}^{\infty} \frac{n!}{m!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots < \frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \frac{1}{(n+1)(n+1) + 1} + \cdots = \frac{1}{1(n+1)} = \frac{1}{n} \),

then \( 0 < \sum_{m=n+1}^{\infty} \frac{n!}{m!} < \frac{1}{n} \) and \( \lim_{n \to +\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \to +\infty} \sum_{m=n+1}^{\infty} \frac{n!}{m!} = 0 \).

On the other hand, we have \( \frac{1}{n+1} < \sum_{m=n+1}^{\infty} \frac{n!}{m!} < \frac{1}{n} \Rightarrow \frac{n}{n+1} < n \cdot \sum_{m=n+1}^{\infty} \frac{n!}{m!} < 1 \), and \( \lim_{n \to +\infty} \frac{n}{n+1} = 1 \Rightarrow \lim_{n \to +\infty} n \cdot \sum_{m=n+1}^{\infty} \frac{n!}{m!} = 1 \).

For more details about irrational numbers, we refer the reader and our students to [1] and to [2].
Theorem 2. We have \( \lim_{n \to +\infty} n \cdot \sin(2\pi n! e) = 2\pi \).

**Proof.** Indeed, \( \lim_{n \to +\infty} n \cdot \sin(2\pi n! e) = \lim_{n \to +\infty} n \cdot \sin(2\pi n! \sum_{m=0}^{+\infty} \frac{1}{m!}) \)

\[
= \lim_{n \to +\infty} n \cdot \sin(2\pi \sum_{m=0}^{n} \frac{n!}{m!} + 2\pi \sum_{m=n+1}^{+\infty} \frac{n!}{m!}),
\]

we put \( a_n = \sum_{m=0}^{n} \frac{n!}{m!} \in \mathbb{N}^* \) and \( b_n = \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \to 0 \) and \( nb_n \to 1 \) when \( n \to +\infty \),

then \( \lim_{n \to +\infty} n \cdot \sin(2\pi n! e) = \lim_{n \to +\infty} n \cdot \sin(2\pi a_n + 2\pi b_n) \)

\[
= \lim_{n \to +\infty} n \cdot \sin(2\pi b_n) = \lim_{n \to +\infty} n \cdot 2\pi b_n \cdot \frac{\sin(2\pi b_n)}{2\pi b_n}
\]

\[
= \lim_{n \to +\infty} 2\pi \cdot nb_n \cdot \frac{\sin(2\pi b_n)}{2\pi b_n} = 2\pi \cdot 1 \cdot 1 = 2\pi.
\]

**Main Theorem 2.** \( e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \pi \in \mathbb{R} \setminus \mathbb{Q} \).

*In other words, \( e \) and \( \pi \) both are irrational numbers.*

**Proof.** An argument by contradiction. First, we prove that \( e \) is irrational. Suppose that \( e \in \mathbb{Q} \), and as \( e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( e = \frac{p}{q} \) and \( p \wedge q = 1 \). Then, \( \lim_{n \to +\infty} n \cdot \sin(2\pi n! e) = \lim_{n \to +\infty} n \cdot \sin(2\pi n! \frac{p}{q}) \),

we put \( a_n = \frac{p}{q} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( n \cdot \sin(2\pi a_n) = 0 : n \geq q \),

this implies that \( \lim_{n \to +\infty} n \cdot \sin(2\pi a_n) = 0 \), and we get a contradiction according to **Theorem 2**. Thus, \( e \) is an irrational number. Another proof presented by Dimitris Koukolopoulos and was found by Fourier in 1815 is available at [3, Théorème 15.2]. Second, we prove that \( \pi \) is irrational. A simple proof that \( \pi \) is irrational made by Ivan Niven in 1947 is available at [4] and Lambert’s proof of the irrationality of \( \pi \) in 1760 is available at [5].

The sine function (or \( \sin(x) \)) is defined, continuous, odd and \( 2\pi \)-periodic on \( \mathbb{R} \) and \( \forall \theta \in \mathbb{R} \) we have \( \sin(\theta) = 0 \iff \theta \in \{k\pi : k \in \mathbb{Z}\} \). For more details about sine function and its properties, we refer the reader and our students to [6, page 101].
\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers} \]

**Theorem 3.** We have

\[
\lim_{n \to +\infty} \sin(n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}) = 0
\]
\[
\lim_{n \to +\infty} \sin(n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}) = 0
\]
\[
\lim_{n \to +\infty} \sin (n! \pi e - \pi \sum_{m=0}^{n} \frac{n!}{m!}) = 0
\]
\[
\lim_{n \to +\infty} \sin (n! pe - p \sum_{m=0}^{n} \frac{n!}{m!}) = 0
\]

**Proof.** First,

\[
\lim_{n \to +\infty} \sin(n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}) = \lim_{n \to +\infty} \sin(n! \pi - n! e + \sum_{m=0}^{n} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin(n! \pi - \sum_{m=0}^{+\infty} \frac{n!}{m!} + \sum_{m=0}^{n} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin(n! \pi - \sum_{m=n+1}^{+\infty} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} -\sin(\sum_{m=n+1}^{+\infty} \frac{n!}{m!}) = -\sin(0) = 0.
\]

Second,

\[
\lim_{n \to +\infty} \sin(n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}) = \lim_{n \to +\infty} \sin(n! \pi + n! e - \sum_{m=0}^{n} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin(n! \pi + \sum_{m=0}^{+\infty} \frac{n!}{m!} - \sum_{m=0}^{n} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin(n! \pi + \sum_{m=n+1}^{+\infty} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin(\sum_{m=n+1}^{+\infty} \frac{n!}{m!}) = \sin(0) = 0.
\]

Third,

\[
\lim_{n \to +\infty} \sin (n! \pi e - \pi \sum_{m=0}^{n} \frac{n!}{m!}) = \lim_{n \to +\infty} \sin (\pi \sum_{m=0}^{+\infty} \frac{n!}{m!} - \pi \sum_{m=0}^{n} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin (\pi \sum_{m=n+1}^{+\infty} \frac{n!}{m!}) = \sin(0) = 0.
\]

Fourth, let \( p \in \mathbb{N}^* \) we have

\[
\lim_{n \to +\infty} \sin (n! pe - p \sum_{m=0}^{n} \frac{n!}{m!}) = \lim_{n \to +\infty} \sin (p \sum_{m=0}^{+\infty} \frac{n!}{m!} - p \sum_{m=0}^{n} \frac{n!}{m!})
\]
\[
= \lim_{n \to +\infty} \sin (p \sum_{m=n+1}^{+\infty} \frac{n!}{m!}) = \sin(0) = 0.
\]
Main Theorem 3. \( \pi - e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \pi + e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \pi e \in \mathbb{R} \setminus \mathbb{Q} \). In other words, \( \pi - e \), \( \pi + e \), \( \pi e \) and \( \frac{\pi}{e} \) all are irrational numbers.

Proof. An argument by contradiction. First, suppose that \( \pi - e \in \mathbb{Q} \), and as \( \pi - e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi - e = \frac{p}{q} \) and \( p \wedge q = 1 \).

We recall that, \( \forall n \in \mathbb{N}^* : n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} > 0 \).

Then, \( \lim_{n \to +\infty} \sin \left( n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n! \frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!} \right) \), we put \( a_n = n! \frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k \pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \), this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].

Second, suppose that \( \pi + e \in \mathbb{Q} \), and as \( \pi + e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi + e = \frac{p}{q} \) and \( p \wedge q = 1 \).

We recall that, \( \forall n \in \mathbb{N}^* : n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} > 0 \).

Then, \( \lim_{n \to +\infty} \sin \left( n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n! \frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!} \right) \), we put \( a_n = n! \frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k \pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \), this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].

Third, suppose that \( \pi e \in \mathbb{Q} \), and as \( \pi e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi e = \frac{p}{q} \) and \( p \wedge q = 1 \).

Then, \( \lim_{n \to +\infty} \left( n! \pi e - \pi \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n! \frac{p}{q} - \pi \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} (-1)^{n+1} \sin \left( n! \frac{p}{q} \right) \), we put \( a_n = n! \frac{p}{q} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k \pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \), this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \) and \( \lim_{n \to +\infty} (-1)^{n+1} \sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].
\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers} \]

Fourth, suppose that \( \frac{\pi}{e} \in \mathbb{Q} \), and as \( \frac{\pi}{e} > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \frac{\pi}{e} = \frac{p}{q} \) and \( p \land q = 1 \), then \( q \pi = pe \) and \( \forall n \in \mathbb{N}^* : n!q\pi = n!pe \).

Then, \( \lim_{n \to +\infty} \sin\left( n!pe - p \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin\left( n!q\pi - p \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} -\sin\left( p \sum_{m=0}^{n} \frac{n!}{m!} \right), \)

we put \( a_n = p \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \in \mathbb{N}^*\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}, \)

this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \) and \( \lim_{n \to +\infty} -\sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].

Thus, we conclude that \( \pi - e, \pi + e, \pi e \) and \( \frac{\pi}{e} \), all are irrational numbers.

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**References**


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